ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MCS 352 Complex Analysis

First Midterm
SOLUTIONS
April 1, 2009
16:40-18:10

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	20	105

1. Suppose $z = \left(\dfrac{1+i}{\sqrt{2}}\right)^{2009}$.

a) Find $|z|$.

b) Find $\operatorname{Arg}(z)$. Here $\operatorname{Arg}(z)$ denotes the principal argument.

Solution:

a)

$$|z| = \left(\frac{|1+i|}{\sqrt{2}}\right)^{2009} = \left(\frac{\sqrt{2}}{\sqrt{2}}\right)^{2009} = 1^{2009} = 1.$$

b)

$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$. Thus $z^{2009} = e^{i\frac{2009\pi}{4}}$. To find $\operatorname{Arg}(z)$, we need to modify $\dfrac{2009\pi}{4}$ by an integer multiple of 2π, to get it to be in the interval $(-\pi, \pi]$:

$\dfrac{2009\pi}{4} = \dfrac{(1+2008)}{4}\pi = \dfrac{\pi}{4} + 502\pi$.

We conclude that $\operatorname{Arg}(z) = \dfrac{\pi}{4}$.

2.

a) Find all of the roots for

$$(-27)^{1/3}$$

in rectangular coordinates, exhibit them as vertices of certain regular polygon, and identify the principal root.

b) Find the limit, if it exists,

$$\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2}.$$

Solution:

a) We need only write

$$-27 = 27 \exp\left[i\left(-\pi + 2k\pi\right)\right] \qquad (k = 0, \pm 1, \pm 2, \cdots)$$

to see that the desired roots are

$$c_k = 3 \exp\left[i\left(-\frac{\pi}{3} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2).$$

They lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$, and are equally spaced around that circle every $\frac{2\pi}{3}$ radians, starting with the principal root.

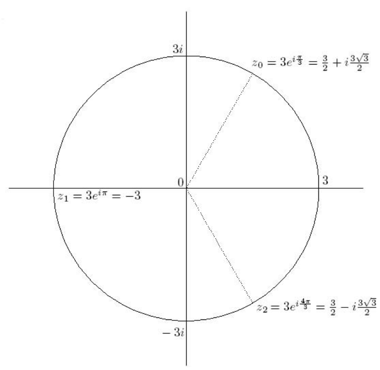
$$c_0 = 3 \exp\left[i\left(-\frac{\pi}{3}\right)\right] = 3\left(\cos\frac{\pi}{3} - i \sin\frac{\pi}{3}\right) = \frac{3 - i3\sqrt{3}}{2}$$

Then $c_1 = 3 \exp\left[i\left(-\frac{\pi}{3} + \frac{2\pi}{3}\right)\right] = 3 \exp\left[i\left(\frac{\pi}{3}\right)\right] = 3\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right) = \frac{3 + i3\sqrt{3}}{2}$ and similarly

$$c_2 = 3 \exp\left[i\left(-\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = 3 \exp\left[i\left(-\frac{\pi}{3} + \frac{4\pi}{3}\right)\right] = 3 \exp\left(i\pi\right) = -3.$$

These roots can of course be written

$$c_0, c_0\omega_3, c_0^2\omega_3^2 \qquad \text{where} \qquad \omega_3 = \exp\left(i\frac{-\pi}{3}\right)$$



b)

$$\begin{aligned}
\lim_{z \to 1+i} \frac{z^2 - 2i}{z^2 - 2z + 2} &= \lim_{z \to 1+i} \frac{z^2 - (1+i)^2}{(z-1)^2 + 1} \\
&= \lim_{z \to 1+i} \frac{(z - 1 - i)(z + 1 + i)}{(z - 1 - i)(z - 1 + i)} = \lim_{z \to 1+i} \frac{(z + 1 + i)}{(z - 1 + i)} \\
&= \frac{1 + i + 1 + i}{1 + i - 1 + i} = \frac{2 + 2i}{2i} = 1 - i
\end{aligned}$$

3. Suppose

$$u(x,y) = x^3 - 3xy^2 + y + 2.$$

a) Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$.
b) Find the analytic function $f(z) = u(x,y) + iv(x,y)$ with $f(1-i) = -1$

Solution:

Since all partial derivatives of u are continuous and u satisfies the Laplace's equation $u_{xx} + u_{yy} = 6x - 6x = 0$, u must be harmonic. To find a harmonic conjugate v,

$$\begin{aligned}
v_y &= u_x = 3x^2 - 3y^2 \\
&\Longrightarrow v = 3x^2y - y^3 + \phi(x) \\
v_x &= -u_y \Longrightarrow 6xy + \phi'(x) = 6xy - 1 \\
&\Longrightarrow \phi'(x) = -1 \\
&\Longrightarrow \phi(x) = -x + C \\
&\Longrightarrow v(x,y) = 3x^2y - y^3 - x + C
\end{aligned}$$

b) Therefore,

$$\begin{aligned}
f(z) &= \left(x^3 - 3xy^2 + y + 2\right) + i\left(3x^2y - y^3 - x + C\right) \\
-1 &= f(1-i) \\
&= (1 - (3)(1) - 1 + 2) + i((3)(1)(-1) - (-1) - 1 + C) \\
&= -1 + i(C - 3) \\
&\Longrightarrow C = 3.
\end{aligned}$$

Hence

$$\begin{aligned}
f(z) &= \left(x^3 - 3xy^2 + y + 2\right) + i\left(3x^2y - y^3 - x + 3\right) \\
&\quad \left(= z^3 - iz + 2 + 3i\right).
\end{aligned}$$

4. Determine where
$$f(z) = 3x^2y^2 - i6x^2y^2$$

(a) is differentiable and,
(b) is analytic.
Solution:
a) Here $u(x,y) = 3x^2y^2$ and $v(x,y) = -6x^2y^2$. We check the Cauchy-Riemann equations:
$$\begin{aligned} u_x &= v_y \longrightarrow 6xy^2 = -12x^2y \longrightarrow 6xy(y+2x) = 0 \\ u_y &= -v_x \longrightarrow 6x^2y = 12xy^2 \longrightarrow 6xy(x-2y) = 0. \end{aligned}$$
These both hold for either $x = 0$ or $y = 0$, i.e., on the real and imaginary axes. In addition, the partials u_x, u_y, v_x, v_y are continuous there. Therefore, $3x^2y^2 - i6x^2y^2$ is *differentiable on the real and imaginary axes.*
b) However, there is no ϵ-neighborhood of a point on the axes which contains only points where the function is differentiable. Therefore it is *nowhere analytic.*

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5. Give all possible values of the following in the form $x + iy$.

a) $\log(-3i)$.

b) $\operatorname{Log} e^{2+i\frac{7}{2}\pi}$.

c) $\dfrac{4i}{(1-i)(2-i)(3-i)}$.

Solution:

a)

$\log(-3i) = \ln|-3i| + i\arg(-i) = \ln 3 + i\left(-\frac{\pi}{2} + 2k\pi\right), k = 0, \pm1, \pm2, \cdots$

b)

$\operatorname{Log} e^{2+i\frac{7}{2}\pi} = \ln\left|e^{2+i\frac{7}{2}\pi}\right| + i\operatorname{Arg}\left(e^{2+i\frac{7}{2}\pi}\right) = \ln\left(e^2\right) + i\left(-\frac{\pi}{2}\right) = 2 - i\frac{\pi}{2}.$

c)

$$\frac{4i}{(1-i)(2-i)(3-i)} = \frac{4i}{(1-3i)(3-i)} = \frac{4i}{-10i} = -\frac{2}{5}.$$

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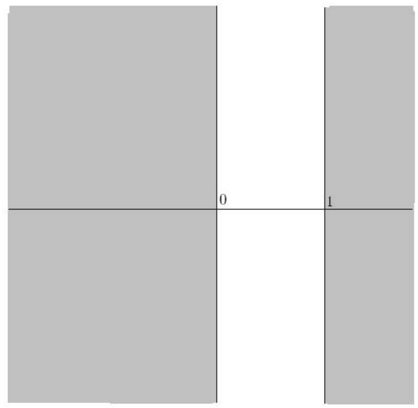
6. Sketch each of the following sets Ω.Determine if Ω is open, closed, connected, domain.
a) $\Omega = \{z = x + iy \mid x \geq 1 \text{ or } x \leq 0\}$.
b) $\Omega = \{z \mid -1/2 \leq x \leq 1/2 \text{ and } |z| \geq 1\}$.

Solution:

a) The set

$$\Omega = \{z = x + iy \mid x \geq 1 \text{ or } x \leq 0\}$$

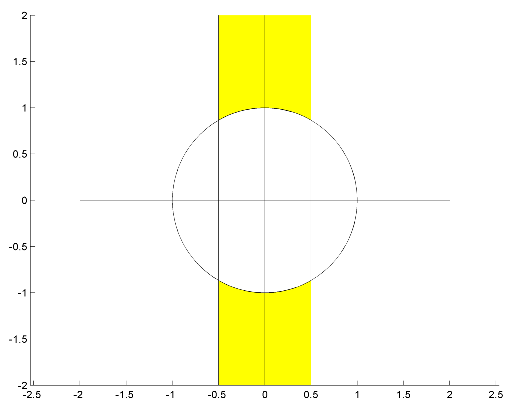
is closed, not open, disconnected and not a domain.



b) The set

$$\Omega = \{z \mid -1/2 \leq x \leq 1/2 \text{ and } |z| \geq 1\}$$

is closed, not open, not connected and not a domain.



ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MCS 352 Complex Analysis
Second Midterm
SOLUTIONS
May 6, 2009
16:40-18:10

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

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GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	15	100

1. Find all values of z satisfying

$$\tan\left(\frac{z}{2}\right) = 2i$$

Solution:

$$\begin{aligned}
\tan\left(\frac{z}{2}\right) &= 2i \Longleftrightarrow \frac{\sin\left(\frac{z}{2}\right)}{\cos\left(\frac{z}{2}\right)} = 2i \\
&\Longleftrightarrow \frac{\left(e^{iz/2} - e^{-iz/2}\right)/2i}{\left(e^{iz/2} + e^{-iz/2}\right)/2} = 2i \\
&\Longleftrightarrow \frac{e^{-iz/2}\left(e^{iz} - 1\right)}{e^{-iz/2}\left(e^{iz} + 1\right)} = -2 \\
&\Longleftrightarrow \frac{e^{iz} - 1}{e^{iz} + 1} = -2 \\
&\Longleftrightarrow 3e^{iz} = -1 \\
&\Longrightarrow e^{iz} = -\frac{1}{3} \Longrightarrow iz = \log\left(-\frac{1}{3}\right) = \ln\frac{1}{3} + i\left(\pi + 2\pi n\right) \\
z_n &= \left(1 + 2n\right)\pi + i\ln 3, \;\; n = 0, \pm 1, \pm 2, \cdots
\end{aligned}$$

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2. Evaluate the integral Res

$$\int_C \operatorname{Log} z \; dz$$

along the contour

$$C = \{z \in \mathbb{C} : z = (1+i)\,t, 1 \le t \le 3\}\,.$$

Solution:
Method I:

$$\begin{aligned}
\int_{1+i}^{3+3i} \operatorname{Log} z \; dz &= [z \operatorname{Log} z]_{z=1+i}^{z=3+3i} - [z]_{z=1+i}^{z=3+3i} \\
&= (3+3i)\operatorname{Log}(3+3i) - (1+i)\operatorname{Log}(1+i) - (2+2i) \\
&= (3+3i)\left(\ln\left(3\sqrt{2}\right) + i\frac{\pi}{4}\right) - (1+i)\left(\ln\left(\sqrt{2}\right) + i\frac{\pi}{4}\right) - (2+2i) \\
&= (1+i)\left(\ln 2 + 3\ln 3 - 2\right) - (1-i)\frac{\pi}{2}
\end{aligned}$$

Method II:

$$\begin{aligned}
\int_C \operatorname{Log} z \; dz &= \int_C \left(\ln|z| + i\frac{\pi}{4}\right)\; dz = \int_1^3 \left(\ln\left(\sqrt{2}t\right) + i\frac{\pi}{4}\right)(1+i)\; dt \\
&= \int_1^3 \left(\ln\left(\sqrt{2}t\right) - \frac{\pi}{4}\right)\; dt + i\int_1^3 \left(\ln\left(\sqrt{2}t\right) + \frac{\pi}{4}\right)\; dt \\
&= (1+i)\int_1^3 \ln\left(\sqrt{2}t\right)\; dt - (1-i)\int_1^3 \frac{\pi}{4}\; dt.
\end{aligned}$$

For the integral $\int_1^3 \ln\left(\sqrt{2}t\right)\; dt$, we have

$$\begin{aligned}
\int_1^3 \ln\left(\sqrt{2}t\right)\; dt &= \int_1^3 \frac{1}{2}\ln 2\; dt + \int_1^3 \ln t\; dt = \ln 2 + [t\ln t]_1^3 - \int_1^3 t\cdot\frac{1}{t}\; dt \\
&= \ln 2 + 3\ln 3 - 2.
\end{aligned}$$

Therefore

$$\int_C \operatorname{Log} z \; dz = (1+i)\left(\ln 2 + 3\ln 3 - 2\right) - (1-i)\frac{\pi}{2}.$$

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3. Evaluate

$$\int_C f(z)\ dz$$

where

$$f(z) = x^2 + iy^2$$

and C is the boundary of the triangle with vertices

$$(0,0),(1,0),(1,1)$$

Solution:

$$\int_C f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz + \int_{C_3} f(z)\ dz$$

where we parametrize the legs as follows

$$\begin{aligned}
\text{on } C_1 &: \quad z = t, \quad f(z) = t^2, \quad 0 \le t \le 1 \\
\text{on } C_2 &: \quad z = 1 + it, \quad f(z) = 1 + it^2, \quad 0 \le t \le 1 \\
\text{on } C_3 &: \quad z = (1-t)(1+i), \quad f(z) = (1-t)^2(1+i), \quad 0 \le t \le 1
\end{aligned}$$

Integating

$$\begin{aligned}
\int_{C_1} f(z)\ dz &= \int_0^1 t^2\ dt = \frac{1}{3} \\
\int_{C_2} f(z)\ dz &= \int_0^1 (1+it)^2\ idt = i - \frac{1}{3} \\
\int_{C_3} f(z)\ dz &= \int_0^1 (1-t)^2(1+i)(-1-i)\ dt = -\frac{2i}{3}
\end{aligned}$$

and therefore,

$$\int_C f(z)\ dz = \frac{1}{3} + i - \frac{1}{3} - \frac{2i}{3} = \frac{i}{3}.$$

---

4. Evaluate the following integrals

(a) $\displaystyle\int_C \frac{z+i}{z^3+2z^2}\,dz,$

where C is the circle $|z-2i|=1$.

(b) $\displaystyle\int_C \frac{e^{iz}}{(z^2+1)^2}\,dz,$

where C is $|z|=3$,

(c) $\displaystyle\int_C \frac{e^{2z}}{z^4}\,dz,$

where C is the counterclockwise contour formed by the line $x=1$ and the parabola $y^2=x+1$.

Solution:

a)

$$\int_C \frac{z+i}{z^3+2z^2}\,dz = 0$$

by the Cauchy-Goursat theorem, since $\dfrac{z+i}{z^3+2z^2}$ is analytic inside and on C.

b)

$$\begin{aligned}
\int_C \frac{e^{iz}}{(z^2+1)^2}\,dz &= \int_C \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz \\
&= \int_{C_1} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz + \int_{C_2} \frac{e^{iz}}{(z+i)^2(z-i)^2}\,dz \\
&= 2\pi i\left[\frac{d}{dz}\frac{e^{iz}}{(z-i)^2}\right]_{z=-i} + 2\pi i\left[\frac{d}{dz}\frac{e^{iz}}{(z+i)^2}\right]_{z=i} \\
&= 2\pi i\left[\frac{(z-i)^2 ie^{iz} - e^{iz}2(z-i)}{(z-i)^4}\right]_{z=-i} + 2\pi i\left[\frac{(z+i)^2 ie^{iz} - e^{iz}2(z+i)}{(z+i)^4}\right]_{z=i} \\
&= 2\pi i\left[\frac{-4ie^{-ii}+4ie^{-ii}}{(-i-i)^4}\right] + 2\pi i\left[\frac{-4ie^{ii}-4ie^{ii}}{(i+i)^4}\right] \\
&= \frac{\pi}{e}
\end{aligned}$$

c) There is only one singularity of the integrand that is $z=0$ and it is inside C.

$$\begin{aligned}
\int_C \frac{e^{2z}}{z^4}\,dz &= \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}\left(e^{2z}\right)\right]_{z=0} \\
&= \left[\frac{2\pi i}{6}2^3 e^{2z}\right]_{z=0} \\
&= \frac{16\pi i}{6} \\
&= \frac{8\pi i}{3}
\end{aligned}$$

---

5. Show that

$$\left| \int_C \frac{z}{z^3 + i} \, dz \right| \leq \frac{3\pi}{13},$$

where C is the circle $|z| = 3$ traversed once.

Solution:

As z traverses the circle

$$|z| = 3$$

once in the positive direction,

$$w = z^3$$

will traverse the circle

$$|w| = 27$$

twice in the positive direction.

The point on this circle that is closest to

$$-i$$

is

$$w = 27,$$

and therefore

$$\max_{|z|=3} \left| \frac{z}{z^3 + i} \right| = \frac{3}{\min |z^3 + i|} = \frac{3}{26}.$$

It follows by the ML- inequality that (by knowing that $L = 2\pi (3) = 6\pi$)

$$\left| \int_C \frac{dz}{z^2 + i} \right| \leq \frac{3}{26} 2\pi (3) = \frac{18\pi}{26} = \frac{9\pi}{13}.$$

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6. Represent the function

$$f(z) = \frac{7z-3}{z(z-1)}$$

by its Laurent series in the domain $0 < |z-1| < 1$.
Solution:

$$\begin{aligned} f(z) &= \frac{7z-3}{z(z-1)} = \frac{3}{z} + \frac{4}{z-1} = \frac{3}{1+(z-1)} + \frac{4}{z-1} \\ &= 3\sum_{n=0}^{\infty} (-1)^n (z-1)^n + \frac{4}{z-1}. \end{aligned}$$

Therefore

$$\frac{7z-3}{z(z-1)} = \sum_{n=-1}^{\infty} c_n (z-1)^n,$$

and so

$$c_{-1} = 4, c_n = 3(-1)^n, n = 0, 1, 2, \cdots.$$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MCS 352 Complex Analysis
Final Exam
SOLUTIONS
June 12, 2009
9:00-11:00
(B-308)

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
21	15	20	25	25	24	105

1. Classify all the singularities (removable, poles, isolated essential, branch points, non-isolated essential) of the following functions.

(a) $f(z) = \dfrac{z}{z^2+1}$

(b) $f(z) = z^3 \sin\left(\dfrac{2}{z}\right)$

(c) $f(z) = \dfrac{1}{z^3} \cos 2z$

Solution:

(a) We factor the denominator to see that *there are simple poles at* $z = \pm i$.

$$\frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)} = \frac{1/2}{z-i} + \frac{1/2}{z+i}$$

(b) We have

$$\begin{aligned} f(z) &= z^3 \sin\left(\frac{2}{z}\right) \\ &= z^3 \left(\frac{2}{z} - \frac{1}{3!}\left(\frac{2}{z}\right)^3 + \frac{1}{5!}\left(\frac{2}{z}\right)^5 - \frac{1}{7!}\left(\frac{2}{z}\right)^7 + \cdots\right) \\ &= 2z^2 - \frac{1}{3!}2^3 + \frac{1}{5!}\frac{2^5}{z^2} - \frac{1}{7!}\frac{2^7}{z^4} + \cdots . \end{aligned}$$

Since the principal part has infinitely many nonzero terms, we have $z_0 = 0$ is an isolated *essential singular point* of the given function.

(c) Since

$$\begin{aligned} f(z) &= \frac{1}{z^3}\cos 2z \\ &= \frac{1}{z^3}\left(1 - \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} - \frac{(2z)^6}{6!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{2}{z} + \frac{2^4}{4!}z - \frac{2^6}{6!}z^3 + \cdots , \end{aligned}$$

we see that $z_0 = 0$ is a pole of order $m = 3$ of the given function.

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2. Evaluate the integral

$$\int_C z^3 \cos\left(\frac{1}{z}\right)\ dz$$

where C is the positively oriented circle $|z+1+i|=4$.
Solution: We know that

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \cdots$$

Then from the substitution $w = \dfrac{1}{z}$, we get

$$z^3 \cos\left(\frac{1}{z}\right) = z^3 - \frac{1}{2!}z + \frac{1}{4!z} - \frac{1}{6!z^3} + \cdots .$$

Therefore

$$\operatorname*{Res}_{z=0} z^3 \cos\left(\frac{1}{z}\right) = \frac{1}{4!}.$$

Hence

$$\int_C z^3 \cos\left(\frac{1}{z}\right)\ dz = 2\pi i \left[\operatorname*{Res}_{z=0} z^3 \cos\left(\frac{1}{z}\right)\right] = 2\pi i \frac{1}{4!} = \frac{\pi i}{12}.$$

.

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3. Short answer.
(a) Find $\operatorname*{Res}_{z=0} \frac{\cos z}{z^3}$
(b) Give an example of a function $f(z)$ which has an essential isolated singularity at $z = 3$.
(c) Find $\operatorname*{Res}_{z=0} \frac{e^{iaz} - e^{ibz}}{z^3}$ where $a, b \in \mathbb{R}$ and $a \neq b$.
Solution:
a) Since

$$\begin{aligned} f(z) &= \frac{\cos z}{z^3} \\ &= \frac{1}{z^3}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{2!z} + \frac{z}{4!} - \frac{z^3}{6!} + \cdots \end{aligned}$$

we have f has a pole of order $m = 3$ at $z_0 = 0$.
Therefore

$$\operatorname*{Res}_{z=0} \frac{\cos z}{z^3} = -\frac{1}{2}.$$

(b) Let

$$f(z) = e^{1/(z-3)}.$$

Then

$$\begin{aligned} f(z) &= e^{1/(z-3)} \\ &= 1 + \frac{1}{z-3} + \frac{1}{2!}\frac{1}{(z-3)^2} + \frac{1}{3!}\frac{1}{(z-3)^3} + \cdots \end{aligned}$$

has an infinite number of nonzero terms in its principal part. We know that $z_0 = 3$ is the only singular point of the given function.
(c)

$$\begin{aligned} \frac{e^{iaz} - e^{ibz}}{z^3} &= \frac{1}{z^3}\left[\left(1 + (iaz) + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \cdots\right) - \left(1 + (ibz) + \frac{(ibz)^2}{2!} + \frac{(ibz)^3}{3!} + \cdots\right)\right] \\ &= \frac{i(a-b)}{1!z^2} + \frac{i^2(a^2-b^2)}{2!z^1} + \frac{i^3(a^3-b^3)}{3!} + + \frac{i^4(a^4-b^4)}{4!} \cdots \end{aligned}$$

Hence

$$\operatorname*{Res}_{z=0} \frac{e^{iaz} - e^{ibz}}{z^3} = -\frac{a^2-b^2}{2!}$$

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4. Evaluate the integral

$$\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\, d\theta.$$

Solution:

$$\begin{aligned} I &= \int_{|z|=1} \frac{\frac{1}{2}(z+1/z)}{5+\frac{3}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{z^2+1}{(10z+3z^2+3)\,z}\, dz \\ &= \frac{1}{3i}\int_{|z|=1} \frac{z^2+1}{\left(z+\frac{10}{3}z^2+1\right) z}\, dz \\ &= \end{aligned}$$

The singularities are at $z=0, z=-\dfrac{1}{3}$ and $z=-3$.
So

$$f(z) := \frac{z^2+1}{\left(z+\frac{10}{3}z^2+1\right) z}$$

and

$$I = \frac{2\pi i}{3i}\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-\frac{1}{3}} f(z)\right],$$

where

$$\operatorname*{Res}_{z=0} f(z) = \left.\frac{\frac{z^2+1}{z^2+\frac{10}{3}z+1}}{1}\right|_{z=0} \left[\frac{\frac{z^2+1}{z^2+\frac{10}{3}z+1}}{1}\right]_{z=0} = 1$$

and

$$\operatorname*{Res}_{z=-\frac{1}{3}} f(z) = \left[\frac{z+1/z}{2z+10/3}\right]_{z=-\frac{1}{3}} = \frac{-1/3-3}{-2/3+10/3} = \frac{-10/3}{8/3} = \frac{-5}{4}.$$

Hence

$$I = \frac{2\pi}{3}\left(1-\frac{5}{4}\right) = -\frac{\pi}{6}$$

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5. Use residues to evaluate the improper integral

$$\int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\, dx.$$

Solution:
We introduce

$$f(z) = \frac{z}{(z^2+1)(z^2+4)}.$$

and observe that the product $f(z)e^{iz}$ is analytic everywhere on and above the real axis except at the points $z=i$ and $z=2i$
Then

$$I := \int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\, dx = \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty f(z)e^{iz}\, dz$$

We integrate $f(z)e^{iz}$ once in the positive direction along the contour C,

$$\begin{aligned}\frac{1}{2}\operatorname{Im}\int_C f(z)e^{iz}\, dz &= \frac{1}{2}(2\pi i)\left[\operatorname*{Res}_{z=i} f(z)e^{iz} + \operatorname*{Res}_{z=2i} f(z)e^{iz}\right] \\ &= \pi\left(\frac{ie^{-1}}{2i(-1+4)} + \frac{2ie^{-2}}{(-4+1)4i}\right) \\ &= \frac{\pi(e^{-1}-e^{-2})}{6}.\end{aligned}$$

With the help of ML-inequality, we have

$$\left|\int_{C_R} f(z)e^{iz}\, dz\right| \le \max_{z\in C_R}\left|f(z)e^{iz}\right|\pi R \le \frac{R}{(R^2-1)(R^2-4)}\pi R \longrightarrow 0 \text{ as } R\to\infty.$$

Therefore, we get

$$\begin{aligned}\frac{\pi(e^{-1}-e^{-2})}{6} &= \frac{1}{2}\operatorname{Im}\int_C f(z)e^{iz}\, dz = \frac{1}{2}\operatorname{Im}\int_{C_R} f(z)e^{iz}\, dz + \frac{1}{2}\operatorname{Im}\int_{I_R} f(z)e^{iz}\, dz \\ &\longrightarrow \frac{1}{2}\operatorname{Im}\int_{-\infty}^\infty \frac{xe^{ix}}{(x^2+1)(x^2+4)}\, dz = I \quad \text{as} \quad R\to\infty.\end{aligned}$$

Hence

$$\int_0^\infty \frac{x\sin x}{(x^2+1)(x^2+4)}\, dx = \frac{\pi(e^{-1}-e^{-2})}{6}.$$

---

6. Determine the number of zeros, counting multiplicities of the polynomials
(a)

$$5z^4 + z^3 + z^2 - 7z + 14 = 0$$

inside the circle $|z| < 3$.
(b)

$$z^5 + 7z^2 + 3z + 2$$

inside the circle $|z| < 1$.
Solution:
The polynomial can be written as the sum of

$$f(z) = 5z^4 \text{ and } g(z) = z^3 + z^2 - 7z + 14.$$

Observe that when z is on the circle $|z| = 3$,

$$|f(z)| = 5|z|^4 = (5)\left(3^4\right) = 405$$

and

$$|g(z)| = \left|z^3 + z^2 - 7z + 14\right| \leq |z|^3 + |z|^2 + 7|z| + 14 = 3^3 + 3^2 + 7(3) + 14 = 71.$$

Since

$$|f(z)| > |g(z)|$$

on the circle and since $f(z)$ has four zeros, counting multiplicities, inside it, Rouche's Theorem tells us that the sum

$$f(z) + g(z) = 5z^4 + z^3 + z^2 - 7z + 14$$

also has *four zeros, counting multiplicities, inside the circle.*
b)
The polynomial

$$z^5 + 7z^2 + 3z + 2$$

can be written as the sum of

$$f(z) = 7z^2 + 2 \text{ and } g(z) = z^5 + 3z^2.$$

Observe that when z is on the circle $|z| = 1$,

$$|f(z)| \geq \left|7|z|^2 - |2|\right| = 7 - 2 = 5$$

and

$$|g(z)| = \left|z^5 + 3z\right| \leq |z|^5 + 3|z| = 1^5 + 3(1) = 4.$$

Since

$$|f(z)| \geq 5 > 4 \geq |g(z)|$$

on the circle and since $f(z)$ has two zeros, inside it, Rouche's Theorem tells us that the sum

$$f(z) + g(z) = z^5 + 7z^2 + 3z + 2$$

also has *two zeros, counting multiplicities, inside the circle.*

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MATH 352 Complex Analysis

Practice Problems

For what values of $z \in \mathbb{C}$ do the following functions have derivatives?

1. $1 + iy$

2. z^6

3. z^{-5}

4. $y + ix$

5. $xy(1+i)$

6. $x^2 + iy$

7. $x + i|y|$

8. $e^x + ie^{2y}$

9. $y - 2xy + i\left(-x + x^2 - y^2\right)$

10. $(x-1)^2 + iy^2 + z^2$

11. $f(z) = \cos x - i \sinh y$

12. $f(z) = \dfrac{1}{z}$ for $|z| > 1$

---

13. Find two functions of z neither of which has a derivative anywhere in the complex plane, but whose nonconstant sum has a derivative everywhere.

---

14. Let $f(z) = u(x,y) + iv(x,y)$. Assume that the second derivative $f''(z)$ exists. Show that

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} \quad \text{and} \quad f''(z) = -\frac{\partial^2 u}{\partial y^2} - i\frac{\partial^2 v}{\partial y^2}$$

15.

(a) Find the derivative of $f(z) = \dfrac{1}{z} + (x-1)^2 + ixy$ at any points where the derivative exists.

(b) Where is this function analytic?

16.

(a) Where is the function $f(z) = z^3 + z^2 + 1$ is analytic?

(b) Find an expression for $f'(z)$ and give the derivative at $1+i$.

17.

(a) Where does the function $f(z) = z^2 + (x-1)^2 + i(y-1)^2$

(b) Where is this function analytic? Expalin:

(c) Derive a formula that will yield the derivative of this expression at points where the derivative exists, and use this formula to find the numerical value of the derivative at the point $z = 1+i$

18.

(a) Show that $f(z) = \dfrac{1}{e^{2x}\cos 2y + ie^{2x}\sin 2y}$ is an entire function.

(b) Obtain an explicit expression for the derivative of this function and find $f'\left(1 + i\dfrac{\pi}{4}\right)$

19.
(a) Show that $z\left[\cos x \cosh y - i \sin x \sinh y\right]$ is an entire function.
(b) Find $f'(i)$.

20. Find the derivative at $z = \pi + 2i$ of $f(z) = \left[\cos x \cosh y - i \sin x \sinh y\right]^5$.

21.(a) Where is the function $f(z) = \dfrac{z}{(1+iz)^4}$ analytic?
(b) Find $f'(-i)$.

22. Find the following limits

$$\text{(a)} \lim_{z \to i} \frac{(z-i) + (z^2+1)}{z^2 - 3iz - 2}, \quad \text{(b)} \lim_{z \to i} \frac{z^3 + i}{(z^2+1)\,z}$$

23. If $g(z)$ has a derivative at z_0 and $h(z)$ does not have a derivative at z_0, explain $g(z)+h(z)$ cannot have a derivative at z_0.

24. Find two functions, each of which is nowhere analytic, but whose sum is an entire function. Thus the sum of two nonanalytic functions can be analytic.

25. Suppose $f(z) = u+iv$ is analytic. Under what circumstances will $g(z) = u-iv$ be analytic?

26. Show that the following functions are nowhere analytic:

$$\text{(a)} \left(\overline{z}+1\right)^2, \quad \text{(b)} \left(\overline{z}\right)^3.$$

27. Where in the complex plane will the function

$$\phi(x,y) = \sin(xy)$$

satisfy Laplace's equation? Is this a harmonic function?

28. Consider the function

$$\phi(x,y) = e^{ky} \sin(mx).$$

Assuming this function is harmonic throughout the complex plane, what must be the relationship between $k \in \mathbb{R}$ and $m \in \mathbb{R}$? Assume $m \neq 0$.

29. Find the value of the integer n if $x^n - y^n$ is harmonic.

30. Putting $z = x+iy$, show that $Im\left(\dfrac{1}{z}\right)$ is harmonic throughout any domain not containing $z = 0$.

31. Putting $z = x+iy$, show that $Re\left(z^3\right)$ is harmonic in any domain.

32. Find two values of k such that $\cos x\left[e^y + e^{ky}\right]$ is harmonic.

33. If $g(x)$ is harmonic, $g(0) = 0, g'(0) = 1$ find $g(x)$.

34.

a) Consider $\phi(x,y) = x^3y - y^3x + y^2 - x^2 + x$. Show that this can be the real part or the imaginary part of an analytic function.
b) Assuming the preceding is the real part of an analytic function, find the imaginary part.
c) Assuming that $\phi(x,y)$ is the imaginary part of an analytic function find the real part. Compare your answer to that in part (b).
d) If $\psi(x,y) + iv(x,y)$ is an analytic function and if $u(x,y) + i\phi(x,y)$ is also analytic, where $\phi(x,y)$ is an arbitrary harmonic function, prove that, neglecting constants, $u(x,y)$ and $v(x,y)$ must be negatives of each other. Is this confirmed by your answers to parts b) and c)?

---

35. Suppose that $f(z) = u + iv$ is analytic and that $g(z) = v + iu$ is also. Show that u and v must be constants.

---

36. Find the harmonic conjugate of $e^x \cos y + e^y \cos x + xy$.

---

37. Find the harmonic conjugate of $\tan^{-1}\left(\dfrac{x}{y}\right)$ where $-\pi < \tan^{-1}\left(\frac{x}{y}\right) \leq \pi$.

---

38. Show if $u(x,y)$ and $v(x,y)$ are harmonic, that $u+v$ must be harmonic but that uv need not be harmonic. Is $e^u e^v$ harmonic?

If $v(x,y)$ is the harmonic conjugate of $u(x,y)$, show that the following are harmonic functions.
39. uv
40. $e^u \cos v$
41. $\sin u \cosh v$

Express each of the following in the form $a + ib$ where $a, b \in \mathbb{R}$
42. $e^{\frac{1}{2}+2i}$
43. $e^{\frac{1}{2}-2i}$
44. e^{-i}
45. $e^{\frac{1}{2}+2i}e^{\frac{1}{2}-2i}$
46. $e^{(-i)^7}$
47. $\left(e^{-i}\right)^7$
48. $e^{\frac{1}{1+i}}$
49. $e^{(-i)^7}$
50. $e^{e^{-i}}$
51. $e^{e^{-i}}$
52. $e^{i \arctan 1}$
53. $e^{(-2)^{1/2}}$
54. $\left(e^{-2}\right)^{1/2}$

---

55. Find all solutions of $e^z = e$ by equating corresponding parts (reals and imaginaries) on both sides of the equation.

State the domain of analyticity of each of the following functions. Find the real and imaginary parts. Find $f'(z)$ in terms of z.
56. $f(z) = e^{iz}$
57. $f(z) = e^{1/z}$

Find the following limits:
58. $\displaystyle\lim_{z \to i} \frac{z - i}{e^z - e^i}$

59. $\lim_{\theta \to \pi} \dfrac{1+e^{i\theta}}{1-e^{2i\theta}}$ where θ is real.

For the following closed regions, R, where does the given $|f(z)|$ achieve its maximum and minimum values, and what are these values?

60. R is $|z-1-i| \leq 2$ and $f(z) = e^z$

61. R is $|z| \leq 1$ and $f(z) = e^{(z)^2}$

---

62. $\sin(2+3i)$

63. $\cos(-2+3i)$

64. $\tan(2+3i)$

65. $(\sin i)^{1/2}$

66. $\sin\left(i^{1/2}\right)$

67. $\sin\left(e^i\right)$

68. $\cos(2i \arg(2i))$

69. $\sin(\cos(1+i))$

70. $\tan\left(i \arg\left(1+\sqrt{3}i\right)\right)$

71. $\arg(\tan i)$

72. $e^{i\cos i} + e^{-i\cos i}$

73. $\tan\left(i \arg\left(1+\sqrt{3}i\right)\right)$

---

74. Prove the identity $\sin^2 z + \cos^2 z = 1$ by the following two methods:

(a) Use the definitions of sine and cosine.

(b) Use $\cos^2 z + \sin^2 z = (\cos z + i\sin z)(\cos z - i\sin z)$.

Prove the following.

75. $\cos^2 z = \dfrac{1}{2} + \dfrac{1}{2}\cos 2z$

76. $\sin(z+2\pi) = \sin z$

77. $\cos(z+2\pi) = \cos z$

---

78. Show that the equation $\sin z = 0$ has solutions in the complex z-plane only where $z = n\pi$ and $n = 0, \pm1, \pm2, \cdots$. Thus $\cos z$ and $\sin z$ has zeros only on the real axis.

---

79. Let $f(z) = \sin\left(\dfrac{1}{z}\right)$.

(a) Express this function in the form $u(x,y) + iv(x,y)$. Where in the complex plane is this function analytic?

(b) What is $f'(z)$? Where in the complex plane is $f'(z)$ analytic?

---

80. Show that $|\cos z| = \sqrt{\sinh^2 y + \cos^2 x}$.

Hint: Recall that $\cosh^2\theta - \sinh^2\theta = 1$.

---

81. Show that $|\sin z| = \sqrt{\sinh^2 y + \sin^2 x}$.

---

82. Show that $|\sin z|^2 + |\cos z|^2 = \sinh^2 y + \cosh^2 y$

83. Show that

$$\tan z = \frac{\sin(2x) + i\sinh(2y)}{\cos(2x) + \cosh(2y)}.$$

Find all values of the logarithm of each of the following numbers and state in each case the principal value. Put answers in the form $a+ib$.

84. e

85. $1-i$

86. $-ie^2$

87. $-\sqrt{3}+i$

88. e^i

89. e^{1+4i}

90. $\left(-\sqrt{3}+i\right)^4$

91. $e^{\log(i\sinh 1)}$

92. e^{e^i}

93. $Log\left(Log\left(i\right)\right)$

---

94. For what values of z is the equation $Logz = \overline{Log\overline{z}}$ true?

Give solutions to the following equations in Cartesian form.

95. $Logz = 1+i$

96. $(Logz)^2 + Logz = -1$

Use logarithms to find all solutions of the following equations.

97. $e^z = e$

98. $e^z = e^{-z}$

99. $e^z = e^{iz}$

100. $(e^z-1)^2 = e^{2z}$

101. $(e^z-1)^2 = e^z$

102. $(e^z-1)^3 = 1$

103. $e^{4z}+e^{2z}+1=0$

104. $e^{e^z}=1$

Is the set of values of $\log i^2$ the same as the set of $2\log i$? Explain.

105. $e^{e^{e^z}}=1$

Prove that if θ is real, then

106. $Re\left[\log\left(1+e^{i\theta}\right)\right] = Log\left|2\cos\theta\left(\frac{\theta}{2}\right)\right|$ if $e^{i\theta}\neq 1$.

107. $Re\left[\log\left(re^{i\theta}-1\right)\right] = \frac{1}{2}Log\left(1-2r\cos\theta+r^2\right)$ if $r\geq 0$ and $re^{i\theta}\neq 1$.

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Practice Problems-2

For what values of $z \in \mathbb{C}$ do the following functions have derivatives?
1. If $f(z)$ has a branch point at z_0, does $1/f(z)$ necessarily have a branch point at z_0? Expalin.

---

2. If two functions each have a branch point at z_0, does their product necessarily have a branch point at z_0? Illustrate with with an example.

Suppose a branch of $\left(z^2-1\right)^{1/3}$ equals -1 when $z=0$. There are branch cuts defined by $y=0, |x| \geq 1$. What value does this branch assume at the following points?
3. i
4. $-i$
5. $1+i$

---

6. Consider the multi-valued function $z^{1/3}(z-1)^{1/3}$.
a) Show, by encircling each of them, that this function has branch points at $z=0$ and $z=1$.
b) Show that the line segment $y=0, 0 \leq x \leq 1$, which connects the two branch points, cannot serve as a branch cut for defining a branch of this function.
c) State suitable branch cuts for defining a branch.

---

7. A branch of $\left(z^2-1\right)^{1/2}$ is defined by means of a branch cut consisting of the line segment $-1 \leq x \leq 1, y=0$.
a) Prove that this function has branch points at $z=\pm 1$.
b) Show that if we encircle these branch points by moving once around the ellipse $\dfrac{x^2}{2}+y^2=1$, we do not pass to a new branch of the function.

A branch of $(z-1)^{2/3}$ is defined by means of the branch cut $x=1, y \leq 0$. If this branch $f(z)$ equals 1 when $z=0$, what is the value of $f(z)$ and $f'(z)$ at the following points.
8. $1+8i$
9. -1
10. $-i$
11. $1/2-i/2$

A certain branch of $z^{1/2}$ is defined by means of the branch cut $x=0, y \geq 0$. If this branch has the value -3 when $z=9$, what values does $f(z)$ assume at the following points? Also, state the value of $f'(z)$ at each point.
12. 1
13. $-9i$
14. $-1+i$
15. $-9+9i\sqrt{3}$

---

16. Let $f(z)=10^{\left(z^3\right)}$. This function is evaluated such that $f'(z)$ is real when $z=1$. Find $f'(1+i)$. Where in the complex plane is $f(z)$ analytic?

17. Find $f'(i)$ if $f(z) = i^{(e^z)}$ and principal values are used.

---

18. Find $(d/dz)\, 2^{\cosh z}$ using principal values. Where in the complex z-plane is $2^{\cosh z}$ analytic?

---

19. Let $f(z) = z^{\sin z}$, where the principal branch is used. Find $f'(i)$.

Let $f(z) = z^z$, where the principal branch is used. Evaluate the following.
20. $f'(z)$
21. $f'(i)$.

Using the principal branch of the function, evaluate the following.
21. $f'(i)$ if $f(z) = z^{2+i}$
22. $f'(128i)$ if $f(z) = z^{8/7}$
23. $f'(-8i)$ if $f(z) = z^{1/3+i}$.

---

24. What is the flaw in this argument?

$$e^{i\theta} = \left(e^{i\theta}\right)^{2\pi/2\pi} = \left(e^{i2\pi}\right)^{\theta/2\pi} = (1)^{\theta/2\pi} = 1.$$

---

25. Show that
a) if n is an integer, then z^n has only one value;
b) if n and m are integers and n/m is an irreducible fraction, then $z^{n/m}$ has just m values
c) if c is an irrational number, then z^c has an infinity of different values;
d) if $\operatorname{Im} c \neq 0$, then z^c has an infinity of different values.

---

26. Show that all possible values of z^i are real if $|z| = e^{n\pi}$, where n is any integer.

Find all values of the following in the form $a + bi$ and state the principal value.
27. 1^{2i}
28. i^{-1}
29. $\left(\sqrt{3} + i\right)^{1-2i}$
30. $\left(e^i\right)^i$
31. $e^{\left(e^i\right)}$
32. $(1.1)^{1.1}$
33. $\pi^{i/2}$
34. $(\operatorname{Log} i)^{\pi/2}$
35. $(1 + i\tan 1)^{\sqrt{2}}$
36. $\left(\sqrt{2}\right)^{1+i\tan 1}$.

37.
a) Show that $\operatorname{Log}(\operatorname{Log} z)$ is analytic in the domain consisting of the z-plane with a branch cut along the line $y = 0, x \leq 1$.
b) Find $\dfrac{d}{dz}(\operatorname{Log}(\operatorname{Log} z))$ within the domain of analyticity found in part a).

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Practice Problems-2

1. Show that

$$\int_{0.1}^{1} xy\ ds = \int_{0}^{1} x\left(1-x^2\right)\sqrt{1+4x^2}\ dx = \int_{0}^{1} y\sqrt{5/4-y}\ dy.$$

Hint: Recall that $ds = \pm\sqrt{1+(dy/dx)^2}dx = \pm\sqrt{1+(dx/dy)^2}dy$, and that $ds \geq 0$. Evaluate the contour integral by integrating either on x or y. One is slightly easier.

Let C be that portion of the curve $y = x^2$ lying between $(0,0)$ and $(1,1)$. Let $F(x,y) = x+y+1$. Evaluate these integrals along C.

2. $\displaystyle\int_{(0,0)}^{(1,1)} F(x,y)\ dx$

3. $\displaystyle\int_{(0,0)}^{(1,1)} F(x,y)\ dy$

Let C be that portion of the curve $x^2+y^2 = 1$ lying in the first quadrant. Let $F(x,y) = x^2y$. Evaluate these integrals along C.

4. $\displaystyle\int_{(0,1)}^{(1,0)} F(x,y)\ dx$

5. $\displaystyle\int_{(0,1)}^{(1,0)} F(x,y)\ dy$

6. $\displaystyle\int_{(0,1)}^{(1,0)} F(x,y)\ ds$

---

7. Show that

$$\int_{(0,-1)}^{(0,1)} y\ dx = -\frac{\pi}{2}$$

The integration is along that portion of the circle $x^2+y^2 = 1$

---

8. Evaluate

$$\int_{(3,0)}^{(0,-1)} x\ dy$$

along the portion of the ellipse $x^2+9y^2 = 9$ lying in the first, second, and third quadrants.

---

9. Find the value of

$$\int_{0+i0}^{1+i} (z+1)\ dz$$

taken along the contour $y = x^2$. $y - 2xy + i\left(-x+x^2-y^2\right)$

---

10. Find the value of

$$\int_{0+i0}^{1+2i} z\, dz$$

performed along the contour $y = 2x\,(2-x)$.

---

11. Find the value of

$$\int_{0+i0}^{1+2i} dz$$

performed along the contour $y = 2x\,(2-x)$.

Evaluate

$$\int_{i}^{1} \overline{z}\, dz$$

along the contour C, where C is
12. the straight line segment lying along $x + y = 1$;
13. the parabola $y = (1-x)^2$;
14. the portion of the circle $x^2 + y^2 = 1$ in the first quadrant. Compare the answers Exercises 12,13, and 14.

---

15. Evaluate

$$\int e^{z}\, dz$$

(a) from $z = 0$ to $z = 1$ along the line $y = 0$;
(b) from $z = 1$ to $z = 1 + i$ along the line $x = 1$;
(c) from $z = 1 + i$ to $z = 0$ along the line $y = x$. Verify that the sum of your three answers is zero.

Perform the following integrations

16. $\displaystyle\int_{1}^{-1} \frac{1}{z}\, dz$ along $|z| = 1$, upper half plane.

17. $\displaystyle\int_{1}^{-1} \frac{1}{z}\, dz$ along $|z| = 1$, lower half plane.

18. $\displaystyle\int_{1}^{i} \overline{z}^{4}\, dz$ along $|z| = 1$, first quadrant.

---

19. Evaluate

$$\int_{2}^{i} \overline{z}\, dz$$

along the portion of the ellipse $\dfrac{x^2}{4} + y^2 = 1$ in the first quadrant..

---

20. Evaluate

$$\int_{1+i}^{2+4i} z^2\, dz$$

along the parabola $y = x^2$.

21.
a) Parametrize the shorter of the two arcs lying along $(x-1)^2 + (y-1)^2 = 1$ that connects $z = 1$ with $z = i$.

b) Evaluate

$$\int_1^i \overline{z}\ dz$$

along the arc of **a)** using the parametrization you have found.

---

22. Consider

$$I = \int_{0+i0}^{2+i} e^{z^2} dz$$

taken along the line $x = 2y$. Without actually doing the integration, show that $|I| \le \sqrt{5}e^3$.

---

23. Consider

$$I = \int_1^i \frac{1}{\overline{z}^4}\ dz$$

taken along the line $x + y = 1$. Without actually doing the integration, show that $|I| \le 4\sqrt{2}$.

---

24. Consider

$$I = \int_i^1 e^{i\,\mathrm{Log}\,\overline{z}}\ dz$$

taken along the parabola $y = 1 - x^2$. Without doing the integration, show that $|I| \le 1.479e^{\pi/2}$.

25.

a) Let $g(t)$ be a complex function of the real variable t. Show that for $b > a$ we have

$$\left| \int_a^b g(t)\ dt \right| \le \int_a^b |g(t)|\ dt.$$

b) Prove that

$$\left| \int_0^1 \sqrt{t}e^{it}\ dt \right| \le \frac{2}{3}.$$

26.

a) Let C be an arbitrary simple closed contour. Use Green's theorem to find a simple interpretation of the integral

$$\frac{1}{2}\int_C (-y\ dx + x\ dy)$$

b) Consider

$$\int_C (\cos y\ dx + \sin x\ dy)$$

performed around the square C with corners at $(0,0), (0,1), (1,0), (1,1)$. Evaluate this integral by doing an equivalent integral over the area enclosed by the square.

c) Suppose you know the area enclosed by a simple closed contour C. Show with the aid of Green's theorem that you can easily evaluate

$$\int_C \overline{z}\ dz$$

To which of the following is the Cauchy-Goursat theorem directly applicable.

27.

$$\int_{|z|=1} \frac{\sin z}{z+2i}\ dz$$

28.

$$\int_{|z+3i|=1} \frac{\sin z}{z+2i}\ dz$$

29.

$$\int_{|z-3i|=6} e^{\overline{z}}\ dz$$

30.

$$\int_{|z+i|=1} \operatorname{Log} z\ dz$$

31.

$$\int_{|z-1-i|=1} \operatorname{Log} z\ dz$$

32.

$$\int_{|z|=1/2} \frac{1}{(z-1)^4+1}\ dz$$

33.

$$\int_{|z|=3} \frac{dz}{1-e^z}$$

34.

$$\int_{|z|=b} \frac{dz}{z^2+bz+1}, 0<b<1$$

35.

$$\int_0^{1+i} z^3\ dz$$

along $y=x$.

Prove the following results.

36.

$$\int_0^{2\pi} e^{\cos\theta}\left[\cos\left(\sin\theta+\theta\right)\right]\ d\theta = 0$$

37.

$$\int_0^{2\pi} e^{\cos\theta}\left[\sin\left(\sin\theta+\theta\right)\right]\ d\theta = 0$$

Prove that the following identities hold for any integer $n \geq 0$.

38.

$$\int_0^{2\pi} e^{\sin n\theta}\cos\left(\theta-\cos n\theta\right)\ d\theta = 0$$

39.

$$\int_0^{2\pi} e^{\sin n\theta} \sin(\theta - \cos n\theta)\ d\theta = 0$$

---

40. Show that for real a, where $|a| > 1$, we have

$$\int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}\ d\theta = 0.$$

Hint: Consider $\oint\limits_{|z|=1} \frac{1}{z-a}\ dz$ and represent the circle parametrically.

41. Let n be any integer, r a positive real number, and z_0 a complex constant. Show that for $r > 0$,

$$\int\limits_{|z-z_0|=r} (z - z_0)^n\ dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$$

Hint: Consider the change of variable $z = z_0 + re^{i\theta}$.

Evaluate the following integrals. The contour C is the square centered at the origin with corners $\pm(2 \pm 2i)$. The result contained in the previous problem as well as the principle of deformation of contours will be useful.

42.

$$\int\limits_C \frac{dz}{z - i}$$

43.

$$\int\limits_C \frac{dz}{(z - i)^4}$$

44.

$$\int\limits_C \frac{z\ dz}{z - i}$$

45.

$$\int\limits_C \frac{(z+1)^m\ dz}{z^m}, \quad m \geq 0 \text{ an integer}$$

46.

$$\int\limits_C \frac{z^m\ dz}{(z-1)^m}, \quad m \geq 0 \text{ an integer}$$

47. Show that

$$\int\limits_{|z-3|=2} \frac{\operatorname{Log} z}{(z+1)(z-3)}\ dz = \int\limits_{|z-3|=2} \frac{\operatorname{Log} z}{4(z-3)}\ dz$$

Hint: Write $\frac{1}{(z+1)(z-3)}$ as a sum of partial fractions.

48. Consider the n-tuply connected domain D whose nonoverlapping boundaries are the simple closed contours $C_0, C_1, \cdots C_{n-1}$. Let $f(z)$ be a function that is analytic in D and its boundaries. Show that

$$\int\limits_{C_0} f(z)\ dz = \int\limits_{C_1} f(z)\ dz + \int\limits_{C_2} f(z)\ dz + \cdots + \int\limits_{C_{n-1}} f(z)\ dz.$$

Hint: Consider the derivation of the principle of deformation of contours. Make a set of cuts in order to link up the boundaries.

49. Use the result derived in Exercise 48 to show

$$\int\limits_{|z|=2} \frac{\sin z}{z^2-1}\,dz = \int\limits_{|z-1|=1/2} \frac{\sin z}{z^2-1}\,dz + \int\limits_{|z+1|=1/2} \frac{\sin z}{z^2-1}\,dz.$$

Evaluate the following integrals along the curve $y = \sqrt{x}$.

50.

$$\int_0^{4+2i} e^{iz}\,dz$$

51.

$$\int_0^{4+2i} \left(1+z^2\right)\,dz$$

52.

$$\int_{1+i}^{4+2i} \left(z+z^{-2}\right)\,dz$$

53.

$$\int_0^{4+2i} e^z \sinh z\,dz$$

54.

$$\int_0^{4+2i} e^z \cosh e^z\,dz$$

55.

$$\int_{1+i}^{4+2i} \frac{z}{z^2-1}\,dz$$

---

56. a) What, if anything, is incorrect about the following two integrations? The integrals are both along the line $y = x$.

$$\begin{aligned}\int_{0+i0}^{1+i} z\,dz &= \left[\frac{z^2}{2}\right]_{0+i0}^{1+i} = \frac{(1+i)^2}{2} = i,\\ \int_{0+i0}^{1+i} \overline{z}\,dz &= \left[\frac{\overline{z}^2}{2}\right]_{0+i0}^{1+i} = \frac{(1-i)^2}{2} = -i\end{aligned}$$

b) What is the correct value of each of the above integrals?

57. Find the value of

$$\int_e^i \operatorname{Log} z\,dz$$

taken along the line connecting $z = e$ with $z = i$. Why is it necessary to specify the contour?

---

58. Find

$$\int_{1+i}^{-1-i} \frac{\operatorname{Log} z}{z}\,dz,$$

where the integral is along a contour not intersecting the branch cut for $\operatorname{Log} z$.

59. Find

$$\int_1^i z^{1/2}\,dz.$$

The principal branch is used. The contour does not pass through any point satisfying $y = 0$ and $x \le 0$.

---

60. Find
$$\int_1^i z^{1/2}\,dz.$$
The branch of $z^{1/2}$ used equals -1 when $z=1$. The branch cut lies along $y=0, x\le 0$ and the contour does not pass through the branch cut.

61. Find
$$\int_1^i i^z\,dz.$$
Use principal values. Why is it not necessary to specify the contour?

62. Evaluate the integral
$$\int_0^i \cos z \cosh z\;dz$$
Evaluate the following integrals using the Cauchy integral formula, its extension or the Cauchy-Goursat theorem where appropriate.

63.
$$\int \frac{\sin z}{z-2}\,dz \quad \text{around} \quad |z|=3$$

64.
$$\int \frac{\sin z}{z-2}\,dz \quad \text{around} \quad |z|=1$$

65.
$$\int \frac{\cosh z}{(z-3)(z-1)}\,dz \quad \text{around} \quad |z|=2$$

66.
$$\frac{1}{2\pi i}\int \frac{\cosh e^z}{z^2-4z+3}\,dz \quad \text{around the square with corners at } z=2, z=4, \text{ and } 3\pm i$$

67.
$$\int \frac{e^{iz}}{z^2+z+1}\,dz \quad \text{around} \quad \left|z+\frac{1}{2}-2i\right|=2$$

68.
$$\frac{1}{2\pi i}\int \frac{\operatorname{Log} z}{z^2+9}\,dz \text{ around } |z-4i|=3$$

69.
$$\frac{1}{2\pi i}\int \frac{e^{iz}}{(z-i)^2}\,dz \quad \text{around} \quad |z-1|=2$$

70.
$$\frac{1}{2\pi i}\int \frac{ze^z}{(z-i)^2}\,dz \quad \text{around} \quad |z-1|=2$$

71.
$$\frac{1}{2\pi i}\int \frac{1}{(z+2)(z-i)^2}\,dz \quad \text{around} \quad |z-1|=2$$

72.
$$\frac{1}{2\pi i}\int \frac{\cos z}{(z-i)^3}\,dz \quad \text{around} \quad |z-1|=2$$

73.
$$\frac{1}{2\pi i}\int \frac{\sin 2z}{z^{15}}\,dz \quad \text{around} \quad |z|=2$$

74.
$$\int \frac{\sin 2z}{z^{16}}\,dz \quad \text{around} \quad |z|=2$$

75. a) Show that
$$\int_{|z|=1} \frac{e^{az}}{z^{n+1}}\,dz = \frac{a^n 2\pi i}{n!}$$
b) Use part a) to show, when θ is real, that,
$$\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta - n\theta)\ d\theta = \frac{2\pi a^n}{n!},\ \text{and}\ \int_0^{2\pi} e^{a\cos\theta}\sin(a\sin\theta - n\theta)\ d\theta = 0$$

76.a) Consider
$$\int_{|z|=1} \frac{dz}{z-a},$$
where a is any nonconstant such that $|a| \neq 1$. Evaluate this integral for the cases $|a| > 1$ and $|a| < 1$.
b) Evaluate this integral for the two cases given above by noticing that on the unit circle we have $\overline{z} = 1/z$.

---

77. If a is a real number and $|a| < 1$ show that
$$\int_0^{2\pi} \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}\ d\theta = 2\pi.$$
Hint: Consider $\displaystyle\int_{|z|=1} \frac{dz}{z-a}$

Prove the following:
78.
$$\frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}}\,d\theta = 1$$
79.
$$\int_{-\pi}^{\pi} e^{\cos\theta}\cos(\sin\theta)\ d\theta = 2\pi, \quad \text{(Do Exercise 78 first.)}$$
80.
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \cos^2\left(\frac{\pi}{6} + ae^{i\theta}\right)\ d\theta = \frac{3}{4}$$
81.
$$\int_{-\pi}^{\pi} \frac{a + \cos n\theta}{a^2 + 1 + 2a\cos n\theta}\ d\theta = \frac{2\pi}{a},\ \text{where}\ a > 1,\ n\ \text{an integer}$$
(*Hint*: $f(z) = \dfrac{1}{z^n + a}$)
82.
$$\int_0^{2\pi} \operatorname{Log}\left[a^2 + 1 + 2a\cos(n\theta)\right]\ d\theta = 4\pi\operatorname{Log} a,\ \text{where}\ a > 1,\ n\ \text{an integer}$$
Hint: $a^2 + 1 + 2a\cos(n\theta) = \left|a + e^{in\theta}\right|^2$

For the following closed regions R and functions $f(z)$, find the values z in R where $|f(z)|$ achieves its maximum and minimum values. Give the values of $|f(z)|$ at its maximum and minimum.
83. $f(z) = z$, R is $|z - 1 - i| \leq 1$
84. $f(z) = z^2$, R is $|z - 1 - i| \leq 2$
85. $f(z) = e^z$, R same as in Exercise 83

86. $f(z) = \sin z$ and R is the rectangle $1 \le y \le 2, 0 \le x \le \pi$.

---

87. Let $u(x, y)$ be real, nonconstant, and continuous in a closed bounded region R. Let $u(x, y)$ be harmonic in the interior of R. Prove that the maximum of $u(x, y)$ in this region occurs on the boundary. This is known as the *maximum principle*.
Hint: Consider $F(z) = u(x, y) + iv(x, y)$, where v is the harmonic conjugate of u. Let $f(z) = e^{F(z)}$. Explain why $|f(z)|$ has its maximum value on the boundary. How does it follow that $u(x, y)$ has its maximum value on the boundary?

---

88. For $u(x, y)$ described in Exercise 87 show that the minimum value of this function occurs on the boundary. This is known as the *minimum principle*.
Hint: Follow the suggestions given in Exercise 87 but show that $|f(z)|$ has its minimum value on the boundary.

---

89. Consider the closed region R given by $0 \le x \le 1, 0 \le y \le 1$. Now $u = \left(x^2 - y^2\right)$ is harmonic in R. Find the maximum and minimum values of u in R.

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Practice Problems-3

Use the nth term test to prove that the following series are divergent in the indicated regions.

1. $\displaystyle\sum_{n=1}^{\infty} (2iz)^n$ for $|z| \geq \dfrac{1}{2}$

2. $\displaystyle\sum_{n=1}^{\infty} (n+1)(i+1)^n (z+1)^n$ for $|z+1| \geq \dfrac{1}{\sqrt{2}}$

3. $\displaystyle\sum_{n=1}^{\infty} \frac{(n)(i-1)^n}{(z-2i)^n}$ for $|z-2i| \leq \sqrt{2}$

4. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+2}{n}\right)^n$ for $|z+1+i| \geq \dfrac{1}{2}$

Use the ratio test to prove the absolute convergence, in the indicated domains, of the following series. Where does the ratio test assert that each series diverges?

5. $\displaystyle\sum_{n=1}^{\infty} n^2 \left(z+\frac{1}{2}\right)^n$ for $|z+1/2| < 1$

6. $\displaystyle\sum_{n=0}^{\infty} n! e^{n^2 z}$ for $\operatorname{Re}(z) < 0$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{(2+i)^n}{(z+i)^n (n+i)^2}$ for $|z+i| > \sqrt{5}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{n}{z}\right)^n$ for $|z| > e$

In Exercises 9 and 10, find the sum of the seres. In each case, state where in the complex plane the series converges to the sum.

9. $1 + (z-1)^2 + (z-1)^4 + (z-1)^6 + \cdots$

10. $1 + 1/z + 1/z^2 + 1/z^3 + \cdots$

11.

a) Prove that

$$\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2} \text{ for } |z| < 1$$

by using series multiplication.

b) Using part a), show that

$$\sum_{n=1}^{\infty} \frac{n(n+1)}{2} z^{n-1} = \frac{1}{(1-z)^3} \text{ for } |z| < 1$$

The identity $\displaystyle\sum_{j=1}^{n} j = n(n+1)/2$ can be helpful here.

State the first three nonzero terms in the following Taylor series expansions. Give also the nth (general term) in the series and state the circle within which the series representation is valid. Use the principal branch of any multivalued functions.

12. $\frac{1}{z}$, $z = i$

13. e^z, $z = 2 + i$

14. $\operatorname{Log} z$, $z = e$

15. $\frac{1}{z^2}$, $z = 1 + i$

16. $\cosh z - \cos z$, $z = 0$

17. z^i, $z = 1$

18. i^z, $z = 0$.

---

19. a) Find all the coefficients in the expansion of z^5 about z_0 (a constant) and write out the entire series in terms of z and z_0.

b) For what values of z is the preceding series a valid expansion of the function?

c) Explain how you could obtained the result in a) by using the binomial theorem.

Note that $z = (z - z_0) + z_0$.

20.a) Explain why $z^{1/2}$(principal branch) cannot be expanded in a Maclaurin series.

b) Explain whether this same branch of $z^{1/2}$ can be expanded in a Taylor series about 1. If so, find the first three terms and state the circle within which the series is valid..

---

21. Consider the two infinite series

$$1 + z + z^2 + z^3 + \cdots$$

and

$$1 + \frac{z}{\ln 2} + \frac{z^2}{\ln 3} + \frac{z^3}{\ln 4} + \cdots$$

Both of these series will converge to

$$\frac{1}{1-z}.$$

Yet the second series is not the Taylor expansion of $\frac{1}{1-z}$. Does this not contradict the Theorem which asserts that the Taylor series expansion of a function is the power series of the function? Explain.

Without actually obtaining the coefficients in the following Taylor series, determine the center and radius of the circle within which each converges to the function on the left. Use the principal branch of any multi-valued functions.

21.

$$\frac{1}{z-i} = \sum_{n=0}^{\infty} c_n \, (z+1)^n$$

22.

$$\frac{1}{z^3+1} = \sum_{n=0}^{\infty} c_n \, (z-i)^n$$

23.

$$\frac{1}{\cos z} = \sum_{n=0}^{\infty} c_n \, (z-1-i)^n$$

24.

$$\frac{1}{\operatorname{Log} z} = \sum_{n=0}^{\infty} c_n (z - 1 - 2i)^n$$

25.

$$\frac{1}{z^{1/2} - 1} = \sum_{n=0}^{\infty} c_n (z - 2)^n$$

Without obtaining the series, setermine the interval along the x-axis for which the indicated Taylor series converges to the given real function. Convergence at the endpoints of the interval need not be considered.

26.

$$\frac{1}{1-x} \text{ expanded about } x = -1$$

27.

$$\frac{1}{x^2+9} \text{ expanded about } x = 2$$

28.

$$\frac{1}{\sin x} \text{ about } x = 1/4$$

29.

$$\frac{1}{\sin x} \text{ about } x = 2$$

30.

$$\tan x \text{ expanded about } x = 2$$

31.

$$\sqrt{x} \text{ expanded about } x = e$$

32.

$$\frac{1}{x^3+1} \text{ expanded about } x = 1$$

---

33. Prove that in the disk $|z| \leq 1$, we have

$$|e^z - 1| \leq (e-1)\,|z|\,.$$

34.a) Let $z^N - z_0^N = \sum_{n=1}^{N} c_n (z - z_0)^n$ valid for all z. N is a positive integer. show that

$$c_n = \frac{N!\, z_0^{N-n}}{n!\,(N-n)!}.$$

b) Replace z in the above with $z + z_0$ and show that

$$(z + z_0)^n = \sum_{n=0}^{N} \frac{N!\, z_0^{N-n}}{n!\,(N-n)!} z^n$$

This is the familiar binomial expansion.

The following exercises involve our generating a new Taylor series through a change of variables in the geometric series or some other familiar expansion. Here a is any constant. Explain how the following are derived.

35.

$$\frac{1}{1+az} = 1 - az + a^2 z^2 - a^3 z^3 + \cdots, \quad |z| < \left|\frac{1}{a}\right|$$

36.

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - \cdots, \; |z| < 1$$

37.

$$\frac{1}{1+a+z} = 1 - (z+a) + (z+a)^2 - \cdots, \; |z+a| < 1$$

38.a)

$$e^{-z^2} = 1 - z^2 + \frac{z^4}{2} - \frac{z^6}{6} + \cdots \text{ for all } z$$

b) Use the preceding result to find the 10th derivative of e^{-z^2} at $z = 0$.

---

39. By a suitable differentiation, show that

$$\frac{1}{z^3} = 1 - \frac{3 \cdot 2}{2}(z-1) + \frac{4 \cdot 3}{2}(z-1)^2 - \frac{5 \cdot 4}{2}(z-1)^3 + \cdots, |z-1| < 1$$

---

40. By a suitable differentiation, find c_n in the expansion

$$\frac{1}{(1-z)^4} = \sum_{n=0}^{\infty} c_n z^n, \; |z| < 1$$

---

41. Show that, for $N \geq 0$,

$$\frac{1}{(1-z)^N} = \sum_{n=0}^{\infty} c_n z^n, \; c_n = \frac{(N-1+n)!}{n!\,(N-1)!}, \; |z| < 1.$$

---

42. Show that

$$\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z| < 1.$$

43.a) Show that

$$\sum_{n=1}^{\infty} n^2 z^n = \frac{z+z^2}{(1-z)^3} \text{ for } |z| < 1.$$

b) Use your result to evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Use the series multiplication to find a formula for c_n in these Maclaurin expansions. In what circle is each series valid?

44.

$$\frac{\cosh z}{1-z} = \sum_{n=0}^{\infty} c_n z^n$$

45.

$$\frac{\operatorname{Log}(1-z)}{1-z} = \sum_{n=0}^{\infty} c_n z^n$$

Obtain the following Taylor expansions. Give a general formula for the nth coefficient, and state the circle within your expression is valid

46.

$$\frac{z}{(z-1)(z+2)} \text{ expanded about } z = 0$$

47.

$$\frac{z}{(z+1)(z+2)} \text{ expanded about } z=1$$

48.

$$\frac{1}{z^2} \text{ expanded about } z=1+i$$

49.

$$\frac{1}{(z-1)^2(z+1)^2} \text{ expanded about } z=2$$

50.

$$\frac{e^z}{(z-2)(z+1)} \text{ expanded about } z=0$$

51. Use the answer to Exercise 49 to find the value of the 10th derivative of

$$\frac{1}{(z-1)^2(z+1)^2} \text{ at } z=2.$$

52.

$$\frac{z^3+2z^2+z-1}{z^2-4} \text{ expanded about } z=1$$

Hint: First use long division.

---

53. Let

$$h(z)=\frac{f(z)}{g(z)},$$

where

$$f(z)=\sum_{n=0}^{\infty} a_n (z-z_0)^n \text{ and } g(z)=\sum_{n=0}^{\infty} b_n (z-z_0)^n$$

and $g(z_0)=b_0 \neq 0$. We seek a Taylor expansion of $h(z)$ of the form

$$h(z)=\sum_{n=0}^{\infty} c_n (z-z_0)^n .$$

Find c_0, c_1, c_2 by long division of the series for $f(z)$ by the series for $g(z)$.

---

54. Find the coefficients c_0, c_1, c_2, c_3 in the Maclaurin expansion

$$\frac{\operatorname{Log}(1+z)}{\cos z}=\sum_{n=0}^{\infty} c_n z^n, \quad |z|<1.$$

55. Obtain all the coefficients in the following Maclaurin expansion by doing a long division:

$$\frac{1+z}{1+z+z^2+z^3+\cdots}=\sum_{n=0}^{\infty} c_n z^n, \quad |z|<1.$$

Explain why there are only two nonzero coefficients in your result.
Obtain the following Laurent expansions. State the first four nonzero terms. State explicitly the nth term in the series, and state the largest possible annular domain in which your series is a valid representation of the function.

56.

$$\frac{\sinh z}{z^3} \text{ expanded in powers of } z$$

57.

$$\sin\left(1+\frac{1}{z-1}\right) \text{ expanded in powers of } z-1$$

58.

$$\frac{\cos(1/z)}{z^3} \text{ expanded in powers of } z$$

59.

$$\operatorname{Log}\left[1+\frac{1}{z-1}\right] \text{ expanded in powers of } z-1$$

60.

$$\left(z+\frac{1}{z}\right)^7 \text{ expanded in powers of } z \text{ (Give all the terms.)}$$

Obtain the indicated Laurent expansions of

$$\frac{1}{z+i}.$$

State the nth term of the series.
61. An expansion valid for $|z|>1$
62. An expansion valid for $|z-i|>2$.
63.

$$\frac{1}{z-1} \text{ expanded in powers of } z+3$$

64.

$$\frac{1}{z+2} \text{ expanded in powers of } z-i$$

65.

$$\frac{z}{z-i} \text{ expanded in powers of } z-1$$

66.a) Consider

$$f(z)=\frac{1}{z(z-1)(z+3)}.$$

This function is expanded in three different Laurent series involving powers of z. State the three domains in which Laurent series are available.
b) Find each series and give an explicit formula for the nth term.

For the following functions, find the Laurent series valid in an annular domain that contains the point $z=2+2i$. The center of the annulus is at $z=1$. State the domain in which series is valid, and give an explicit formula for the nth term of your series
67.

$$f(z)=\frac{1}{z(z-2)}$$

68.

$$f(z)=\frac{1}{z(z-4)}$$

69.

$$f(z)=\frac{1}{(z-1)(z-3)}$$

70.

$$f(z)=\frac{z-i}{z-1}$$

71.

$$f(z)=\frac{1}{(z-1)^3}+\frac{1}{z}$$

72.

$$f(z)=\frac{1}{(z-1)^3}+z^3$$

73.
74.

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Practice Problems-4

Using the method of residues, evaluate the integral

$$\int_C \left(\frac{1}{(z-1)^2} + \frac{i}{z-1} + 2(z-1) + \frac{3}{z-4} \right) dz$$

where the contour C is given below.

1. $|z-1| = 2$

2. $|z-5| = 2$

3. The rectangle with corners at $\pm(5 \pm i)$

Evaluate the following integrals by using the method of residues. In problems 6-8 use Laurent expansions valid in deleted neighborhoods of the singular points to get the residue.

4. $\displaystyle\int_C \sum_{n=-\infty}^{\infty} e^{-n^2} (n-1)(z-1)^n \; dz$ around $|z| = 2$

5. $\displaystyle\int_C \sum_{n=-5}^{\infty} \frac{1}{(z+i)(n+6)!} \; dz$ around $|z-i| = 3$

6. $\displaystyle\int_C \cosh(1/z) \; dz$ around the square with at $\pm(1 \pm i)$

7. $\displaystyle\int_C z \sin\left(\frac{1}{z-1}\right) \; dz$ around $|z| = 2$

8. $\displaystyle\int_C \frac{1}{\sin z} \; dz$ around $|z| = 2$

---

9. Show that if $n \geq 1$ is an integer, then

$$\int_C \left(z + \frac{1}{z}\right)^n dz = \begin{cases} \dfrac{2\pi i n!}{\left(\frac{n-1}{2}\right)! \left(\frac{n+1}{2}\right)!}, & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

where C is any simple closed contour encircling the origin.
Hint: Use the binomial theorem.

---

10. We wish to evaluate

$$\int_{\substack{|z|=R \\ R>1}} \left(z^2 - 1\right)^{1/2} \; dz$$

where we employ a branch of the integrand defined by a straight branch cut connecting $z = 1$ and $z = -1$, and $\left(z^2 - 1\right)^{1/2} > 0$ on the line $y = 0, x > 1$. Note that the singularities enclosed by the path of integration are not isolated.

Show by means of a Laurent series expansion

$$\sum_{n=-\infty}^{\infty} c_n \left(z - z_0\right)^n$$

that the following functions have essential singularities at the points stated. State the residue and give c_{-2}, c_{-1}, c_0, c_1.

11. $\sinh\left(\dfrac{1}{z}\right)$ at $z = 0$

12. $(z-1)^3 \cosh \dfrac{1}{z-1}$ at $z = 1$

13. $e^{1/z} \sin \dfrac{1}{z}$ at $z = 0$

14. $2^{i/z}$ (princ. value) at $z = 0$

15. $e^{1/(z-i)} e^{z-i}$ at $z = i$.

---

16. Does the function

$$e^{\mathrm{Log}(1/z)}$$

have an essential singularity at $z = 0$? What is the nature of the singularity and what is the residue?.

Use series expansions or L'Hopital's Rule to show that the following functions possess removable singularities at the indicated singular points. You must show that

$$\lim_{z \to z_0} f(z)$$

exists and is finite at the singular point. Also, state how $f(z_0)$ should be defined at each point in order to rempve the singularity. Use principal branches where there is any ambiguity.

17. $\dfrac{e^z - 1}{z}$ at $z = 0$

18. $\dfrac{e^z - z}{\mathrm{Log}\, z}$ at $z = 1$

19. $\dfrac{\sinh z}{z^2 + \pi^2}$ at the two singular points

20. $\dfrac{z^2 - 1}{z^i - i^{z-1}}$ at $z = 1$

21. $\dfrac{\mathrm{Log}\, z}{z^z - 1}$ at $z = 1$

22. $\dfrac{1}{z} \mathrm{Log} \dfrac{1}{z-1}$ at $z = 0$

23. $\dfrac{1}{z^2} - \dfrac{\cos z}{z^2}$ at $z = 0$.

---

24. Let

$$f(z) = g(z) + h(z).$$

Prove that the residue of $f(z)$ is the sum of the residues of $g(z)$ and $h(z)$ at z_0. Assume that z_0 is an isolated singular point of both $g(z)$ and $h(z)$.

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25. Can a function have a residue of zero at a simple pole? Can a function have a residue of zero at a higher order pole? Can a function have a residue of zero at an essential singularity? Explain.

For each of the following functions state the location and the order of each pole and find the corresponding residue. Use the principal branch of any multi-valued function given below.

26. $\dfrac{\cos z}{z^2+z+1}$

27. $\dfrac{1}{z} - \dfrac{e^z}{z(z+1)} + \dfrac{1}{(z-1)^4}$

28. $\dfrac{1}{z^{1/2}(z^2-9)^2}$

29. $\dfrac{\cos\left(\frac{\pi}{2}z\right)}{z^2(z-1)^2}$

30. $\dfrac{1}{(\operatorname{Log} z)(z^2+1)^2}$

31. $\dfrac{\sin z - z}{z\sinh z}$

32. $\dfrac{z^8+1}{z^4}$

33. $\dfrac{1}{(\operatorname{Log}(z/e)-1)^2}$

34. $\dfrac{1}{\sin z^2}$

35. $\dfrac{1}{10^z - e^z}$

36. $\dfrac{\cos(1/z)}{\sin z}$

37. $\dfrac{1}{e^{2z}+e^z+1}$..

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38. Consider the analytic function

$$f(z) = \frac{g(z)}{h(z)}$$

having a pole at z_0. Let $g(z_0) \neq 0$, $h(z_0) = h'(z_0) = 0$, $h''(z_0) \neq 0$. Thus $f(z)$ has a pole of second order at $z = z_0$. Show that

$$\operatorname{Res}_{z=z_0} f(z) =$$

Find the residue of the following functions at the indicated point.

39.

$$\frac{z+1}{z}\sin\left(\frac{1}{z}\right) \text{ at } 0$$

40.

$$\frac{1}{z^2-1} \text{ (principal branch) at } 1$$

41.

$$\frac{1}{(z+i)^5} \text{ at } -i$$

42.

$$\frac{\sin z}{(z+i)^5} \text{ at } -i$$

43.

$$\frac{z^{12}}{(z-1)^{10}} \text{ at } 1$$

44.

$$\frac{1}{\sinh(2\,\mathrm{Log}\, z)} \text{ at } i$$

45.

$$\frac{1}{\cos\left(\frac{\pi}{2}e^z + \sin z\right)} \text{ at } 0$$

46.

$$\frac{1}{\sin\left[z\left(e^z - 1\right)\right]} \text{ at } 0$$

47.

$$\frac{\cos(z-1)}{z^{10}} + \frac{2}{z-1} \text{ at } z = 1$$

48.

$$\frac{\cos(z-1)}{z^{10}} + \frac{2}{z-1} \text{ at } z = 0$$

49.a) Let $n \geq 1$ be an integer. Show that the poles of

$$\frac{1}{z^n + z^{n-1} + z^{n-2} + \cdots + 1}$$

are at

$$\exp i\left(\frac{2k\pi}{n+1}\right), \quad k = 1, 2, \cdots n.$$

b) Show that the poles are simple.

c) Show that the residue at $\exp i\left(\dfrac{2k\pi}{n+1}\right)$ is

$$\frac{\exp i\left(\frac{2k\pi}{n+1}\right) - 1}{(n+1)\exp i\left(\frac{2k\pi}{n+1}\right)}.$$

Use residues to evaluate the following integrals. Use the principal branch of the multi-valued functions.

50.

$$\int \frac{dz}{\sin z} \text{ around } |z-6| = 4$$

51.

$$\int \frac{\sinh(1/z)}{z-1}\, dz \text{ around } |z| = 2$$

52.

$$\int \frac{\sin z}{\sinh^2 z}\, dz \text{ around } |z| = 3$$

53.

$$\int \frac{dz}{[\mathrm{Log}(\mathrm{Log}\, z) - 1]} \text{ around } |z-16| = 5$$

54.

$$\int \frac{e^{1/z}}{z^2-1}\, dz \text{ around } |z-1| = 3/2$$

55.

$$\int \frac{dz}{\sin z - 2e^z} \text{ around } |z+1| = 2$$

56.

$$\int \frac{dz}{\sin\left(z^{1/2}\right)} \text{ around } |z-9| = 5$$

57.

$$\int \frac{dz}{\overline{z}-b} \text{ around } |z| = a > 0$$

Note that the integrand is not analytic. Consider $a > |b|$ and $a < |b|$.
Hint: Multiply both numerator and denominator by z.
Using residue calculus, establish the following identities.

58.

$$\int_0^{2\pi} \frac{d\theta}{k-\sin\theta} = \frac{2\pi}{\sqrt{k^2-1}} \text{ for } k > 1$$

Does your result hold for $k < -1$? Explain.

59.

$$\int_{-\pi}^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}} \text{ for } a > b \geq 0$$

60.

$$\int_{-\pi/2}^{3\pi/2} \frac{\cos\theta}{a+b\cos\theta}\, d\theta = \frac{2\pi}{b}\left[1-\frac{a}{\sqrt{a^2-b^2}}\right] \text{ for } a > b > 0$$

61.

$$\int_0^{2\pi} \sin^4\theta\ d\theta = \frac{3\pi}{4}$$

62.

$$\int_0^{2\pi} \cos^m\theta\ d\theta = \frac{2\pi}{2^m}\frac{m!}{\left[\left(\frac{m}{2}\right)!\right]^2} \text{ for } m \geq 0 \text{ even.}$$

Show that the preceding integral is zero when m is odd.

63.

$$\int_0^{2\pi} \frac{d\theta}{(a+b\sin\theta)^2} = \frac{2\pi a}{\left(\sqrt{a^2-b^2}\right)^3} \text{ for } a > b \geq 0$$

64.

$$\int_0^{2\pi} \frac{d\theta}{a+\sin^2\theta} = \frac{2\pi}{\sqrt{a\,(a+1)}} \text{ for } a > 0$$

65.

$$\int_{-\pi}^{\pi} \frac{\cos\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{2\pi}{a\,(a^2-1)} \text{ for } a \text{ real, } |a| > 1$$

66.

$$\int_0^{2\pi} \frac{\cos\theta}{1-2a\cos\theta+a^2}\, d\theta = \frac{2\pi a}{1-a^2} \text{ for } a \text{ real, } |a| < 1$$

67.

$$\int_0^{2\pi} \frac{\cos n\theta}{\cosh\theta+\cos\theta}\, d\theta = \frac{2\pi\,(-1)^n\, e^{-n\theta}}{\sinh a},\ n \geq 0 \text{ is an integer, } a > 0$$

68.

$$\int_0^{2\pi} \frac{d\theta}{a\sin^2\theta+b^2\cos^2\theta} = \frac{2\pi}{ab}, \text{ for a,b real,} ab > 0$$

For which of the following integrals does the Cauchy principal value exist?

69.

$$\int_{-\infty}^{\infty} e^{-x}\ dx$$

70.

$$\int_{-\infty}^{\infty} e^{-|x|}\ dx$$

71.

$$\int_{-\infty}^{\infty} \frac{x^2+x}{1+x^2}\ dx$$

72.

$$\int_{-\infty}^{\infty} \frac{x-1}{1+x^2}\ dx$$

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73. The standard ordinary definition of $\int_{-\infty}^{\infty} f(x)\ dx$ is given by

$$\int_{-\infty}^{\infty} f(x)\ dx = \lim_{b\to\infty}\int_0^b f(x)\ dx + \lim_{a\to\infty}\int_{-a}^0 f(x)\ dx,$$

where the two limits must exist independently of one another. Work the following without using complex variables.
a) Show that

$$\int_{-\infty}^{\infty} \sin x\ dx$$

fails to exist according to the standard definition.
b) Show that the Cauchy principal value of the integral in part a) does exist and is zero.
c) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

for both the standard definition and the Cauchy principal value.
d) Show that if the ordinary value of

$$\int_{-\infty}^{\infty} f(x)\ dx$$

exists, then the Cauchy principal value must also exist and that the two results agree.

Using the symmetry properties of the integrand, but without evaluating the integrals, state which of the following must be true, based entirely on symmetry arguments.
74.

$$\int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$$

75.

$$\int_0^{\infty} \frac{dx}{x^2+x+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$$

76.

$$\int_0^{\infty} \frac{\cos x\ dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x\ dx}{x^2+1}$$

77.

$$\int_0^{\infty} \frac{\tanh x\ dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\tanh x\ dx}{x^2+1}$$

78.

$$\int_{-\infty}^{\infty} \frac{x\ dx}{x^4+1} = 0$$

79.

$$\int_{-\infty}^{\infty} \frac{x+1}{x^4+1}\ dx = 0$$

80.

$$\int_{-\infty}^{\infty} \frac{1+\sin x}{x^4+x^2+1}\, dx = 0$$

81.

$$\int_{-\infty}^{\infty} \frac{x\sin x^2}{x^4+x^2+1}\, dx = 0$$

82.

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+x^2+1}\, dx = \int_{-\infty}^{\infty} \frac{ix\sin x}{x^4+x^2+1}\, dx$$

Evaluate the following integrals by means of the residue calculus. Use the Cauchy principal value.

83.

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+x+1}$$

84.

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+x+1)(x^2+1)}$$

85.

$$\int_{0}^{\infty} \frac{x^4\, dx}{x^6+1}$$

86.

$$\int_{-\infty}^{\infty} \frac{x^3+x^2+x+1}{x^4+1}\, dx$$

87.

$$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2},\ a>0$$

88.

$$\int_{-\infty}^{\infty} \frac{dx}{(x+a)^2+b^2},\quad a, b \text{ real, } b>0$$

Evaluate the following integrals by residue calculus. Use Cauchy principal values and take advantage of even and odd symmetries where appropriate.

89.

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+9}\, dx$$

90.

$$\int_{-\infty}^{\infty} \frac{x\cos 2x}{x^2+3}\, dx$$

91.

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{(x-1)^2+9}\, dx$$

92.

$$\int_{0}^{\infty} \frac{x^3\sin 2x}{x^4+16}\, dx$$

93.

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2+x+1}\, dx$$

94.

$$\int_{-\infty}^{\infty} \frac{(x-1)\cos 2x}{x^2+x+1}\, dx$$

95.

$$\int_{-\infty}^{\infty} \frac{(x^3+x^2)\cos\left(\sqrt{2}x\right)}{x^4+1}\, dx$$

96.

$$\int_{-\infty}^{\infty} \frac{xe^{ix/3}}{(x-i)^2+4}\,dx$$

97.

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+x^2+1}\,dx$$

98.

$$\int_{0}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+16)}\,dx$$

99.

$$\int_{0}^{\infty} \frac{x^2\cos x}{(x^2+1)(x^2+16)}\,dx$$

100.

$$\int_{-\infty}^{\infty} \frac{(x^3+x^2+x)\sin(x/2)}{(x^2+1)(x^2+4)}\,dx$$

101.

$$\int_{0}^{\infty} \frac{\cos x}{(x^2+4)^2}\,dx$$

102.

$$\int_{0}^{\infty} \frac{\sin mx \sin nx}{a^2+x^2}\,dx = \frac{\pi}{2a}e^{-ma}\sinh na \text{ for } m \geq n \geq 0.$$

Assume a is positive.
Hint: Express $\sin mx \sin nx$ as a sum involving $\cos(m+n)x$ and $\cos(m-n)x$.

Find the Cauchy principal value of the following integrals:
103.

$$\int_{-\infty}^{\infty} \frac{\sin(2x)}{x+4}\,dx$$

104.

$$\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2-16}\,dx$$

105.

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x-\pi/2)(x-\pi)}\,dx$$

106.

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x-\pi)(x^2+1)}\,dx$$

107.

$$\int_{-\infty}^{\infty} \frac{\cos\left(\frac{\pi}{2}x\right)}{(x-1)^2}\,dx$$

Hint: Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{e^{i\pi x/2}}{(x-1)^2}\,dx$
Prove the following, where the Cauchy principal value is used:
108.

$$\int_{-\infty}^{\infty} \frac{\cos(mx)}{ax^2+bx+c}\,dx = \frac{-2\pi\cos\frac{mb}{2a}\sin\frac{m\sqrt{b^2-4ac}}{2a}}{\sqrt{b^2-4ac}}, \text{ where } m \geq 0,\ a,b,c \text{ are real, } b^2 > 4ac, \text{ and } a \neq 0$$

109.

$$\int_{-\infty}^{\infty} \frac{\cos(mx)}{x^4-b^4}\,dx = \frac{-\pi}{2b^3}\sin mb - \frac{\pi e^{-mb}}{2b^3}, \text{ where } m \geq 0,\ b > 0$$

110.

$$\int_{-\infty}^{\infty} \frac{\sin(bx)}{\sinh ax}\, dx = \frac{\pi}{a} \tanh \frac{\pi b}{2a}, \text{ where } a > 0$$

Prove the following for $a > \dot{0}$

111.

$$\int_{0}^{\infty} \frac{\ln x}{x^2 + a^2}\, dx = \frac{\pi}{2a} \ln a$$

112.

$$\int_{0}^{\infty} \frac{\ln x}{(x^2 + a^2)^2}\, dx = \frac{\pi}{4a^3} \ln\left(\frac{a}{e}\right)$$

113.

$$\int_{0}^{\infty} \frac{x^2 \ln x}{x^4 + a^4}\, dx = \frac{\pi\sqrt{2}}{4a} \ln a + \frac{\pi^2\sqrt{2}}{16a}$$

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114. Derive the two results

$$\int_{0}^{\infty} \frac{\ln x}{x^4 + x^2 + 1}\, dx = \frac{-\pi^2}{12} \text{ and } \int_{0}^{\infty} \frac{1}{x^4 + x^2 + 1}\, dx = \frac{\pi}{6}\sqrt{3}.$$
