ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

1st Midterm

March 29, 2007

11:40-13:30

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for

(a) $|z| < 1$. (5 points)
(b) $1 < |z| < 2$. (5 points)
(c) $|z| > 2$. (5 points)
(d) $|z-1| > 1$. (5 points)

Answer 1.

$$\frac{z}{(z-1)(2-z)} = \frac{A}{z-1} + \frac{B}{2-z} \Rightarrow A(2-z)+B(z-1) = z, \qquad z=2 \Rightarrow$$

$B=2$ and $z=1 \Rightarrow A=1$. So $f(z) = \dfrac{1}{z-1} + \dfrac{2}{2-z}$

(a) $$f(z) = \frac{-1}{1-z} + \frac{2}{2(1-\frac{z}{2})} = -\sum_{k=0}^{\infty} z^k + \sum_{k=0}^{\infty} \frac{z^k}{2^k}$$

(b) $$f(z) = \frac{1}{z(1-\frac{1}{z})} + \frac{2}{2(1-\frac{z}{2})} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} + \sum_{k=0}^{\infty} \frac{z^k}{2^k}$$

(c) $$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{2}{z(1-\frac{2}{z})} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} - \frac{2}{z}\sum_{k=0}^{\infty} \frac{2^k}{z^k}$$

(d) $$f(z) = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{z-1}\,\frac{1}{1-\frac{1}{z-1}}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\sum_{k=0}^{\infty} \frac{1}{(z-1)^k}$$

Question 2. Let $f(z) = \dfrac{1-e^z}{z^3}$.

(a) Locate the zeros of f and determine the order of each zero. (10 points)

(b) Locate the poles of f and determine the order of each pole, and find the corresponding residue. (10 points)

Answer 2.

$1-e^z = 0 \Rightarrow e^z = 1 \Rightarrow z = 2n\pi i$ for $n \in \mathbb{Z}$ and

$\frac{d}{dz}(1-e^z)\Big|_{z=2n\pi i} = -e^z\Big|_{z=2n\pi i} = -1 \neq 0$. Therefore $1-e^z$ has simple zeros at $z = 2n\pi i$, $n \in \mathbb{Z}$.

Clearly, z^3 has a zero of order 3 at the origin.

(a) By previous discussion, f has simple zeros at $z = 2n\pi i$, $n \in \mathbb{Z}\setminus\{0\}$.

(b) Similarly, f has a double pole at the origin, and

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1-e^z}{z^3}\cdot z^2\right) = \lim_{z\to 0} \frac{d}{dz}\frac{1-e^z}{z}$$

$$= \lim_{z\to 0} \frac{-e^z\cdot z - (1-e^z)}{z^2} = \lim_{z\to 0} \frac{-ze^z + e^z - 1}{z^2}$$

$$\overset{\text{L.R.}}{=} \lim_{z\to 0} \frac{-e^z - ze^z + e^z}{2z} = \lim_{z\to 0} -\frac{e^z}{2} = -\frac{1}{2}.$$

OR

$$\frac{1-e^z}{z^3} = \frac{1-\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)}{z^3}$$

$$= -\frac{1}{z^2} - \frac{1}{2z} - \frac{1}{6} - \cdots$$

$$\Rightarrow \operatorname*{Res}_{z=0} f(z) = -\frac{1}{2}.$$

Question 3. Show that $\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \frac{5\pi}{32}$. (20 points)

Answer 3.

Put $z = e^{i\theta}$, then $dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$, and

$$\frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{1}{2i}(\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)) = \sin\theta \Rightarrow \sin\theta = \frac{z^2-1}{2iz}.$$

Then

$$I = \int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)^2} = \oint_{|z|=1} \frac{dz}{\left(5 - \frac{3(z^2-1)}{2iz}\right)^2 iz}$$

$$= \oint_{|z|=1} \frac{4iz}{(3z^2-10iz-3)^2} dz = \oint_{|z|=1} \frac{4iz}{\left[3\left(z-\frac{i}{3}\right)(z-3i)\right]^2} dz$$

$$= 2\pi i \operatorname{Res}_{z=\frac{i}{3}} \frac{4iz}{\left[3\left(z-\frac{i}{3}\right)(z-3i)\right]^2}$$

$$= 2\pi i \lim_{z\to\frac{i}{3}} \frac{d}{dz} \frac{4iz}{(3(z-3i))^2}$$

$$= 2\pi i \lim_{z\to\frac{i}{3}} \frac{4i(3(z-3i))^2 - 18(z-3i)4iz}{(3(z-3i))^4}$$

$$= 2\pi i \left[\frac{4i}{\left(3\left(\frac{i}{3}-3i\right)\right)^2} - \frac{6\cdot 4i\cdot\frac{i}{3}}{\left(3\left(\frac{i}{3}-3i\right)\right)^3}\right] = 2\pi i\left(-\frac{1}{16}i + \frac{1}{64i}\right)$$

$$= \frac{\pi}{8} + \frac{\pi}{32} = \frac{5\pi}{32}.$$

Question 4. Choose one of the integrals below and evaluate. If you evaluate more than one integral, 10 more points will be given for each extra solution.

(20 points)

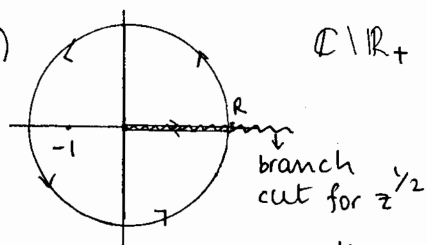
(a) $\displaystyle\int_0^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}\,dx.$

(b) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4-1}\,dx.$

(c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\,dx.$

Answer 4.

(a)



consider the branch of $z^{1/2}$ defined on $\mathbb{C}\setminus\mathbb{R}_+$ by $z^{1/2} = |z|^{1/2} e^{i\frac{1}{2}\arg z}$, $0<\arg z<2\pi$

and the contour $L_R = [0,R]\cup\underbrace{\{z:|z|=R\}}_{C_R}\cup[R,0]$

Let $f(z) = \dfrac{z^{1/2}}{(1+z)^2}$. Then

$$2\pi i \operatorname{Res}_{z=-1} f(z) = \oint_C f(z) = \int_0^R \frac{x^{1/2}}{(1+x)^2}dx + \int_{C_R} \frac{z^{1/2}}{(1+z)^2}dz + \int_R^0 \frac{x^{\frac{1}{2}}e^{i\pi}}{(1+x)^2}dx$$

$$2\pi i \lim_{z\to -1}\frac{d}{dz} z^{1/2} = 2\int_0^R \frac{x^{1/2}}{(1+x)^2}dx + \int_{C_R}\frac{z^{1/2}}{(1+z)^2}dz.$$

Note that $\left|\displaystyle\int_{C_R}\frac{z^{1/2}}{(1+z)^2}dz\right| \le \dfrac{\sqrt{R}}{(R-1)^2}\cdot 2\pi R \to 0$ as $R\to\infty$.

Then $\displaystyle\int_0^\infty \frac{x^{\frac{1}{2}}}{(1+x)^2}dx = \pi i\cdot \frac{1}{2z^{\frac{1}{2}}}\Big|_{z=-1} = \frac{\pi i}{2\cdot 1\cdot e^{\frac{1}{2}\pi i}} = \frac{\pi}{2}.$

(b) $\deg(x^2)+2 \le \deg(x^4-1)$ and $x^4-1 = \underbrace{(x-1)(x+1)}_{\text{on } \mathbb{R}}(x-i)(x+i)$ ← $(x-i)$ on the upper half plane

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty}\frac{x^2}{x^4-1}dx = 2\pi i \operatorname{Res}_{z=i}\frac{z^2}{z^4-1} + \pi i\left(\operatorname{Res}_{z=1}\frac{z^2}{z^4-1} + \operatorname{Res}_{z=-1}\frac{z^2}{z^4-1}\right)$$

$$= 2\pi i\cdot\frac{i^2}{(i^2-1)2i} + \pi i\frac{1}{2(1+1)} + \pi i\frac{1}{-2(1+1)} = \frac{\pi}{2}.$$

(c) $\deg(x)+1 \le \deg(x^2+2x+5)$ and $x^2+2x+5 = \underbrace{(x+1-2i)}_{\text{on the upper half plane}}(x+1+2i)$

$$\Rightarrow \text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin\pi x}{x^2+2x+5}dx = 2\pi\, \mathrm{Re}\left(\operatorname{Res}_{z=-1+2i}\frac{e^{i\pi z}z}{z^2+2z+5}\right)$$

$$= 2\pi\,\mathrm{Re}\frac{e^{i\pi(-1+2i)}(-1+2i)}{4i} = 2\pi\,\mathrm{Re}\left(\frac{1-2i}{4i}\right)e^{-2\pi} = -\pi e^{-2\pi}.$$

Question 5.

(a) Prove that all the roots of

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$. (10 points)

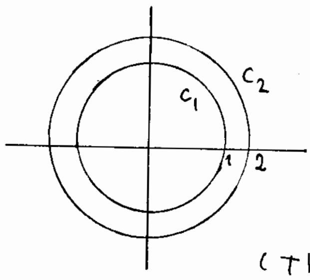
(b) Let $f(z) = \dfrac{(z^2+1)^2}{(z^2+2z+2)^3}$. Evaluate

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz$$

where C is the circle $|z| = 4$. (10 points)

Answer 5.

(a)



On C_1, let $f_1 = 12$ $f_2 = z^7 - 5z^3$

Clearly $|f_1| = 12$ on C_1 and $|f_2| \leq 1 + 5 = 6$ on C_1

and $Z_{f_1} = Z_{f_1+f_2} = 0$ inside and on C_1.

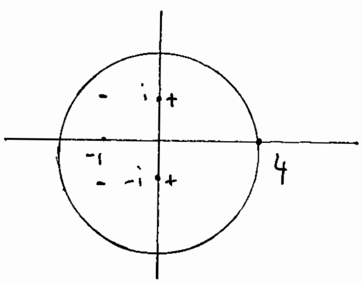
(That is, $z^7 - 5z^3 + 12$ has no zeros inside and on C_1)

On C_2, let $f_1 = z^7$ $f_2 = -5z^3 + 12$. $|f_1| = 128$ and

$|f_2| \leq 5.8 + 12 = 52$ on C_2 So $Z_{f_1} = Z_{f_1+f_2} = 7$ inside and on C_2.

Since $z^7 - 5z^3 + 12$ has 7 zeros, all of them lie between the circles $|z| = 1$ and $|z| = 2$.

(b)



$$f(z) = \frac{(z-i)^2(z+i)^2}{(z+1-i)^3(z+1+i)^3}$$

f has double zeros at i and $-i$

and f has triple pole at $-1+i$ and $-1-i$

all of them are included inside the circle $|z| = 4$.

Then by the Argument principle

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = Z_f - P_f = 4 - 6 = -2.$$

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2nd Midterm

May 17, 2007

11:40-13:30

Surname : ____________________

Name : ____________________

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Department : ____________________

Section : ____________________

Instructor : ____________________

Signature : ____________________

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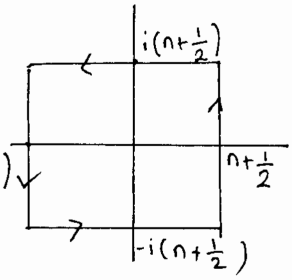
Question 1. Show that $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. (20 points)

Answer 1.

Let $f(z) = \frac{1}{z^4} \cot \pi z = \frac{1}{z^4} \frac{\cos \pi z}{\sin \pi z}$ and L_n :



Then $$\oint_{L_n} f(z)dz = 2\pi i \sum_{k=-n}^{n} \operatorname{Res}_{z=k} f(z) = 2\pi i \Big(\sum_{\substack{k=-n \\ k\neq 0}}^{n} \operatorname{Res}_{z=k} f(z) + \operatorname{Res}_{z=0} f(z) \Big)$$

We have proved in class that $\exists C>0$ (not depending on n) such that $|\cot \pi z| < C$ for all $z \in L_n$. Therefore

$$\Big| \oint_{L_n} f(z)dz \Big| \leq \frac{C}{(n+\frac{1}{2})^4} 4(2n+1) \to 0 \quad \text{as } n \to \infty .$$

f has a pole of order 5 at $z=0$ and simple poles at $-n \leq k \leq n$, $k \neq 0$. Thus,

$$\operatorname{Res}_{\substack{z=k \\ k\neq 0}} f(z) = \lim_{z\to k} \frac{1}{z^4} \frac{\cos \pi z}{\sin \pi z}(z-k) = \frac{\cos \pi k}{k^4} \lim_{z \to k} \frac{z-k}{\sin \pi z}$$

$$\overset{L.R.}{=} \frac{\cos \pi k}{k^4} \lim_{z\to k} \frac{1}{\pi \cos \pi z} = \frac{\cos \pi k}{k^4 \pi \cos \pi k} = \frac{1}{\pi k^4}.$$

To find the residue at $z=0$, we can use the series division

$$\frac{1}{z^4} \frac{\cos \pi z}{\sin \pi z} = \frac{1}{z^4} \frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \frac{\pi^6 z^6}{6!} + \cdots}{\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \frac{\pi^7 z^7}{7!} + \cdots} = \frac{1}{\pi z^5} \frac{1 - \frac{\pi^2 z^2}{2!} + \cdots}{1 - \frac{\pi^2 z^2}{3!} + \cdots}$$

$$= \frac{1}{\pi z^5} (1 + a_1 z + a_2 z^2 + \cdots) \quad \text{and} \quad \operatorname{Res}_{z=0} f(z) = \frac{a_4}{\pi}.$$

So, $$\sum_{k=1}^{\infty} \frac{1}{n^4} = \frac{1}{2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^4} = \frac{1}{2}(-\pi) \operatorname{Res}_{z=0} f(z) = -\frac{a_4}{2}.$$ It only remains to find a_4. Note that

$$\Big(1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \cdots \Big) = \Big(1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \cdots \Big)(1 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots)$$

implies $a_1 = 0$, $-\frac{\pi^2}{2} = a_2 - \frac{\pi^2}{6}$, $0 = a_3 - \frac{\pi^2 a_1}{6}$ and

$\frac{\pi^4}{24} = a_4 - \frac{\pi^2}{6} a_2 + \frac{\pi^4}{120}$. Solving these equations we get $a_4 = -\frac{\pi^4}{45}$,

and so $$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Question 2. Evaluate $\displaystyle\oint_{|z|=\frac{37}{2}} \frac{z^{19}\sin\frac{1}{z}}{(z-1)(z-2)(z-3)\cdots(z-19)}\,dz.$ (20 points)

Answer 2.

$$I = \oint_{|z|=\frac{37}{2}} \frac{z^{19}\sin\frac{1}{z}}{(z-1)(z-2)\cdots(z-19)}\,dz = 2\pi i\left(\sum_{k=0}^{18} \operatorname*{Res}_{z=k} f(z)\right)$$

$$= -2\pi i\left(\operatorname*{Res}_{z=19} f(z) + \operatorname*{Res}_{z=\infty} f(z)\right).$$

f has a simple pole at $z=19$, and

$$\operatorname*{Res}_{z=19} f(z) = \lim_{z\to 19} \frac{z^{19}\sin\frac{1}{z}}{(z-1)\cdots(z-19)}(z-19) = \frac{19^{19}\sin\frac{1}{19}}{18.17.\cdots 1}$$

$$= \frac{19^{19}\sin\frac{1}{19}}{18!}.$$

Note that $\displaystyle\lim_{z\to\infty} \frac{z^{19}\sin\frac{1}{z}}{(z-1)(z-2)\cdots(z-19)} = \lim_{z\to\infty} \sin\frac{1}{z} = 0$

and so

$$\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} z f(z) = -\lim_{z\to\infty} \frac{z^{20}\sin\frac{1}{z}}{(z-1)\cdots(z-19)}$$

$$= -\lim_{z\to\infty} \frac{\sin\frac{1}{z}}{\frac{1}{z}} = -1.$$

Therefore,
$$I = -2\pi i\left(\frac{19^{19}\sin\frac{1}{19}}{18!} - 1\right).$$

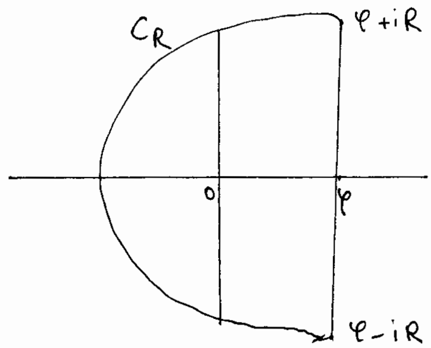
Question 3. Find the inverse Laplace transform of $F(s) = \dfrac{5s-7}{s^3-2s^2-s+2}$. (20 points)

Answer 3.

One can easily show that

$$s^3-2s^2-s+2 = (s-1)(s-2)(s+1)$$

Let $F(s) = \dfrac{5s-7}{s^3-2s^2-s+2}$. Let C_R be the circular part of the contour shown below.



On C_R,

$$|F(s)| \le \frac{5|s|+7}{(|s|-1)(|s|-2)(|s|-1)}$$

$$\le \frac{5(\varphi+R)+7}{(R-\varphi-1)(R-\varphi-2)(R-\varphi-1)}$$

$$\to 0 \quad \text{as} \quad R\to\infty.$$

Thus,

$$f(t) = \operatorname*{Res}_{s=1} \frac{5s-7}{s^3-2s^2-s+2}e^{st} + \operatorname*{Res}_{s=2} \frac{5s-7}{s^3-2s^2-s+2}e^{st} + \operatorname*{Res}_{s=-1} \frac{5s-7}{s^3-2s^2-s+2}$$

$$= A+B+C.$$

Clearly,

$$A = \lim_{s\to 1} \frac{5s-7}{(s-2)(s+1)}e^{st} = \frac{-2e^t}{(-1)(2)} = e^t,$$

$$B = \lim_{s\to 2} \frac{5s-7}{(s-1)(s+1)}e^{st} = \frac{3e^{2t}}{1.3} = e^{2t}$$

$$C = \lim_{s\to -1} \frac{5s-7}{(s-1)(s-2)}e^{st} = \frac{-12e^{-t}}{(-2)(-3)} = -2e^{-t},$$

and so

$$f(t) = e^t + e^{2t} - 2e^{-t}.$$

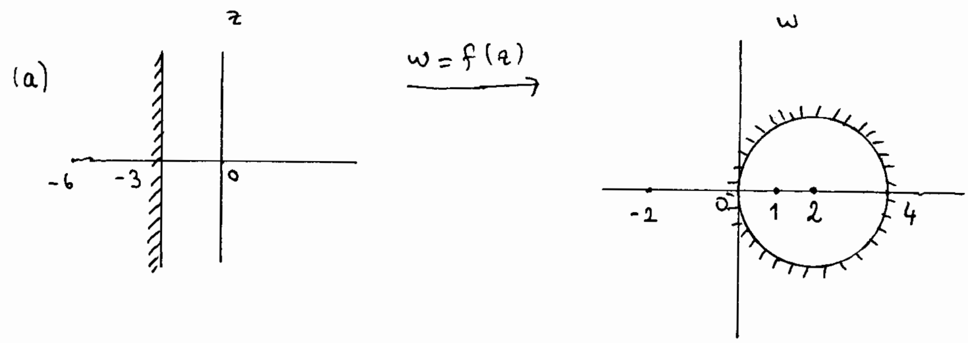
Question 4. Find the bilinear transformation $w = f(z)$ that maps

(a) the half plane $\{z : \operatorname{Re} z > -3\}$ onto the disk $\{w : |w-2| < 2\}$ in such a way that $f(-3) = 0$ and $f(0) = 1$. (10 points)

(b) the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the unit disk $\{w : |w| < 1\}$ in such a way that $f(i) = 0$ and $f'(i) = \frac{1}{2}$. (10 points)

Answer 4.

(a)



By symmetry, -6 is mapped onto -2 (-6 is symmetric to 0 with respect to the line $\operatorname{Re} z = -3$, so the image of -6 should be symmetric to the image of 0, which is 1 with respect to the circle $|w-2| = 2$, and clearly, 1 and -2 are symmetric with respect to the circle $|w-2|=2$.)

Thus, $-3, 0, -6$ are mapped onto $0, 1, -2$ respectively.

Then, a formula seen in class gives the desired bilinear map

$$\frac{z+3}{z+6} \cdot \frac{6}{3} = \frac{w}{w+2} \cdot \frac{3}{1} \Rightarrow$$

$$(2z+6)(w+2) = (z+6)3w \Rightarrow \quad w = \frac{4z+12}{z+12}.$$

(b) The general form of the bilinear map which maps the upper half plane onto the unit disk and α to the origin is:

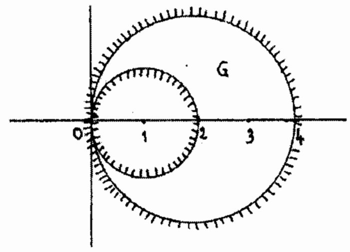
$$f(z) = e^{i\gamma} \frac{z-\alpha}{z-\bar{\alpha}}$$

So $\alpha = i$, we will use the second condition to find γ.

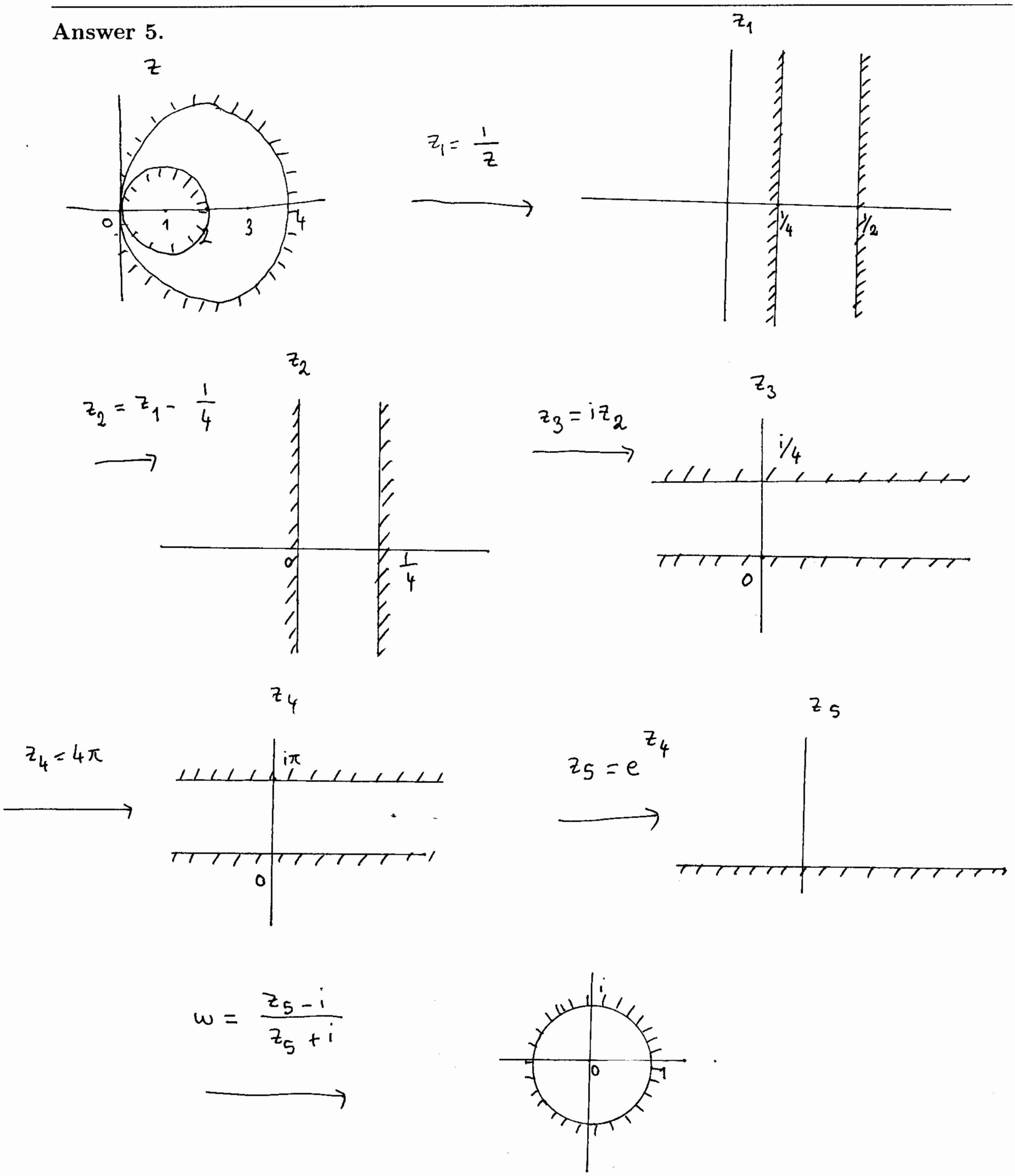
$$f'(z) = e^{i\gamma} \frac{(z+i)-(z-i)}{(z+i)^2} \Rightarrow f'(i) = \frac{e^{i\gamma}}{2i} = \frac{1}{2} \Rightarrow e^{i\gamma} = i \text{ and}$$

$$f(z) = i\,\frac{z-i}{z+i} = \frac{iz+1}{z+i}.$$

Question 5. Map the region $G = \{z : |z-2| < 2\} \backslash \{z : |z-1| \le 1\}$, which is shown below, onto the unit disk.



Answer 5.



ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Final
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Surname : ______________________
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Question 1. Find and classify all the singularities of the functions below, including the singularity at infinity and find the corresponding residues.

(a) $e^{\frac{1}{z-2}}$. (6 points)

(b) $\dfrac{3z^3}{z(z-2)(z^2+1)}$. (8 points)

(c) $\dfrac{\sin z}{z^3}$. (6 points)

Answer 1.

(a) Since $e^{\frac{1}{z-2}} = 1 + \frac{1}{z-2} + \frac{1}{2!(z-2)^2} + \cdots\cdots, \quad |z-2|>0$

and $\lim_{z\to\infty} e^{\frac{1}{z-2}} = 1$, f has an essential singularity at 2 and a removable singularity at ∞. f has no other singularities. Clearly $\operatorname{Res}_{z=2} e^{\frac{1}{z-2}} = 1$ and so

$\operatorname{Res}_{z=\infty} e^{\frac{1}{z-2}} = -1$.

(b) f has simple poles at $2, i, -i$ and removable singularity at ∞ (Because $\lim_{z\to\infty} f(z)$ exists)

And,

$$\operatorname{Res}_{z=2} f(z) = \lim_{z\to 2} \frac{3z^2}{z^2+1} = \frac{12}{5}$$

$$\operatorname{Res}_{z=i} f(z) = \lim_{z\to i} \frac{3z^2}{(z-2)(z+i)} = \frac{-3}{(i-2)2i} = \frac{3-6i}{10}.$$

$$\operatorname{Res}_{z=-i} f(z) = \lim_{z\to -i} \frac{3z^2}{(z-2)(z-i)} = \frac{-3}{(-i-2)(-2i)} = \frac{3+6i}{10}$$

$$\operatorname{Res}_{z=\infty} f(z) = -\left(\frac{12}{5} + \frac{3-6i}{10} + \frac{3+6i}{10}\right) = -3$$

(c)
$$\frac{\sin z}{z^3} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z^3}$$

$$= \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \cdots\cdots, \quad |z|>0$$

$\Rightarrow$ f has a pole of order 2 with residue 0 at the origin and an essential singularity at the infinity with residue 0.

Question 2. Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (20 points)

Answer 2.

Put $z = e^{i\theta}$, then $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z^2+1}{2z}$.

Moreover,

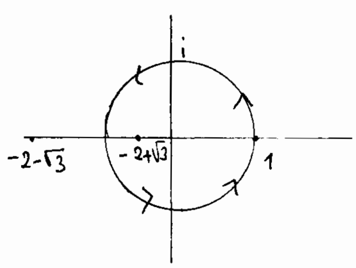
$$I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_{|z|=1} \frac{1}{2+\frac{z^2+1}{2z}} \frac{dz}{iz}$$

$$= \frac{2}{i} \oint_{|z|=1} \frac{dz}{z^2+4z+1}$$

$$= \frac{2}{i} \cdot 2\pi i \operatorname*{Res}_{z=-2+\sqrt{3}} \frac{1}{z^2+4z+1}$$

$$= 4\pi \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}}$$

$$= 4\pi \frac{1}{2\sqrt{3}} = \frac{2\pi}{\sqrt{3}}.$$



$z^2+4z+1=0 \Rightarrow$

$$z = \frac{-4+(16-4.1.1)^{1/2}}{2}$$

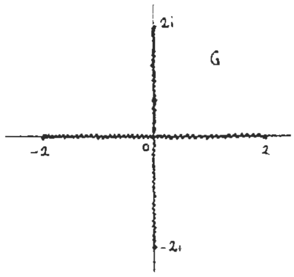
$$= \frac{-4 \pm 2\sqrt{3}}{2}$$

$$= -2 \pm \sqrt{3}$$

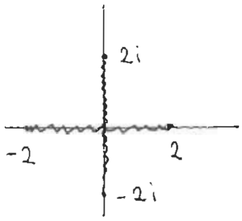
Question 3. Map the region

$$G = \mathbb{C} \setminus (\{z : \operatorname{Re} z = 0,\ -2 \le \operatorname{Im} z \le 2\} \cup \{z : -2 \le \operatorname{Re} z \le 2,\ \operatorname{Im} z = 0\})$$

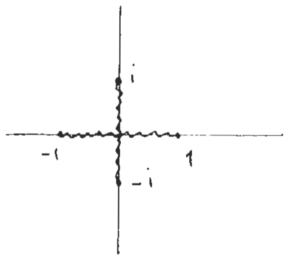
shown in the figure below onto the upper half plane. (20 points)



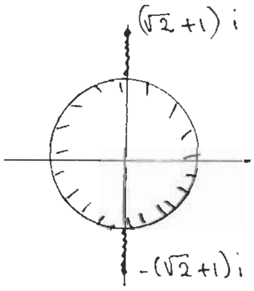
Answer 3.



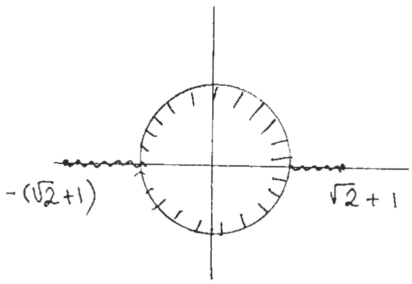
$z_1 = \frac{z}{2}$ $\longrightarrow$



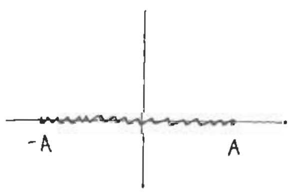
$z_2 = z_1 + \sqrt{z_1^2 - 1}$ $\longrightarrow$



$z_3 = i z_2$ $\longrightarrow$

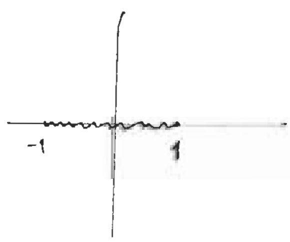


$z_4 = \frac{1}{2}\left(z_3 + \frac{1}{z_3}\right)$ $\longrightarrow$

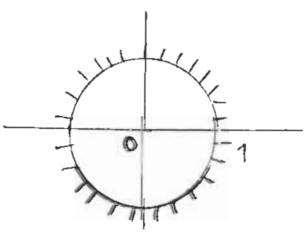


$A = \frac{1}{2}\left(\sqrt{2}+1+\frac{1}{\sqrt{2}+1}\right)$

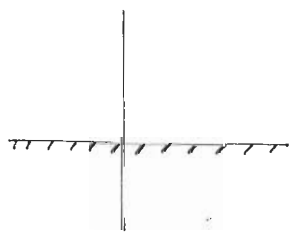
$z_5 = \frac{z_4}{A}$ $\longrightarrow$



$z_6 = z_5 - \sqrt{z_5^2 - 1}$ $\longrightarrow$



$w = \frac{i z_6 + i}{1 - z_6}$



(For the last map: $w = \frac{z-i}{z+i}$ map the upper half plane onto the unit disk, so the inverse of this map maps the unit disk onto the upper half plane: $w(z+i) = z - i \Rightarrow zw + iw = z - i$ $\Rightarrow z(w-1) = -i - iw \Rightarrow z = \frac{iw+i}{1-w}$.)

Question 4.

(a) Evaluate P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^3+1}\,dx.$ (10 points)

(b) Evaluate $\displaystyle\oint_{|z|=3} \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz.$ (10 points)

Answer 4.

(a) $x^3+1 = (x-x_1)(x-x_2)(x-x_3)$ where $x_1 = -1$ (on the real axis)

$x_2 = \frac{1-\sqrt{3}i}{2}$

$x_3 = \frac{1+\sqrt{3}i}{2}$ (on the upper half plane)

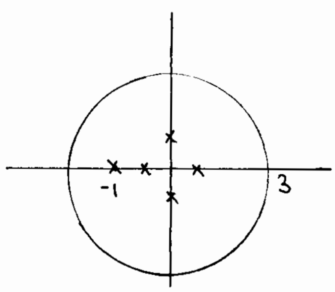
Since $\deg(x^3+1) \geqslant \deg(x)+2$,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x}{x^3+1}\,dx = 2\pi i \operatorname*{Res}_{z=\frac{1+\sqrt{3}i}{2}} \frac{z}{z^3+1} + \pi i \operatorname*{Res}_{z=-1} \frac{z}{z^3+1}.$$

$$= 2\pi i \lim_{z\to\frac{1+\sqrt{3}i}{2}} \frac{z}{(z+1)\left(z-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)} + \pi i \lim_{z\to -1} \frac{z}{z^2-z+1}$$

$$= 2\pi i \frac{\frac{1}{2}+\frac{\sqrt{3}}{2}i}{\left(\frac{3}{2}+\frac{\sqrt{3}}{2}i\right)i\sqrt{3}} + \pi i \frac{-1}{3} = \frac{\pi\sqrt{3}}{3}.$$

(b)



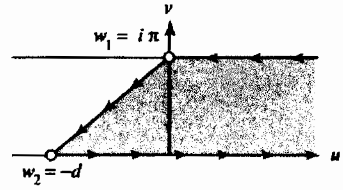
All singularities are inside the contour, thus we can use single residue theorem

$$\oint_{|z|=3} \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} \frac{\frac{1}{z^3}\left(1-\frac{3}{z}\right)}{\left(1+\frac{1}{z}\right)\left(1+\frac{2}{z^4}\right)}$$

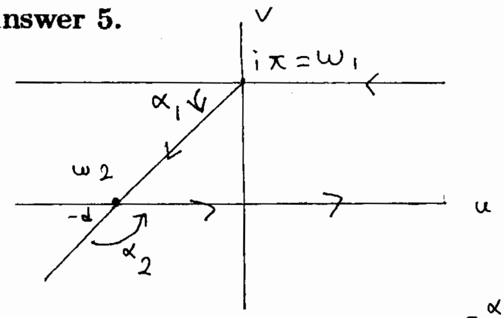
$$= 2\pi i \operatorname*{Res}_{z=0} \frac{z-3}{z(z+1)(z^4+2)}$$

$$= 2\pi i \frac{-3}{1\cdot 2} = -3\pi i.$$

Question 5. Use the Schwarz-Christoffel formula to show that the function $w = f(z) = \operatorname{Log} z$ maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the infinite strip $\{w : 0 < \operatorname{Im} w < \pi\}$. *Hint:* Set $x_1 = -1$, $x_2 = 0$, $w_1 = i\pi$, and $w_2 = -d$ and let $d \to \infty$. The figure below may help. (20 points)



Answer 5.



$$g(z) = A \int (z+1)^{-\frac{\alpha_1}{\pi}} z^{-\frac{\alpha_2}{\pi}} dz + B$$

(maps the shaded region above to the upper half plane)

let $d \to -\infty$

$$f(z) = A \int (z+1)^{-0} z^{-\frac{\pi}{\pi}} dz + B = A \int \frac{1}{z} dz + B$$

$$= A \operatorname{Log} z + B = A(\ln|z| + i \operatorname{Arg} z) + B$$

f maps -1 to $i\pi$ and 0 to ∞

Clearly $A = 1$, $B = 0$ gives the desired conditions $f(-1) = i\pi$ and $f(0) = \infty$.

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make up for the 1st Midterm
June 13, 2007
15:00-17:00

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Expand $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ in a Laurent series valid for

(a) $0 < |z| < 1$. (5 points)
(b) $1 < |z| < \infty$. (5 points)
(c) $0 < |z - i| < 1$. (5 points)
(d) $1 < |z - i| < 2$. (5 points)

Answer 1.

Question 2. Let $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2-4)}$.

(a) Locate the zeros of f and determine the order of each zero. (5 points)

(b) Locate the poles of f and determine the order of each pole, and find the corresponding residue. (15 points)

Answer 2.

Question 3. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta}\, d\theta = \frac{\pi}{12}$. (20 points)

Answer 3.

Question 4. Choose one of the integrals below and evaluate. If you evaluate more than one integral, 10 more points will be given for each extra solution.

(20 points)

(a) $\displaystyle\int_0^\infty \frac{x^{\frac{1}{3}}}{(1+x)^2}\,dx.$

(b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}.$

(c) $\displaystyle\int_0^\infty \frac{\sin 2x}{x(x^2+5)}\,dx.$

Answer 4.

Question 5.

(a) Determine the number of roots of the equation $z^8 - z^3 + z + 18 = 0$ on

(i) $|z| < 1$, (4 points)

(ii) $1 \leq |z| < 2$. (4 points)

(ii) $|z| \geq 2$. (2 points)

(b) Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the value of $\triangle_C \arg f(z)$ when

(i) $f(z) = \dfrac{z^3 + 2}{z}$. (5 points)

(ii) $f(z) = \dfrac{(2z-1)^7}{z^3}$. (5 points)

Answer 5.

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make up for the 1st and 2nd Midterms

June 13, 2007

15:00-17:00

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Choose one of the integrals below and evaluate. If you solve two of them, 20 more points will be given.

(20 points)

(a) $\displaystyle\int_0^\infty \frac{x^{\frac{1}{3}}}{(1+x)^2}\, dx.$

(b) $\displaystyle\int_0^\infty \frac{\sin 2x}{x(x^2+5)}\, dx.$

Answer 1.

Question 2.

(a) Determine the number of roots of the equation $z^8 - z^3 + z + 18 = 0$ on

(i) $|z| < 1$, (4 points)

(ii) $1 \leq |z| < 2$. (4 points)

(ii) $|z| \geq 2$. (2 points)

(b) Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the value of $\triangle_C \arg f(z)$ when

(i) $f(z) = \dfrac{z^3 + 2}{z}$. (5 points)

(ii) $f(z) = \dfrac{(2z - 1)^7}{z^3}$. (5 points)

Answer 2.

Question 3. Find the inverse Laplace transform of $F(s) = \dfrac{s+3}{(s-2)(s^2+1)}$. (20 points)

Answer 3.

Question 4. Find the bilinear transformation $w = f(z)$ that maps

(a) the half plane $\{z : \operatorname{Re} z > -3\}$ onto the disk $\{w : |w - 2| < 2\}$ in such a way that $f(-3) = 0$ and $f(0) = 1$. (10 points)

(b) the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the unit disk $\{w : |w| < 1\}$ in such a way that $f(i) = 0$ and $f'(i) = \dfrac{1}{2}$. (10 points)

Answer 4.

Question 5. Expand $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ in a Laurent series valid for

(a) $0 < |z| < 1$. (5 points)
(b) $1 < |z| < \infty$. (5 points)
(c) $0 < |z - i| < 1$. (5 points)
(d) $1 < |z - i| < 2$. (5 points)

Answer 5.

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 352 Complex Analysis II

Make up for the Final
June 13, 2007
15:00-17:00

Surname	:	______________
Name	:	______________
ID #	:	______________
Department	:	______________
Section	:	______________
Instructor	:	______________
Signature	:	______________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Let $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2-4)}$.

(a) Find the zeros of f and determine the order of each zero. (5 points)

(b) Find and classify the isolated singularities of f (including the singularity at infinity), and the corresponding residues. (15 points)

Answer 1.

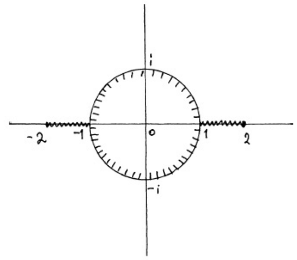
Question 2. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. (20 points)

Answer 2.

Question 3. Map the region

$$G = \{z \in \mathbb{C} : \; |z| > 1\} \backslash \{z \in \mathbb{C} : \; -2 \le \operatorname{Re} z \le 2, \; \operatorname{Im} z = 0\}$$

shown in the figure below onto the upper half plane. (20 points)



Answer 3.

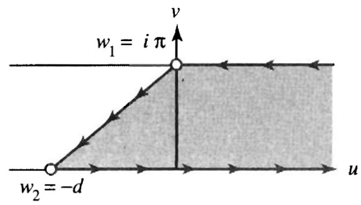
Question 4.

(a) Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$. (10 points)

(b) Evaluate $\displaystyle\oint_{|z|=3} \frac{(3z+2)^2}{z(z-1)(2z+5)}\,dz$. (10 points)

Answer 4.

Question 5. Use the Schwarz-Christoffel formula to show that the function $w = f(z) = \operatorname{Log} z$ maps the upper half plane $\{z : \operatorname{Im} z > 0\}$ onto the infinite strip $\{w : 0 < \operatorname{Im} w < \pi\}$. *Hint*: Set $x_1 = -1$, $x_2 = 0$, $w_1 = i\pi$, and $w_2 = -d$ and let $d \to \infty$. The figure below may help. (20 points)



Answer 5.