ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

1st Midterm

March 13, 2008

12:40-14:30

Surname	:	________________
Name	:	________________
ID #	:	________________
Department	:	________________
Section	:	________________
Instructor	:	________________
Signature	:	________________

- The exam consists of 4 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

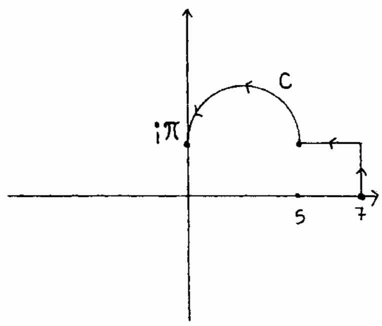
Please do not write below this line.

Q1	Q2	Q3	Q4	TOTAL
25	25	25	25	100

Question 1.

(a) Evaluate $\displaystyle\oint_C \frac{\sin\frac{1}{z+5}}{e^z(z^2+8)}\,dz$ where $C = \{z : |z-1| = 1\}$. (12.5 points)

(b) Evaluate $\displaystyle\int_C \cosh z\,dz$ where C is the contour which connects 7 and $i\pi$ as shown in the figure below. (12.5 points)



Answer 1.

(a) $\dfrac{\sin\frac{1}{z+5}}{e^z(z^2+8)}$ is analytic everywhere except at the points $-5, 2\sqrt{2}i, -2\sqrt{2}i$ and none of them belong to C and are inside C, then by Cauchy-Goursat theorem,

$$\oint_C \frac{\sin\frac{1}{z+5}}{e^z(z^2+8)}\,dz = 0.$$

(b) Since $\cosh z$ is analytic in a region containing C (actually, $\cosh z$ is analytic everywhere) and $\frac{d}{dz}\sinh z = \cosh z$, by the fundamental theorem of integration, we have

$$\int_C \cosh z\,dz = \int_7^{i\pi} \cosh z\,dz = \sinh z\Big|_7^{i\pi}$$

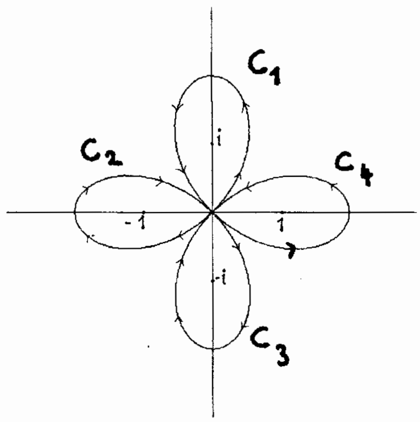
$$= \frac{e^{i\pi} - e^{-i\pi}}{2} - \frac{e^7 - e^{-7}}{2}$$

$$= \frac{(-1)-(-1)}{2} + \frac{e^{-7} - e^{7}}{2}$$

$$= \frac{e^{-7} - e^{7}}{2}.$$

Question 2. Evaluate

(a) $\oint\limits_C \frac{dz}{z^2+1}$ where C is the contour shown below. (12.5 points)



(b) $\oint\limits_C \frac{e^z}{(z^2-2z+1)(z^2+9)}\,dz$ where $C = \{z : |z| = 2\}$. (12.5 points)

Answer 2.

(a) Clearly, $I = \oint\limits_C \frac{dz}{z^2+1} = \oint\limits_{C_1} \frac{dz}{z^2+1} + \oint\limits_{C_4} \frac{dz}{z^2+1} + \oint\limits_{C_2} \frac{dz}{z^2+1} + \oint\limits_{C_3} \frac{dz}{z^2+1}$

Since $\frac{1}{z^2+1}$ has singularities only at i, and $-i$, $\oint\limits_{C_2} \frac{dz}{z^2+1} = 0$,

and $\oint\limits_{C_4} \frac{dz}{z^2+1} = 0$. Since $\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{i}{2}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$,

$\oint\limits_{C_1} \frac{dz}{z^2+1} = \frac{i}{2}\left(\oint\limits_{C_1} \frac{1}{z-i}\,dz - \oint\limits_{C_1} \frac{1}{z+i}\,dz\right) = \frac{i}{2}.2\pi i = -\pi$, and

$\oint\limits_{C_3} \frac{dz}{z^2+1} = \frac{i}{2}\left(\oint\limits_{C_3} \frac{1}{z-i}\,dz - \oint\limits_{C_3} \frac{1}{z+i}\,dz\right) = \frac{i}{2}.2\pi i = -\pi$, and so,

$I = -2\pi$.

(b) Let $f(z) = \frac{e^z}{z^2+9}$. Clearly, f is analytic inside and on C,

Then by Cauchy integral formula for derivatives,

$$f'(1) = \frac{1}{2\pi i}\oint\limits_C \frac{f(z)}{(z-1)^2}\,dz = \frac{1}{2\pi i}\oint\limits_C \frac{e^z}{(z^2-2z+1)(z^2+9)}\,dz = \frac{I}{2\pi i}.$$

Thus, $I = 2\pi i\, f'(1) = 2\pi i . \left.\frac{e^z(z^2+9) - 2ze^z}{(z^2+9)^2}\right|_{z=1} = 2\pi i . \frac{8e}{100} = \frac{4\pi e i}{25}$.

Question 3.

(a) Prove the *minimum modulus theorem*: Let $f(z)$ be analytic inside and on a simple closed curve C. Prove that if $f(z) \neq 0$ inside C, then $|f(z)|$ must assume its minimum value on C. (12.5 points)

(b) Give an example to show that if $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ at some point inside C, then $|f(z)|$ need not assume its minimum value on C. (12.5 points)

Answer 3.

(a) If $f(z_0) = 0$ for some $z_0 \in C$, then there is nothing to prove. If $f(z) \neq 0$ $\forall z \in C$, set $g(z) = \frac{1}{f(z)}$. Then g is analytic inside and on C. And by maximum modulus theorem, $|g(z)|$ takes its maximum value on C and hence $|f(z)|$ takes its minimum value on C ∎

(b) Let $f(z) = z$ and $C = \{z \in \mathbb{C} : |z| = 1\}$. Clearly, f is analytic inside and on C and $f(0) = 0$. Now, $|f(z)|$ assumes its minimum value at 0 which is inside C not on C.

Question 4.

(a) Find the Taylor series centered at 2 for

$$f(z) = \frac{2-z}{z-4}$$

and state where it converges to $f(z)$. (12.5 points)

(b) What is the largest circle within which the Maclaurin series for the function $\tan z$ converges to $\tan z$? Write the first two nonzero terms of that series. (12.5 points)

Answer 4.

(a) Remember that $\displaystyle\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$, $|w|<1$. $(*)$

So, $\displaystyle f(z) = \frac{2-z}{z-4} = \frac{2-z}{z-2-2} = \frac{z-2}{2-(z-2)} = \frac{z-2}{2}\cdot\frac{1}{1-\frac{z-2}{2}}$

Set $w = \frac{z-2}{2}$ in $(*)$. Then, $\displaystyle f(z) = \frac{z-2}{2}\sum_{n=0}^{\infty}\left(\frac{z-2}{2}\right)^n$ if $\left|\frac{z-2}{2}\right|<1$.

Therefore $\displaystyle f(z) = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}(z-2)^{n+1}$ for $|z-2|<2$.

(b) $\tan z = \frac{\sin z}{\cos z}$ is analytic everywhere except at the points $z=(n+\frac{1}{2})\pi$, $n\in\mathbb{Z}$. Thus, the nearest singular points to the center (= origin) are $\pm\frac{\pi}{2}$. Therefore, the Maclaurin series for $\tan z$ converges in $|z|<\frac{\pi}{2}$, and in no larger disk.

Since $\displaystyle f(z)=\tan z = \sum_{n=0}^{\infty} a_n z^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$,

$a_0 = \tan 0 = 0$, $\quad a_1 = \frac{d}{dz}\tan z\Big|_{z=0} = \sec^2 0 = 1 \neq 0$,

$a_2 = \frac{d^2}{dz^2}\tan z\Big|_{z=0} = 2\sec^2 z\tan z\Big|_{z=0} = 2\sec^2 0\tan 0 = 0$, and

$a_3 = 4\sec z\sec z\tan z\tan z + 2\sec^2 z\sec^2 z\Big|_{z=0} = 2\neq 0$. So the first two nonzero terms are $z + 2\frac{z^3}{3!} = z + \frac{z^3}{3}$.

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 352 Complex Analysis II
2^{nd} Midterm
April 24, 2008
12:40-14:30

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
Signature : ______________________

- The exam consists of 5 questions of equal weight.
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GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Expand $f(z) = \dfrac{1}{z^2 - 5z + 6}$ in a Laurent series valid for

(a) $|z| < 2$. (5 points)
(b) $2 < |z| < 3$. (5 points)
(c) $|z| > 3$. (5 points)
(d) $0 < |z - 2| < 1$. (5 points)

Answer 1.

Recall that $$\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}, \quad |w|<1 \qquad (*)$$

And note that $$f(z) = -\frac{1}{z-2} + \frac{1}{z-3}.$$

(a) $$f(z) = \frac{1}{2-z} - \frac{1}{3-z} = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

(b) $$f(z) = -\frac{1}{z-2} - \frac{1}{3-z} = -\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}}$$
$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

(c) $$f(z) = -\frac{1}{z-2} + \frac{1}{z-3} = -\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} + \frac{1}{z}\cdot\frac{1}{1-\frac{3}{z}}$$
$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n$$

(d) $$f(z) = -\frac{1}{z-2} + \frac{1}{(z-2)-1} = -\frac{1}{z-2} - \frac{1}{1-(z-2)}$$
$$= -\frac{1}{z-2} - \sum_{n=0}^{\infty}(z-2)^n.$$

Question 2. Find and classify all the zeros and singularities of f and calculate the residue of f at each singular point.

(a) $f(z) = \sin \frac{1}{z}$. (10 points)

(b) $f(z) = \frac{\tan z}{z}$. (10 points)

Answer 2.

(a) $\sin \frac{1}{z} = 0 \Rightarrow \frac{1}{z} = n\pi$ for some $n \in \mathbb{Z}$ or equivalently,

$z = \frac{1}{n\pi}$, $n \in \mathbb{Z}\setminus\{0\}$. Since $\frac{d}{dz} \sin \frac{1}{z} = -\frac{1}{z^2} \cos \frac{1}{z}$ and

$-(n\pi)^2 \cos(n\pi) \neq 0$ for $n \in \mathbb{Z}\setminus\{0\}$ all zeros are simple.

Since $\sin \frac{1}{z} = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots$, $0<|z|<\infty$, $\sin \frac{1}{z}$ has

an essential singularity at $z=0$ with $\operatorname{Res}_{z=0} \sin \frac{1}{z} = 1$ and

has no other singularity in $\mathbb{C}$.

(b) $f(z) = \frac{\sin z}{z \cos z}$. Clearly, $\sin z$ has simple zeros at $z = n\pi$,

z has a simple zero at $z=0$ and $\cos z$ has simple zeros

at $z = (n+\frac{1}{2})\pi$ for $n \in \mathbb{Z}$. Then f has a removable

singularity at $z=0$, simple zeros at $z = n\pi$, $n \in \mathbb{Z}\setminus\{0\}$ and

simple poles at $z = (n+\frac{1}{2})\pi$, $n \in \mathbb{Z}$. And,

$$\operatorname*{Res}_{z=(n+\frac{1}{2})\pi} \frac{\sin z}{z\cos z} = \lim_{z\to(n+\frac{1}{2})\pi} \frac{(z-(n+\frac{1}{2})\pi)\sin z}{z\cos z}$$

$$= \lim_{z\to(n+\frac{1}{2})\pi} \frac{\sin z}{z} \cdot \lim_{z\to(n+\frac{1}{2})\pi} \frac{(z-(n+\frac{1}{2})\pi)}{\cos z}$$

$$\overset{\text{L.R.}}{=} \frac{(-1)^n}{(n+\frac{1}{2})\pi} \lim_{z\to(n+\frac{1}{2})\pi} \frac{1}{-\sin z} = \frac{(-1)^n}{(n+\frac{1}{2})\pi} \cdot \frac{-1}{(-1)^n}$$

$$= \frac{-1}{(n+\frac{1}{2})\pi}.$$

Question 3. Evaluate

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{3+\sin\theta}$. (10 points)

(b) $\displaystyle\oint_C \frac{dz}{z^2\sinh z}$ where $C = \{z : |z| = 1\}$. (10 points)

Answer 3.

(a) Let $z = e^{i\theta}$, then $d\theta = \dfrac{dz}{iz}$ and $\sin\theta = \dfrac{z^2-1}{2iz}$. Therefore,

$$I = \int_0^{2\pi} \frac{d\theta}{3+\sin\theta} = \oint_{|z|=1} \frac{1}{3+\frac{z^2-1}{2iz}} \frac{dz}{iz} = 2\oint_{|z|=1} \frac{dz}{z^2+6iz-1}$$

$$z^2+6iz-1=0 \Rightarrow z = \frac{-6i+(-36+4)^{1/2}}{2} = \frac{-6i\pm 4\sqrt{2}i}{2} = (-3\pm 2\sqrt{2})i$$

Only $(-3+2\sqrt{2})i$ is inside the unit circle, and so

$$I = 2.\,2\pi i \operatorname*{Res}_{z=(-3+2\sqrt{2})i} \frac{1}{z^2+6iz-1} = 4\pi i \lim_{z\to(-3+2\sqrt{2})i} \frac{z-(-3+2\sqrt{2})i}{z^2+6iz-1}$$

$$\overset{\text{L.R.}}{=} 4\pi i \lim_{z\to(-3+2\sqrt{2})i} \frac{1}{2z+6i} = \frac{4\pi i}{4\sqrt{2}i} = \frac{\pi}{\sqrt{2}}.$$

(b) z^2 has a double zero at $z=0$ and $\sinh z$ has simple zeros at $z = n\pi i$, $n\in\mathbb{Z}$. Thus, inside C, $\dfrac{1}{z^2\sinh z}$ has a triple zero at $z=0$ and no other singularities, and so,

$$I = \oint_C \frac{dz}{z^2\sinh z} = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2\sinh z}.$$

$$\frac{1}{z^2\sinh z} = \frac{1}{z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)} = \frac{1}{z^3}\underbrace{\frac{1}{1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots}}_{\text{analytic at } z=0} = \frac{1}{z^3}(1+a_1z+a_2z^2+\cdots)$$

$\Rightarrow \operatorname*{Res}_{z=0} \dfrac{1}{z^2\sinh z} = a_2$. Since $1 = (1+a_1z+a_2z^2+a_3z^3+\cdots)\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots\right)$

$a_1 = 0$ and $\dfrac{1}{3!}+a_2 = 0 \Rightarrow a_2 = -\dfrac{1}{6}$ and $I = 2\pi i\left(-\dfrac{1}{6}\right) = -\dfrac{\pi i}{3}$.

Question 4. Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{29x^2+4x+1}$. (10 points)

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx$. (10 points)

Answer 4.

(a) $29z^2+4z+1=0 \Rightarrow z = \dfrac{-4+(16-4.29)^{1/2}}{58} = -\dfrac{2}{29} \pm \dfrac{5}{29}i$

$-\frac{2}{9}+\frac{5}{28}i$ belongs to the upper half plane and

$\deg(29z^2+4z+1) \geqslant \deg(1)+2$. so

$$\int_{-\infty}^{\infty} \frac{dx}{29x^2+4x+1} = 2\pi i \operatorname*{Res}_{z=-\frac{2}{29}+\frac{5}{29}i} \frac{1}{29z^2+4z+1} = 2\pi i \lim_{z\to -\frac{2}{29}+\frac{5}{29}i} \frac{(z-(-\frac{2}{29}+\frac{5}{29}i))}{29z^2+4z+1}$$

$$= 2\pi i \lim_{z\to -\frac{2}{29}+\frac{5}{29}i} \frac{1}{58z+4} = 2\pi i \cdot \frac{1}{10i} = \frac{\pi}{5}.$$

(b) $\deg(z)+1 \leqslant \deg(1+z^2)$, $1+z^2=0 \Rightarrow z=\pm i$ and

i belongs to the upper half plane, so

$$\int_{-\infty}^{\infty} \frac{x\sin x}{1+x^2}\,dx = 2\pi \operatorname{Re}\left(\operatorname*{Res}_{z=i} \frac{ze^{iz}}{1+z^2}\right)$$

$$= 2\pi \operatorname{Re}\left(\lim_{z\to i} \frac{ze^{iz}}{z+i}\right)$$

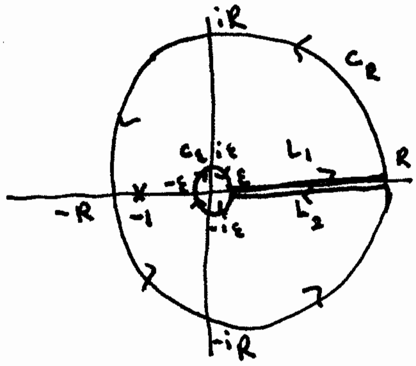
$$= 2\pi \operatorname{Re}\left(\frac{ie^{-1}}{2i}\right) = \frac{\pi}{e}.$$

Question 5. Use residues to evaluate the principal value of $\displaystyle\int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)}$. (20 points)

Answer 5.

Let $$f(z) = \frac{z^{-1/2}}{1+z} = \frac{e^{-\frac{1}{2}\log z}}{1+z} = \frac{e^{-\frac{1}{2}(\ln|z| + i\arg z)}}{1+z}, \quad 0<\arg z<2\pi$$

and $L_{R,\varepsilon}$ is the contour shown below (let $\varepsilon<1<R$)



Then $$\int_{L_{R,\varepsilon}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z)$$

and $$\operatorname*{Res}_{z=-1} f(z) = \lim_{z\to -1} z^{-1/2} = \lim_{z\to-1} e^{-\frac{1}{2}(\ln|z| + i\arg z)} = e^{-\frac{1}{2}(\ln 1 + i\pi)} = e^{-i\frac{\pi}{2}} = -i$$

so, $\int_{L_{R,\varepsilon}} f(z)\,dz = 2\pi$. On the other hand,

$$\int_{L_{R,\varepsilon}} f(z)\,dz = \int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\varepsilon} f(z)\,dz, \text{ and}$$

$$\int_{L_1} f(z)\,dz = \int_\varepsilon^R \frac{e^{-\frac{1}{2}(\ln x + i0)}}{1+x}\,dx = \int_\varepsilon^R \frac{dx}{x^{1/2}(1+x)} \to \int_0^\infty \frac{dx}{x^{1/2}(1+x)} \text{ as } R\to\infty,\ \varepsilon\to 0,$$

$$\int_{L_2} f(z)\,dz = \int_R^\varepsilon \frac{e^{-\frac{1}{2}(\ln x + i2\pi)}}{1+x}\,dx = \int_R^\varepsilon \frac{e^{-i\pi}}{x^{1/2}(1+x)}\,dx = \int_\varepsilon^R \frac{dx}{x^{1/2}(1+x)} \to \int_0^\infty \frac{dx}{x^{1/2}(1+x)} \text{ as } R\to\infty,\ \varepsilon\to 0$$

$$\left|\int_{C_R} f(z)\,dz\right| \le 2\pi R \cdot \frac{1}{R^{1/2}(R-1)} \to 0 \text{ as } R\to\infty,$$

$$\left|\int_{C_\varepsilon} f(z)\,dz\right| \le 2\pi\varepsilon \cdot \frac{1}{\varepsilon^{1/2}(1-\varepsilon)} \to 0 \text{ as } \varepsilon\to 0.$$

Therefore $$2\int_0^\infty \frac{dx}{x^{1/2}(1+x)} = 2\pi \text{ and so}$$

$$\int_0^\infty \frac{dx}{x^{1/2}(1+x)} = \pi.$$

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Final

May 28, 2008

11:00-12:50

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

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GOOD LUCK!

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Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Let $f(z) = \dfrac{\cos z}{z^2}$.

(a) Find the Laurent series representation of f which is valid in $|z| > 0$. (8 points)

(b) Determine the type of the isolated singularity of f at $z = 0$ and find the corresponding residue. (6 points)

(c) Determine the type of the isolated singularity of f at $z = \infty$ and find the corresponding residue. (6 points)

Answer 1.

(a) $$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots \qquad \forall z \in \mathbb{C}$$

So $$f(z) = \frac{\cos z}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots - \right)$$

$$= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \frac{z^4}{6!} + \cdots \qquad \forall z \in \mathbb{C}\setminus\{0\}$$

(b) f has a double pole at $z=0$ and $\operatorname{Res}_{z=0} f(z) = 0$.

(c) f has an essential singularity at $z=\infty$ and $\operatorname{Res}_{z=\infty} f(z) = 0$.

Question 2. Evaluate.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta}$ (10 points)

(b) $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1}$ (10 points)

Answer 2.

(a) Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ and then $\cos\theta = \frac{z^2+1}{2z}$ and $d\theta = \frac{dz}{iz}$.

So,
$$\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = \oint_{|z|=1} \frac{1}{5-\frac{4(z^2+1)}{2z}} \cdot \frac{dz}{iz}$$

$$= -\frac{2}{i}\oint_{|z|=1} \frac{dz}{4z^2-10z+4} = i\oint_{|z|=1}\frac{dz}{2z^2-5z+2}$$

$$= i\oint_{|z|=1}\frac{dz}{(z-2)(2z-1)} = i.2\pi i. \operatorname*{Res}_{z=\frac{1}{2}} \frac{1}{(z-2)(2z-1)}$$

$$= -2\pi \lim_{z\to\frac{1}{2}} \frac{(z-\frac{1}{2})}{(z-2)(2z-1)} = -2\pi.\frac{1}{2(-\frac{3}{2})} = \frac{2\pi}{3}.$$

(b) Let $P(z)=1$, $Q(z)=z^4+1$ and $\deg(Q) \geq \deg(P)+2$. If $z^4+1=0$ then $z = e^{i\frac{\pi+2k\pi}{4}}$, $k=0,1,2,3$, or equivalently $z = \pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$.

Then by a theorem, we have seen in class,

$$\int_{-\infty}^{\infty}\frac{dx}{x^4+1} = 2\int_0^{\infty}\frac{dx}{x^4+1} = 2\pi i\left(\operatorname*{Res}_{z=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{1}{z^4+1} + \operatorname*{Res}_{z=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{1}{z^4+1}\right)$$

Clearly,
$$\operatorname*{Res}_{z=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{1}{z^4+1} = \lim_{z\to\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{(z-(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}))}{z^4+1} \overset{\text{L.R.}}{=} \lim_{z\to\frac{1}{2}+\frac{i}{\sqrt{2}}}\frac{1}{4z^3}$$

$$= \lim_{z\to\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{z}{4z^4} = -\frac{1}{4}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right), \text{ and}$$

similarly,
$$\operatorname*{Res}_{z=-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}} = \lim_{z\to-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}}\frac{z}{4z^4} = -\frac{1}{4}\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right) \text{ and hence}$$

$$\int_0^{\infty}\frac{dx}{x^4+1} = \pi i\left(-\frac{1}{4}\right)\left[\cancel{\frac{1}{\sqrt{2}}}+\frac{i}{\sqrt{2}}-\cancel{\frac{1}{\sqrt{2}}}+\frac{i}{\sqrt{2}}\right] = \frac{\pi}{2\sqrt{2}}.$$

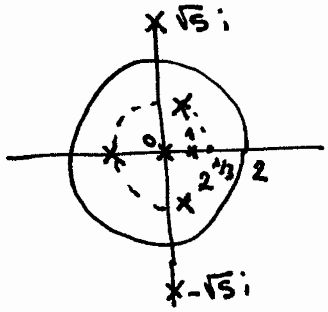
Question 3.

(a) Determine the value of $\Delta_C \arg f(z)$ if C is the circle $|z| = 2$, described in the positive sense and $f(z) = \dfrac{(z^3+2)(z-1)}{z^5(z^2+5)}$. (10 points)

(b) Prove that all the zeros of the polynomial $z^3 + z^2 + 3$ lie in the annulus $1 < |z| < 2$. (10 points)

Answer 3.

(a)



Inside C: $Z_f = 4$

$P_f = 5$

Then

$$\Delta_C \arg f(z) = 2\pi(4-5) = -2\pi$$

(f has simple zeros at $2^{1/3} e^{\frac{i(\pi+2k\pi)}{3}}$, $k=0,1,2$ and $z=1$, and pole of order 5 at $z=0$)

(b) Let $C_1: |z|=1$ and $C_2: |z|=2$

On and inside C_1, let $f(z)=3$ and $g(z)=z^3+z^2$.

If $z\in C_1$, then $|z|=1$ and $|f(z)|=3$, $|g(z)| \le |z|^3+|z|^2 = 2$

Since $|g(z)| < |f(z)|$ on C_1, $Z_{f+g} = Z_f$ inside (and on) C_1. Since f has no zeros inside C_1, $f+g$ has no zeros inside (and on) C_1.

On and inside C_2, let $f(z)=z^3$ and $g(z)=z^2+3$.

If $z\in C_2$, then $|z|=2$ and $|g(z)| \le |z|^2+3 = 7$ and $|f(z)|=8$.

Since $|g(z)|<|f(z)|$ on C_2, $Z_{f+g} = Z_f$ inside and on C_2. Since f has 3 zeros inside C_2, $f+g$ has three zeros inside (and on) C_2. Since $f+g$ has no zeros inside C_1 all three zeros must lie between C_1 and C_2. Since z^3+z^2+3 has exactly three zeros in $\mathbb{C}$, all zeros lie in the annulus, $1<|z|<2$.

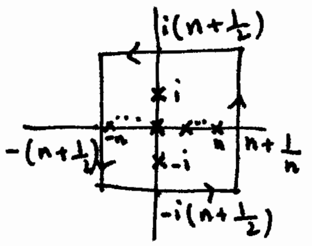
Question 4. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = \pi \coth \pi$$

(20 points)

Answer 4. Let $f(z) = \dfrac{\cot \pi z}{z^2+1} = \dfrac{\cos \pi z}{(z^2+1)\sin \pi z}$ and

L_n:



clearly,

$$\oint_{L_n} f(z)\,dz = 2\pi i \left(\sum_{k=-n}^{n} \operatorname*{Res}_{z=k} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right).$$

On the other hand, $\left| \oint_{L_n} f(z)\,dz \right| \le \dfrac{4(2n+1)}{(n+\frac{1}{2})^2-1} \cdot \max_{z \in L_n} |\cot \pi z| \le \dfrac{4(2n+1)}{(n+\frac{1}{2})^2-1} C$

for some $C > 0$ that does not depend on n. So

$\oint_{L_n} f(z)\,dz \to 0$ as $n \to \infty$. Therefore,

$$\sum_{k=-\infty}^{\infty} \operatorname*{Res}_{z=k} f(z) = -\left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right).$$

Note that, $\operatorname*{Res}_{z=k} f(z) = \lim_{z \to k} \dfrac{(z-k)\cos \pi z}{(z^2+1)\sin \pi z} = \lim_{z\to k} \dfrac{z-k}{\sin \pi z} \lim_{z \to k} \dfrac{\cos \pi z}{z^2+1}$

$$\stackrel{L.R.}{=} \frac{\cos \pi k}{k^2+1} \lim_{z \to k} \frac{1}{\pi \cos \pi z}$$

$$= \frac{\cos \pi k}{k^2+1} \cdot \frac{1}{\pi \cos \pi k} = \frac{1}{\pi(k^2+1)}$$

and so

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+1} = -\pi \left(\operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right)$$

$$= -\pi \left(\lim_{z \to i} \left(\frac{\cot \pi z}{z^2+1} \cdot (z-i) \right) + \lim_{z \to -i} \left(\frac{\cot \pi z}{z^2+1} \cdot (z+i) \right) \right)$$

$$= -\pi \left(\frac{\cot \pi i}{2i} + \frac{\cot(-\pi i)}{-2i} \right)$$

$$= -\pi \left(-i \frac{\coth \pi}{2i} - i \frac{\coth \pi}{2i} \right) = \pi \coth \pi.$$

Question 5. Evaluate

(a) $\displaystyle\oint\limits_{|z|=2} \frac{z^7}{(z^4+1)^2}\,dz$ (10 points)

(b) $\displaystyle\oint\limits_{|z|=2} \frac{1}{(z+1)^4(z^2-9)(z-4)}\,dz$ (10 points)

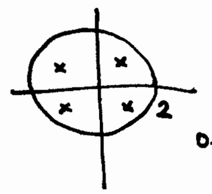
Answer 5.

(a)

$$\oint\limits_{|z|=2} \frac{z^7}{(z^4+1)^2} = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$$

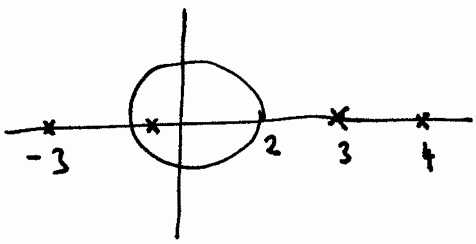
$$= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z(1+z^4)^2}$$

$$= 2\pi i \lim_{z\to 0} \frac{1}{(1+z^4)^2} = 2\pi i$$



all singularities of $\frac{z^7}{z^4+1}$ lie inside the contour.

(b)



let $f(z) = \dfrac{1}{(z+1)^4(z^2-9)(z-4)}$

$$I = \oint\limits_{|z|=2} \frac{1}{(z+1)^2(z^2-9)(z-4)}\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z)$$

$$= -2\pi i \left(\operatorname*{Res}_{z=3} f(z) + \operatorname*{Res}_{z=-3} f(z) + \operatorname*{Res}_{z=4} f(z) + \operatorname*{Res}_{z=\infty} f(z) \right)$$

$$\operatorname*{Res}_{z=3} f(z) = \lim_{z\to 3} \frac{1}{(z+1)^4(z+3)(z-4)} = \frac{1}{4^4 6 (-1)}$$

$$\operatorname*{Res}_{z=-3} f(z) = \lim_{z\to -3} \frac{1}{(z+1)^4(z-3)(z-4)} = \frac{1}{2^4(-6)(-7)}$$

$$\operatorname*{Res}_{z=4} f(z) = \lim_{z\to 4} \frac{1}{(z+1)^4(z^2-9)} = \frac{1}{5^4 \cdot 7}$$

and since $\lim_{z\to\infty} f(z) = 0$, $\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} z f(z) = 0$ and so

$$I = -2\pi i \left(-\frac{1}{6\cdot 4^4} + \frac{1}{42\cdot 2^4} + \frac{1}{7\cdot 5^4} \right)$$

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make-up for the first midterm

June 9, 2008, 10:00-11:50

Questions

(1) (a) Evaluate $\dfrac{1}{\pi i}\displaystyle\oint_C \dfrac{\tan z}{(3z-\pi)^3}\,dz$ where $C = \{z : |z| = 1\}$. (12.5 points)

(b) Evaluate $\displaystyle\int_C (12z^2 - 4iz)\,dz$ where C is the curve $y = x^2$ joining points $(1,1)$ and $(2,4)$, (12.5 points)

(2) Evaluate

(a) $\displaystyle\oint_C \dfrac{z^2+z+1}{z^2(z-1)(z-2)}$ where C is the the circle with radius $\frac{3}{2}$ centered at the origin. (12.5 points)

(b) $\displaystyle\oint_C \dfrac{\cos \pi z}{z^2-1}\,dz$ where C is the rectangle with vertices at $i, -i, 2+i, 2-i$. (12.5 points)

(3) (a) Find all functions $f(z)$ which are analytic in $|z| < 1$ and which satisfy the conditions (a) $f(0) = 1$, (b) $|f(z)| \geq 1$ for $|z| < 1$. (12.5 points)

(b) Find all functions $f(z)$ which are analytic everywhere, satisfy the conditions $|f(z)| \leq 6|z|$ for all z, $f(0) = 0$ and $f(i) = -1$. (12.5 points)

(4) (a) If $\dfrac{z}{e^z+1}$ were expanded into its Maclaurin series, what would be the region of convergence? Do not perform the expansion (12.5 points)

(b) Find the Taylor series centered at $\alpha = 1$ and state where it converges for $f(z) = \dfrac{1-z}{z-3}$. (12.5 points)

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Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make-up for the second midterm

June 9, 2008, 10:00-11:50

Questions

(1) Expand $f(z) = \dfrac{1}{z^2+4z+3}$ in a Laurent series valid for

(a) $1 < |z| < 3$. (5 points)

(b) $|z| > 3$. (5 points)

(c) $0 < |z+1| < 2$. (5 points)

(d) $|z| < 1$. (5 points)

(2) Find and classify all the zeros and singularities of f and calculate the residue of f at each singular point.

(a) $f(z) = (z-3)\sin\dfrac{1}{z+2}$. (10 points)

(b) $f(z) = \dfrac{e^{2z}}{(z-1)^3}$. (10 points)

(3) Evaluate

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5-3\sin\theta}$. (10 points)

(b) $\displaystyle\oint_C \frac{dz}{z^2\sinh z}$ where $C = \{z : |z| = 1\}$. (10 points)

(4) Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+1)}$. (10 points)

(b) $\displaystyle\int_0^{\infty} \frac{\cos 2x}{x^2+1}\,dx$. (10 points)

(5) Use residues to evaluate the principal value of $\displaystyle\int_0^{\infty} \frac{dx}{x^{\frac{1}{3}}(1+x)}$. (20 points)

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Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make-up for the final

June 9, 2008, 10:00-11:50

Questions

(1) Let $f(z) = \dfrac{z - \sin z}{z^3}$.

(a) Find the Laurent series representation of f which is valid in $|z| > 0$. (8 points)

(b) Determine the type of the isolated singularity of f at $z = 0$ and find the corresponding residue. (6 points)

(c) Determine the type of the isolated singularity of f at $z = \infty$ and find the corresponding residue. (6 points)

(2) Evaluate.

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$. (10 points)

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+1)}$. (10 points)

(3) (a) Determine the value of $\triangle_C \arg f(z)$ if C is the circle $|z| = 2$, described in the positive sense and $f(z) = \dfrac{(z^3+2)(z-1)}{z^5(z^2+5)}$. (10 points)

(b) Prove that all the zeros of the polynomial $z^3 + z^2 + 3$ lie in the annulus $1 < |z| < 2$. (10 points)

(4) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+4} = \frac{\pi}{2}\coth(2\pi)$$

(20 points)

(5) Evaluate

(a) $\displaystyle\oint_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\,dz$ (10 points)

(b) Evaluate $\displaystyle\oint_{|z|=\frac{35}{2}} \frac{z^{19}\sin\frac{1}{z}}{(z-1)(z-2)(z-3)\cdots(z-19)}\,dz$. (10 points)

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 352 Complex Analysis II
Make-up for the first and second midterms
June 9, 2008, 10:00-11:50

Questions

(1) (a) Evaluate $\dfrac{1}{\pi i}\oint\limits_C \dfrac{\tan z}{(3z-\pi)^3}\,dz$ where $C=\{z:\,|z|=1\}$. (10 points)

(b) Evaluate $\int\limits_C (12z^2-4iz)\,dz$ where C is the curve $y=x^2$ joining points $(1,1)$ and $(2,4)$, (10 points)

(2) (a) Find all functions $f(z)$ which are analytic in $|z|<1$ and which satisfy the conditions (a) $f(0)=1$, (b) $|f(z)|\geq 1$ for $|z|<1$. (10 points)

(b) Find all functions $f(z)$ which are analytic everywhere, satisfy the conditions $|f(z)|\leq 6|z|$ for all z, $f(0)=0$ and $f(i)=-1$. (10 points)

(3) Evaluate

(a) $\displaystyle\int_0^{2\pi}\frac{d\theta}{5-3\sin\theta}$. (10 points)

(b) $\oint\limits_C \dfrac{dz}{z^2\sinh z}$ where $C=\{z:\,|z|=1\}$. (10 points)

(4) Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+1)(x^2+2x+1)}$. (10 points)

(b) $\displaystyle\int_0^{\infty}\frac{\cos 2x}{x^2+1}\,dx$. (10 points)

(5) Use residues to evaluate the principal value of $\displaystyle\int_0^{\infty}\frac{dx}{x^{\frac{1}{3}}(1+x)}$. (20 points)

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Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make-up for the first and second midterms

June 13, 2008, 10:00-11:50

Questions

(1) (a) Evaluate $\dfrac{1}{\pi i}\oint\limits_C \dfrac{\tan z}{(3z-\pi)^3}\,dz$ where $C = \{z : |z| = 1\}$. (10 points)

(b) Evaluate $\int\limits_C (12z^2 - 4iz)\,dz$ where C is the curve $y = x^2$ joining points $(1,1)$ and $(2,4)$, (10 points)

(2) (a) Find all functions $f(z)$ which are analytic in $|z| < 1$ and which satisfy the conditions (a) $f(0) = 1$, (b) $|f(z)| \geq 1$ for $|z| < 1$. (10 points)

(b) Find all functions $f(z)$ which are analytic everywhere, satisfy the conditions $|f(z)| \leq 6|z|$ for all z, $f(0) = 0$ and $f(i) = -1$. (10 points)

(3) Evaluate

(a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$. (10 points)

(b) $\oint\limits_C \dfrac{e^z}{z^3 + z}\,dz$, where $C = \{z : |z| = 2\}$. (10 points)

(4) Evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$. (10 points)

(b) $\displaystyle\int_0^{\infty} \frac{\cos 2x}{x^2+1}\,dx$. (10 points)

(5) Use residues to evaluate the principal value of $\displaystyle\int_0^{\infty} \frac{dx}{x^{\frac{1}{3}}(1+x)}$. (20 points)

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Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make-up

June 9, 2008, 10:00-11:50

Questions

(1) (a) Evaluate $\displaystyle \frac{1}{\pi i}\oint_C \frac{\tan z}{(3z-\pi)^3}\,dz$ where $C = \{z : |z| = 1\}$. (10 points)

(b) Evaluate $\displaystyle \int_C (12z^2 - 4iz)\,dz$ where C is the curve $y = x^2$ joining points $(1,1)$ and $(2,4)$, (10 points)

(2) Evaluate.

(a) $\displaystyle \int_0^{2\pi} \frac{d\theta}{5 - 3\sin\theta}$. (10 points)

(b) $\displaystyle \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+1)}$. (10 points)

(3) (a) Find all functions $f(z)$ which are analytic in $|z| < 1$ and which satisfy the conditions (a) $f(0) = 1$, (b) $|f(z)| \geq 1$ for $|z| < 1$. (10 points)

(b) Prove that all the zeros of the polynomial $z^3 + z^2 + 3$ lie in the annulus $1 < |z| < 2$. (10 points)

(4) Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+4} = \frac{\pi}{2}\coth(2\pi)$$

(20 points)

(5) Evaluate

(a) $\displaystyle \oint_{|z|=3} \frac{(z-1)^3}{z(z+2)^3}\,dz$ (10 points)

(b) $\displaystyle \oint_C \frac{dz}{z^2\sinh z}$ where $C = \{z : |z| = 1\}$. (10 points)