ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352 Complex Analysis II

First Midterm Exam
07.03.2000

NAME, SURNAME :

NUMBER :

DEPARTMENT :

SECTION :

SIGNATURE :

INSTRUCTOR'S NAME:

TIME : 09:00

TOTAL NUMBERS : 3

IMPORTANT :
1) Write your name and department.
2)Show all your work. Correct answers without the intermediate steps may not get credit.

Math 352

Complex Analysis II

• First Midterm Exam 8.3.2000

Duration: 60 minutes

Q1. Write the Laurent Series expansion of the function $\frac{1}{z-a}$ for the domain $|a|<|z|<\infty$, where a is real and $-1<a<1$. Then write $z=e^{i\theta}$ to obtain the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos(n\theta) = \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2}, \quad \sum_{n=1}^{\infty} a^n \sin(n\theta) = \frac{a\sin\theta}{1-2a\cos\theta+a^2}.$$

Q2. Show that $\int_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3}$, where C is the unit circle $|z|=1$, taken counterclockwise.

Q3. Find all the Laurent Series in powers of z that represent the function $f(z)=\frac{1}{(z-1)(z-3)}$ in certain domains, and specify those domains.

Note: Each question worths 10 points.

Math 352 KEY

• First Midterm Exam Complex Analysis II 8·3·2000

Q1. $$\frac{1}{z-a} = \frac{1}{z}\,\frac{1}{1-\frac{a}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty}\frac{a^n}{z^{n+1}} \qquad (|a|<|z|<\infty).$$

Take $z=e^{i\theta}$ to get $$\frac{1}{e^{i\theta}-a} = \sum_{n=0}^{\infty}\frac{a^n}{(e^{i\theta})^{n+1}}$$ Or

$$\frac{e^{i\theta}}{e^{i\theta}-a} = \sum_{n=0}^{\infty}\frac{a^n}{(e^{i\theta})^n} = \sum_{n=0}^{\infty}\frac{a^n}{e^{in\theta}} = \sum_{n=0}^{\infty} a^n e^{-in\theta}.$$

Now,

$$\frac{e^{i\theta}}{e^{i\theta}-a} = \frac{1}{1-a e^{-i\theta}} = \frac{1}{1-a(\cos\theta - i\sin\theta)} = \frac{1}{(1-a\cos\theta)+i\,a\sin\theta}$$

$$= \frac{(1-a\cos\theta)-i\,a\sin\theta}{(1-a\cos\theta)^2 + a^2\sin^2\theta} = \frac{(1-a\cos\theta)-i\,a\sin\theta}{1-2a\cos\theta + a^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1})}$$

$$\Rightarrow \frac{e^{i\theta}}{e^{i\theta}-a} = \frac{(1-a\cos\theta)}{1-2a\cos\theta+a^2} + i\,\frac{-a\sin\theta}{1-2a\cos\theta+a^2}.$$ Also

$$\sum_{n=0}^{\infty} a^n e^{-in\theta} = \sum_{n=0}^{\infty} a^n(\cos n\theta - i\sin n\theta)$$
$$= \sum_{n=0}^{\infty} a^n\cos n\theta - i\sum_{n=0}^{\infty} a^n\sin n\theta$$
$$= 1 + \sum_{n=1}^{\infty} a^n\cos n\theta - i\sum_{n=1}^{\infty} a^n\sin n\theta$$

It follows that

$$\sum_{n=1}^{\infty} a^n\sin n\theta = \frac{a\sin\theta}{1-2a\cos\theta+a^2}$$ and

$$\sum_{n=1}^{\infty} a^n\cos n\theta = \frac{1-a\cos\theta}{1-2a\cos\theta+a^2} - 1 = \frac{\cancel{1}-a\cos\theta-\cancel{1}+2a\cos\theta-a^2}{1-2a\cos\theta+a^2}$$
$$= \frac{a\cos\theta - a^2}{1-2a\cos\theta+a^2} \qquad \blacksquare$$

Q2. It is known that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$, where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} \quad\text{and}\quad b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}.$$

Take $n=1$, $z_0=0$ and $f(z)=\frac{1}{z^2\sinh z}$ to see that

$$2\pi i\, b_1 = \int_C \frac{dz}{z^2 \sinh z}$$; where C is any smooth simple contour containing the singular point $z_0 = 0$.

Now, the problem is to find b_1. This can be done easily if we write $f(z) = \dfrac{1}{z^2 \sinh z}$ in its Laurent series expansion; that can be found as follows:

Since $$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{3!} - \frac{z^4}{5!} - \frac{z^6}{7!} - \cdots\cdots\right] = \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{7}{360} z + \cdots$$

It is clear that $b_1 = -\frac{1}{6}$. Hence

$$\int_C \frac{dz}{z^2 \sinh z} = -\frac{1}{6} \cdot 2\pi i = -\frac{\pi i}{3}$$

Q3. $f(z) = \dfrac{1}{(z-1)(z-3)} = \dfrac{1}{2}\left[\dfrac{1}{z-3} - \dfrac{1}{z-1}\right]$ has the two singular points $z = 1$ and $z = 3$, is analytic in the domains $|z| < 1$, $1 < |z| < 3$ and $3 < |z| < \infty$. Let us denote them by $D1$, D_2, and D_3, respectively. Using

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (|z| < 1)$$, we write

$$f(z) = \frac{1}{2}\left[\frac{1}{1-z} + \frac{1}{z-3}\right] = \frac{1}{2}\frac{1}{1-z} - \frac{1}{6}\frac{1}{1-\frac{z}{3}} = \frac{1}{2}\sum_{n=0}^{\infty} z^n - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(1 - \frac{1}{3^{n+1}}\right) z^n$$ is the representation in $D1$ because $|z| < 1$ and $\left|\frac{z}{3}\right| < 1$ in D_1. For the second domain D_2, we write

$$f(z) = \frac{1}{2}\left[-\frac{1}{z}\frac{1}{1-\frac{1}{z}} - \frac{1}{3}\frac{1}{1-\frac{z}{3}}\right] = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n$$

because $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$ when $1 < |z| < 3$. Hence

$$f(z) = -\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}\right)$$ is the representation in D_2.

Similarly;

$$f(z) = \frac{1}{2}\cdot\frac{1}{z}\left[\frac{1}{1-\frac{3}{z}} - \frac{1}{1-\frac{1}{z}}\right] = \frac{1}{2}\cdot\frac{1}{z}\left[\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^n - 1}{z^{n+1}}$$ is the representation in D_3

Since $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{3}{z}\right| < 1$ when $3 < |z| < \infty$.

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352; Complex Analysis II

Second Midterm Exam
19.04.2000

NAME, SURNAME :

NUMBER :

DEPARTMENT :

SECTION :

SIGNATURE :

INSTRUCTOR'S NAME: Y. Doç. Dr. Rajeh Eid

TIME : 09:00

IMPORTANT :
1) Write your name and department.
2)Show all your work. Correct answers without the intermediate steps may not get credit.

Math 352

Second Midterm Exam — Complex Analysis II — 19.4.2000

Duration: 75 minutes.

Q1. (10 points) Find the value of the integral

$$\int_C \frac{dz}{z^3(z+4)},$$

taken counterclockwise around the circle $|z+2| = 3$.

Q2. (10 points) With the aid of residues, evaluate

$$\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}.$$

Q3. (10 points) Use residues to evaluate the improper integral

$$\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx, \qquad a \geq 0.$$

Q4. (6 points) Use residues to evaluate the integral

$$\int_0^\pi \frac{d\theta}{(a+\cos\theta)^2}, \qquad a > 1.$$

Second Midterm Exam — Math 352 — Complex Analysis II — KEY — 19.4.2000

Q1. $f(z) = \dfrac{1}{z^3(z+4)}$ has two isolated singular points $z_0 = 0$ and $z_1 = -4$ which are both in the circle $|z+2| = 3$. Also $z_1 = -4$ is a simple pole, but $z_0 = 0$ is a pole of order 3. Hence

$$\int_C f(z)\,dz = \int_C \frac{dz}{z^3(z+4)} = 2\pi i\,(B_1 + B_0),$$ where B_1 is the residue of $f(z)$ at $z_1 = -4$ and B_0 is the residue of $f(z)$ at $z_0 = 0$.

Now, $f(z) = \dfrac{\phi(z)}{z+4}$, where $\phi(z) = \dfrac{1}{z^3}$. Since $\phi(z)$ is analytic at $z = -4$ and $\phi(-4) = -\frac{1}{64} \neq 0$, $z_1 = -4$ is a simple pole of f; the residue there is $B_1 = \phi(z_1) = \phi(-4) = -\frac{1}{64}$. It left to find B_0,

$$f(z) = \frac{1}{z^3(z+4)} = \frac{1}{z^3}\,\frac{1}{4}\,\frac{1}{1+(z/4)} = \frac{1}{4}\,\frac{1}{z^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{4}\right)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^{n-3}}{4^{n+1}}$$

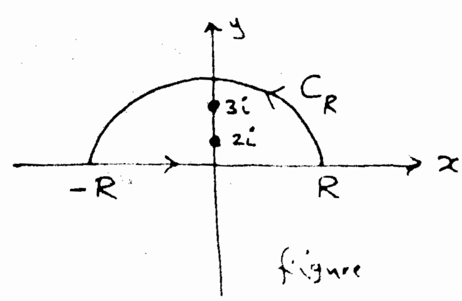
$$\Rightarrow f(z) = \frac{1}{z^3(z+4)} = \frac{1}{4}\cdot\frac{1}{z^3} - \frac{1}{16}\,\frac{1}{z^2} + \frac{1}{64}\,\frac{1}{z} - \frac{1}{256} + \frac{1}{1024}\,z - \cdots$$

which means that $B_0 = \frac{1}{64}$. Therefore,

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i\left(-\frac{1}{64} + \frac{1}{64}\right) = 0.$$

Q2. $f(z) = \dfrac{z^2}{(z^2+9)(z^2+4)^2}$ has isolated singularities at the points $z = \mp 3i$, $z = \mp 2i$ and is analytic everywhere else. When $R > 3$, the singular points of f in the upper half plane lie in the interior of the semicircular region bounded by the segment $z = x$ $(-R \le x \le R)$ of the real axis and the upper half C_R of the circle $|z| = R$ from $z = R$ to $z = -R$ as it's in the figure down. It is clear that

$$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i\,(B_1 + B_2)$$



figure

where B_1 is the residue of f at $z = 3i$ and B_2 is the residue at $z = 2i$.

Q2. Write $f(z)$ as

$$f(z) = \frac{z^2}{(z^2+9)(z^2+4)^2} = \frac{z^2}{(z-3i)(z+3i)(z^2+4)^2} = \frac{\phi(z)}{z-3i}, \text{ where } \phi(z) = \frac{z^2}{(z+3i)(z^2+4)^2}$$

It is clear that $z=3i$ is a simple pole and $B_1 = \phi(3i) = \frac{3i}{50}$. Similarly,

$$f(z) = \frac{z^2}{(z^2+9)(z-2i)^2(z+2i)^2} = \frac{\phi(z)}{(z-2i)^2}, \text{ where } \phi(z) = \frac{z^2}{(z^2+9)(z+2i)^2}.$$

Here $z=2i$ is a pole of order 2 and $B_2 = \phi'(2i)$. But

$$\phi'(z) = \frac{2z[(z^2+9)(z+2i)^2] - z^2[2z(z+2i)^2 + 2(z^2+9)(z+2i)]}{[(z^2+9)(z+2i)^2]^2}$$

$$B_2 = \phi'(2i) = \frac{4i[(-4+9)(-16)] + 4[4i(4i)^2 + 2(-4+9)(4i)]}{[(-4+9)(-16)]^2} = -\frac{13i}{200}$$

$$\Rightarrow \int_{-R}^{R} f(x)\,dx = 2\pi i\left(\frac{-13i+12i}{200}\right) - \int_{C_R} f(z)\,dz = \frac{\pi}{100} - \int_{C_R} f(z)\,dz$$

which is valid for all values of $R>3$. When $|z|=R$;

$$|z^2+9|\,|(z^2+4)^2| = |(z^2+9)(z^2+4)^2| \geq \big||z|^2-9\big|\,\big|(|z|^2-4)^2\big| = (R^2-9)(R^2-4)^2.$$

So, if z is any point on C_R, $|f(z)| \leq M_R$ where $M_R = \frac{R^2}{(R^2-9)(R^2-4)^2}$,

and this means that

$$\left|\int_{C_R} f(z)\,dz\right| \leq M_R \pi R = \frac{\pi R^3}{(R^2-9)(R^2-4)^2},$$

where πR is the length of the semicircle C_R. It is clear that as $R \to \infty$ $\frac{\pi R^3}{(R^2-9)(R^2-4)^2}$ will go to zero. Hence

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0; \text{ Consequently,}$$

$$\lim \int^{R} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{100}, \text{ or } P.V. \int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{100}.$$

Since the integrand is even, then

$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$$

Q3. $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z-i)(z+i)}$, the point $z_1 = i$ is a pole of the function $f(z)e^{aiz}$, with residue $B_1 = \phi(i) = \dfrac{e^{-a}}{2i}$, where $e^{aiz}f(z) = \dfrac{\phi(z)}{z-i}$ and $\phi(z) = \dfrac{e^{aiz}}{z+i}$. When $R>1$ and C_R denote the upper half of the positively oriented circle $|z| = R$,

$$\int_{-R}^{R} \frac{e^{aix}\,dx}{x^2+1} = 2\pi i B_1 - \int_{C_R} f(z)\, e^{aiz}\, dz \text{ ; and this means}$$

that

$$\int_{-R}^{R} \frac{\cos(ax)}{x^2+1}\,dx = Re\left[2\pi i B_1\right] - Re \int_{C_R} f(z)\, e^{aiz}\, dz$$

$$= \pi e^{-a} - Re \int_{C_R} f(z)\, e^{aiz}\, dz .$$

when z is a point on C_R, $|f(z)| \leq M_R = \dfrac{1}{R^2-1}$ $(R>1)$ where $|z^2+1| \geq \left| |z|^2-1 \right| = R^2-1$ and $\left|e^{aiz}\right| \leq 1$. Now

$$\left| Re \int_{C_R} f(z)\, e^{aiz}\, dz \right| \leq \left| \int_{C_R} f(z)\, e^{aiz}\, dz \right| \leq \frac{\pi R}{R^2-1} = M_R \pi R$$

which tends to zero as $R \to \infty$. Hence

$$P.V. \int_{-\infty}^{\infty} \frac{\cos(ax)\,dx}{x^2+1} = \pi e^{-a} \text{ and then}$$

$$\int_{0}^{\infty} \frac{\cos(ax)\,dx}{x^2+1} = \frac{\pi}{2} e^{-a} .$$

Q4. $z = e^{i\theta}$, $(0 \leq \theta \leq 2\pi)$ implies that

$dz = i e^{i\theta} d\theta$ or $d\theta = \dfrac{dz}{iz}$, $\sin\theta = \dfrac{z - z^{-1}}{2i}$ and $\cos\theta = \dfrac{z+z^{-1}}{2}$.

$$\Rightarrow \int_{0}^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} \text{ takes the form } \int_C \frac{dz}{iz\left(a + \frac{z+z^{-1}}{2}\right)^2}$$

where C is the positive oriented circle $|z| = 1$.

Q4. Now $\int_C \frac{dz}{iz\left(a+\frac{z+z^{-1}}{2}\right)^2} = \int_C \frac{-4iz^2\,dz}{z(z^2+2az+1)^2} = \int_C f(z)\,dz$, ..

where $f(z) = \frac{-4iz^2}{z(z-z_1)^2(z-z_2)^2}$ and $z_1 = -a-\sqrt{a^2-1}$
$z_2 = -a+\sqrt{a^2-1}$

$z_0 = 0$, $z_1 = -a-\sqrt{a^2-1}$ and $z_2 = -a+\sqrt{a^2-1}$ are poles of the function $f(z)$ of order, one, two, and two, respectively. But it's not difficult to see that $|z_1| > 1$ which means that

$$\int_C f(z)\,dz = 2\pi i(B_0+B_2), \text{ where}$$

$B_0 = \phi(z_0) = \phi(0) = 0$ with $\phi(z) = \frac{-4iz^2}{(z^2+2az+1)^2}$. For z_2

$B_2 = \phi'(z_2)$, $\phi(z) = \frac{-4iz}{(z-z_1)^2}$, $\phi'(z) = \frac{-4i[z-z_1]^{\cancel{2}}+8iz\cancel{[z-z_1]}}{\cancel{(z-z_1)}(z-z_1)^3}$

$$\Rightarrow \phi'(z) = \frac{-4iz+4iz_1+8iz}{(z-z_1)^3} = 4i\frac{(z+z_1)}{(z-z_1)^3}, \text{ Hence}$$

$$B_2 = \phi'(z_2) = 4i\frac{z_2+z_1}{(z_2-z_1)^3} = 4i\frac{-2a}{(2\sqrt{a^2-1})^3} = -\frac{ia}{(\sqrt{a^2-1})^3}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left[0 - i\frac{a}{(\sqrt{a^2-1})^3}\right] = 2\pi\frac{a}{(\sqrt{a^2-1})^3}.$$

Or
$$\int_0^{2\pi}\frac{d\theta}{(a+\cos\theta)^2} = 2\pi\frac{a}{(\sqrt{a^2-1})^3}. \text{ Therefore,}$$

$$\int_0^{\pi}\frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}.$$

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352 , Complex Analysis II

Final Exam
23.05.2000

- **NAME, SURNAME :**
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME: Y.Doç.Dr.Rajeh Eid
TIME :

- **IMPORTANT :**
1) Write your name and department.
2)Show all your work. Correct answers without the intermediate steps may not get credit.

Math 352

Final Exam — Complex Analysis II — 23.5.2000.

Duration : 90 minutes

Q1. (12+8 points)

a) Use integration through a branch cut to show that

$$\int_0^\infty \frac{x^{-a}}{1+x}\,dx = \frac{\pi}{\sin(a\pi)} \quad , \quad 0<a<1$$

b) The Beta function of two real variables is defined as

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt .$$

Show that $B(p,1-p) = \dfrac{\pi}{\sin(p\pi)}$; $0<p<1$

Q2. (6+6 points)

a) Find the image of the semi-infinite strip $x>0$, $0<y<1$ under the transformation $w=\frac{i}{z}$. Sketch the strip and it's image.

b) Find a linear fractional transformation that maps three given distinct points z_1, z_2 and z_3 onto three specified distinct points w_1, w_2 and w_3, respectively. Then apply it to obtain the linear fractional transformation such that $z_1=0$, $z_2=\infty$, $z_3=2$, $w_1=\infty$, $w_2=0$ and $w_3=1$.

Q3. (8 points) Show that if C is the positively oriented circle $|z|=3$, then

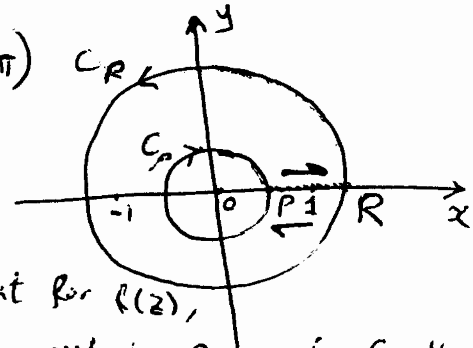
$$\frac{1}{2\pi i}\int_C \frac{z^3 e^{1/z}}{1+z^3}\,dz = 1 .$$

Math 352 KEY

Final Exam Complex Analysis II 23.5.2000

Q1. a) Let C_ρ and C_R denote the circles $|z| = \rho$ and $|z| = R$, respectively, where $\rho < 1 < R$; and we assign them the orientations shown in the following Figure:

Let $f(z) = \dfrac{z^{-a}}{1+z}$ $\quad (|z|>0,\ 0<\arg z<2\pi)$



Consider the closed contour, indicated in the Figure, that's traced out by a point moving from ρ to R along the branch cut for $f(z)$, next around C_R back to R, then along the cut to ρ, and finally around C_ρ back to ρ. Now write $f(z)$ in polar form as

$$f(z) = \frac{e^{-a\log z}}{z+1} = \frac{e^{-a(\ln r + i\theta)}}{re^{i\theta}+1} \qquad (z = re^{i\theta})$$

and use $\theta = 0$ and $\theta = 2\pi$ along the upper and lower edges, respectively, of the cut annulus, we see that

$$f(z) = \frac{e^{-a(\ln r + 0i)}}{r+1} = \frac{r^{-a}}{r+1} \qquad (z = re^{0i},\ \theta = 0)$$

on the upper edge and

$$f(z) = \frac{e^{-a(\ln r + 2\pi i)}}{r+1} = \frac{r^{-a}e^{-2\pi a i}}{r+1} \qquad (z = re^{2\pi i},\ \theta = 2\pi)$$

on the lower edge. Thus

$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{C_R} f(z)\,dz - \int_\rho^R \frac{r^{-a}e^{-2\pi a i}}{r+1}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z)$$

let $f(z) = \dfrac{\phi(z)}{z+1}$, where $\phi(z) = z^{-a} = e^{-a\log z}$, $\begin{pmatrix}|z|>0 \\ 0<\arg z<2\pi\end{pmatrix}$

$$\Rightarrow \phi(z) = e^{-a(\ln r + i\theta)} \Rightarrow \phi(-1) = e^{-a(\ln 1 + i\pi)} = e^{-ia\pi}.$$

Since $z = -1$ is a simple pole of $f(z)$, then

$$\operatorname*{Res}_{z=-1} f(z) = \phi(-1) = e^{-ia\pi}$$

Q1. a) It is not difficult to see that

$$\left| \int_{C_\rho} R(z)dz \right| \le \frac{\rho^{-a}}{1-\rho} 2\pi\rho = \frac{2\pi}{1-\rho} \rho^{1-a} \quad \text{and}$$

$$\left| \int_{C_R} R(z)dz \right| \le \frac{R^{-a}}{R-1} 2\pi R = \frac{2\pi R}{R-1} \cdot \frac{1}{R^a} \text{ . Since } 0<a<1$$

the values of these two integrals tend to zero as $\rho \to 0$ and $R \to \infty$, respectively. Therefore,

$$\int_0^\infty \frac{r^{-a}}{r+1} dr - \int_0^\infty \frac{r^{-a}}{r+1} e^{-2a\pi i} dr = 2\pi i \cdot e^{-\pi a i} \quad \text{or}$$

$$\int_0^\infty \frac{r^{-a}}{r+1} dr = 2\pi i \frac{e^{-\pi a i}}{1-e^{-2a\pi i}} \cdot \frac{e^{\pi a i}}{e^{\pi a i}} = 2\pi i \frac{1}{e^{a\pi i} - e^{-a\pi i}}$$

Or $$\int_0^\infty \frac{x^{-a}}{x+1} dx = \pi \cdot \left(\frac{2i}{e^{a\pi i} - e^{-a\pi i}} \right) = \frac{\pi}{\sin(a\pi)} \quad \blacksquare$$

b) Since $B(p,q) = \int_0^1 t^{p-1} (1-t)^{q-1} dt$,

$$B(p,1-p) = \int_0^1 t^{p-1} (1-t)^{-p} dt = \int_0^1 \frac{t^p}{(1-t)^p} \frac{dt}{t}$$

Let $x = \frac{1-t}{t}$ or $t = \frac{1}{1+x} \Rightarrow dt = -\frac{dx}{(1+x)^2}$

when $t=0 \Rightarrow x \to \infty$
$t=1 \Rightarrow x=0$, this means that

$$B(p,1-p) = \int_0^\infty x^{-p} \cdot \cancel{(1+x)} \cdot \frac{dx}{(1+x)^{\cancel{2}}} = \int_0^\infty \frac{x^{-p}}{1+x} dx = \frac{\pi}{\sin(p\pi)}$$

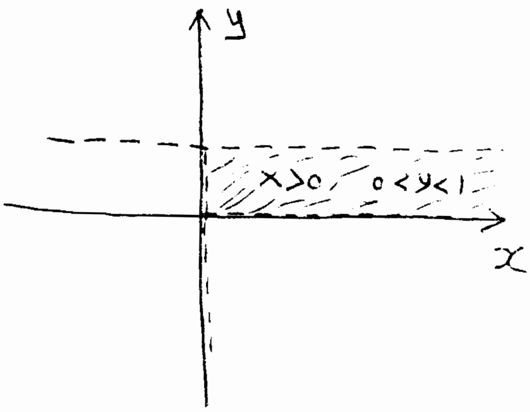
as shown in part a) $\blacksquare$

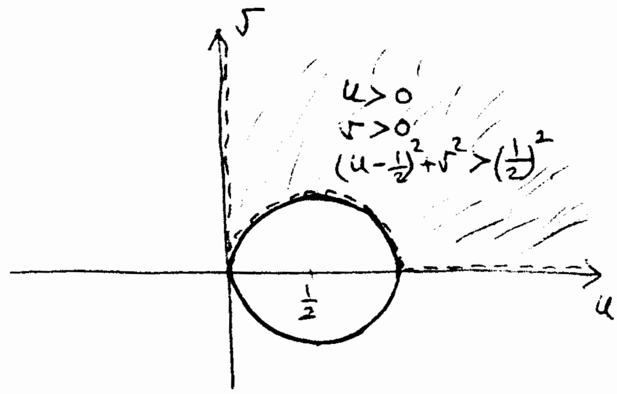
Q2. a) When a point $w = u + iv$ is the image of a nonzero point $z = x + iy$ under the transformation $w = \frac{i}{z}$, we obtain $u = \frac{y}{x^2+y^2}$, $v = \frac{x}{x^2+y^2}$. Similarly $z = \frac{i}{w}$ implies $x = \frac{v}{u^2+v^2}$ and $y = \frac{u}{u^2+v^2}$. Now $x > 0$ implies

$\frac{v}{u^2+v^2} > 0 \Rightarrow v > 0$, $y > 0 \Rightarrow \frac{u}{u^2+v^2} > 0 \Rightarrow u > 0$ and finally

$$y < 1 \Rightarrow \frac{u}{u^2+v^2} < 1 \Rightarrow u^2 + v^2 - u > 0 \Rightarrow u^2 - u + \frac{1}{4} - \frac{1}{4} + v^2 > 0$$

$$\Rightarrow \left(u - \tfrac{1}{2}\right)^2 + v^2 > \left(\tfrac{1}{2}\right)^2.$$





b) The expression

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

is the required transformation for which none of the points z_1, z_2, z_3, w_1, w_2 and w_3 is the point at infinity.

Since z_2 and w_1 are points at infinity, the above expression can be reduced to the form:

$$\frac{w_2 - w_3}{w - w_3} = \frac{z - z_1}{z - z_3}. \text{ Now } w_2 = 0, w_3 = 1, z_1 = 0$$

and $z_3 = 2 \Rightarrow \frac{-1}{w-1} = \frac{z}{z-2} \Rightarrow -w + \cancel{1} = \cancel{1} - \frac{2}{z}$

$\Rightarrow w = \frac{2}{z}$ is the required linear fractional transformation.

Q3. Let $f(z) = \dfrac{z^3 e^{1/z}}{1+z^3}$ and $I = \int_C f(z)\,dz$, where C is the positively oriented circle $|z|=3$.

Since $I = \int_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right)$

$$f\left(\tfrac{1}{z}\right) = \frac{1}{z^3}\frac{e^z}{1+\left(\frac{1}{z}\right)^3} = \cancel{\frac{1}{z^3}}\frac{\cancel{z^3} e^z}{z^3+1} = \frac{e^z}{1+z^3}$$

$$\frac{1}{z^2} f\left(\tfrac{1}{z}\right) = \frac{e^z}{z^2(1+z^3)} = \frac{e^z}{z^2}\frac{1}{1+z^3}$$

$$= \frac{1}{z^2}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right)\left(\sum_{n=0}^{\infty} (-1)^n z^{3n}\right)$$

$$= \frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)\left(1-z^3+z^6-z^9+\cdots\right)$$

$$= \frac{1}{z^2}\left[(1-z^3+z^6-z^9+\cdots)+(z-z^4+z^7-z^{10}+\cdots)+\frac{z^2}{2!}+\cdots\right]$$

$$\Rightarrow \frac{1}{z^2} f\left(\tfrac{1}{z}\right) = \frac{1}{z^2}+\frac{1}{z}+\frac{1}{2}\mp\cdots$$

Hence it's clear that the residue is the coefficient of $\frac{1}{z}$ that is 1, so

$$I = \int_C \frac{z^3 e^{1/z}}{1+z^3}dz = 2\pi i\,(1) = 2\pi i \quad \text{and}$$

$$\frac{1}{2\pi i}\int_C \frac{z^3 e^{1/z}}{1+z^3}dz = 1$$

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 ; Complex Analysis II

Spring Semester First Midterm (Answers)
06.04.2001

- **NAME, SURNAME :**
ID. NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME : Y. Doç. Dr. Mustafa Demirbaş
DURATION : 90 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

- **IMPORTANT :**
1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

1. Let $f(z)=\begin{cases} \dfrac{\sin z-1}{(z-\frac{\pi}{2})^2} & \text{if } z\neq\frac{\pi}{2} \\ \dfrac{-1}{2} & \text{if } z=\frac{\pi}{2} \end{cases}$

(a) Show that $f(z)$ is the sum of the power series

$$(*)\quad \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n-2}=-\frac{1}{2!}+\frac{(z-\frac{\pi}{2})^2}{4!}-\frac{(z-\frac{\pi}{2})^4}{6!}+\cdots$$

for all z whether $z\neq\frac{\pi}{2}$, or $z=\frac{\pi}{2}$.

$$\sin z=\cos\left(\frac{\pi}{2}-z\right)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{\pi}{2}-z\right)^{2n}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n} \text{ for all } z$$

$$\sin z=1+\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n} \text{ for all } z$$

$$\frac{\sin z-1}{(z-\frac{\pi}{2})^2}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n)!}\left(z-\frac{\pi}{2}\right)^{2n-2}=-\frac{1}{2!}+\frac{1}{4!}\left(z-\frac{\pi}{2}\right)^2-\frac{(z-\frac{\pi}{2})^4}{6!}+\cdots$$

for all $z\neq\frac{\pi}{2}$

So $f(z)=\dfrac{\sin z-1}{(z-\frac{\pi}{2})^2}$ is the sum of the series (*) for $z\neq\frac{\pi}{2}$

For $z=\frac{\pi}{2}$, the sum of the series $=\frac{-1}{2}=f\left(\frac{\pi}{2}\right)$.

Therefore $f(z)$ is the sum of the series (*) for all z

(b) Use Part (a) to conclude that f is an entire function.

The sum of a power series is analytic at each point z interior to the circle of convergence of that series. Since $f(z)$ is the sum of a power series for all z f is an entire function.

2. (a) Derive the Maclaurin series representation

$$\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n \quad (|z|<1)$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z|<1$$

$$\frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} n z^{n-1}, \quad |z|<1$$

$$\frac{2}{(1-z)^3} = \sum_{n=2}^{\infty}(n-1)n z^{n-2}, \quad |z|<1$$

$$\frac{1}{(1-z)^3} = \frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n, \quad |z|<1.$$

(b) Derive the Taylor series representation

$$e^z = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \quad \text{for all } z$$

$$e^z = e \cdot e^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} \quad \text{for all } z.$$

(c) Derive the Taylor series representation

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n \quad (|z-1|<1)$$

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n = \sum_{n=0}^{\infty}(-1)^n(z-1)^n, \quad |z-1|<1$$

3.- Represent the function $f(z)=\dfrac{e^z}{z(1-z)^3}$ by its Laurent series (Obtain the first tree terms only)

(a) for the punctured neighborhood $0<|z|<1$

From Question 2(a) we have the power series expansion

(1) $$\frac{1}{(1-z)^3}=\frac{1}{2}\sum_{n=0}^{\infty}(n+1)(n+2)z^n=\frac{1}{2}\left(2+6z+12z^2+20z^3+\cdots\right),\quad |z|<1$$

On the other hand we have

(2) $$\frac{1}{z}e^z=\frac{1}{z}\sum_{n=0}^{\infty}\frac{z^n}{n!}=\sum_{n=0}^{\infty}\frac{z^{n-1}}{n!}=\frac{1}{z}+1+\frac{1}{2}z+\frac{1}{6}z^2+\cdots,\quad 0<|z|<\infty$$

By taking the product of the series in Equations (1) and (2):

$$f(z)=\left(\frac{1}{z}+1+\frac{1}{2}z+\frac{1}{6}z^2+\cdots\right)\left(1+3z+6z^2+10z^3+\cdots\right),\quad 0<|z|<1$$

$$f(z)=\frac{1}{z}+4+\frac{19}{2}z+\frac{53}{3}z^2+\cdots$$

(b) for the punctured neighborhood $0<|z-1|<1$

$$f(z)=\frac{-1}{(z-1)^3}\cdot e^z\cdot\frac{1}{z}=\frac{-1}{(z-1)^3}\cdot e\left[1+(z-1)+\frac{1}{2}(z-1)^2+\frac{1}{6}(z-1)^3+\cdots\right]\cdot\left[1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right]$$

$$f(z)=\frac{-e}{(z-1)^3}\left[1+\frac{1}{2}(z-1)^2-\frac{1}{3}(z-1)^3+\cdots\right]=-\frac{e}{(z-1)^3}-\frac{e}{2(z-1)}+\frac{e}{3}-\cdots,\quad 0<|z-1|<1$$

4. Show that the singular points of the function $f(z)=\dfrac{e^z}{z(1-z)^3}$ are poles. Determine the order m of each pole, and find the corresponding residue B. [Ans. $m_1=1$, $B_1=1$; $m_2=3$, $B_2=-\frac{e}{2}$]

Method I: In Question we have obtained the Laurent series:

(1) $f(z)=\dfrac{1}{z}+4+\dfrac{19}{2}z+\dfrac{53}{3}z^2+\cdots \qquad (0<|z|<1)$

(2) $f(z)=\dfrac{-e}{(z-1)^3}-\dfrac{e/2}{z-1}+\dfrac{e}{3}-\cdots \qquad (0<|z-1|<1)$

From these series representation we conclude that the isolated singular point $z=0$ is a simple pole with $B_1=\underset{z=0}{\operatorname{Res}} f(z)=1$, and the isolated singular point $z=1$ is a pole of order 3 and $B_2=\underset{z=1}{\operatorname{Res}} f(z)=-\dfrac{e}{2}$.

Method II. $f(z)=\dfrac{e^z/(1-z)^3}{z}=\dfrac{\phi(z)}{z}$, where $\phi(z)$ is analytic at $z=0$, and $\phi(0)=1\neq 0$. Therefore the isolated singular point $z=0$ is a simple pole of f and $B_1=\underset{z=0}{\operatorname{Res}} f(z)=1$.

Similarly, $f(z)=\dfrac{-e^z/z}{(z-1)^3}=\dfrac{\psi(z)}{(z-1)^3}$, where $\psi(z)$ is analytic at $z=1$, and $\psi(1)=-e\neq 0$. Therefore the isolated singular point $z=1$ is a pole of order $m=3$.

$$B_2=\underset{z=1}{\operatorname{Res}} f(z)=\frac{\psi^{(m-1)}(1)}{(m-1)!}=\frac{\psi''(1)}{2!}=-\frac{e}{2}$$

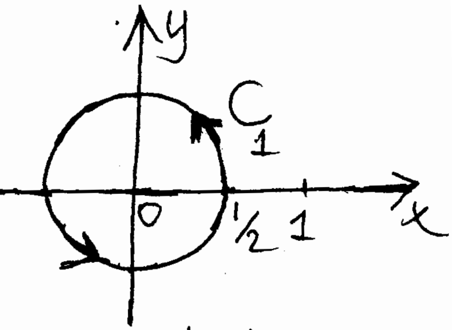
$$\psi(z)=-\frac{e^z}{z},\quad \psi'(z)=-\left(\frac{ze^z-e^z}{z^2}\right)=\frac{(1-z)e^z}{z^2}$$

$$\psi''(z)=\frac{[-e^z+(1-z)e^z]z^2-2z(1-z)e^z}{z^4}$$

$$\psi''(1)=-e$$

5.- Let C_1, C_2 and C_3 be the circles $|z|=\frac{1}{2}$, $|z-1|=\frac{1}{2}$, and $|z|=2$ respectively, all described in the positive sense. Evaluate the following integrals:

(a) $\displaystyle\int_{C_1} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \cdot 1 = 2\pi i$

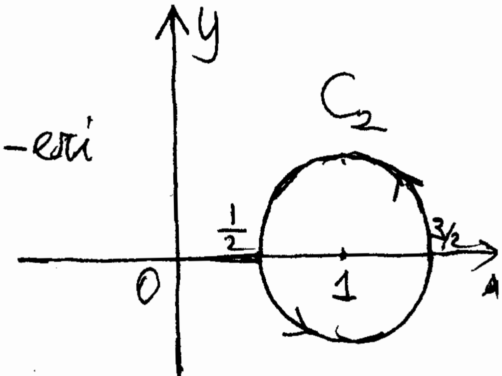


$C_1: |z|=\frac{1}{2}$

$f(z)=\dfrac{e^z}{z(1-z)^3}$ is analytic inside and on C_1 except for $z=0$ inside C_1.

$B_1 = \operatorname*{Res}_{z=0} f(z) = 1$ was obtained in Question 4

(b) $\displaystyle\int_{C_2} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = 2\pi i\left(-\frac{e}{2}\right) = -e\pi i$

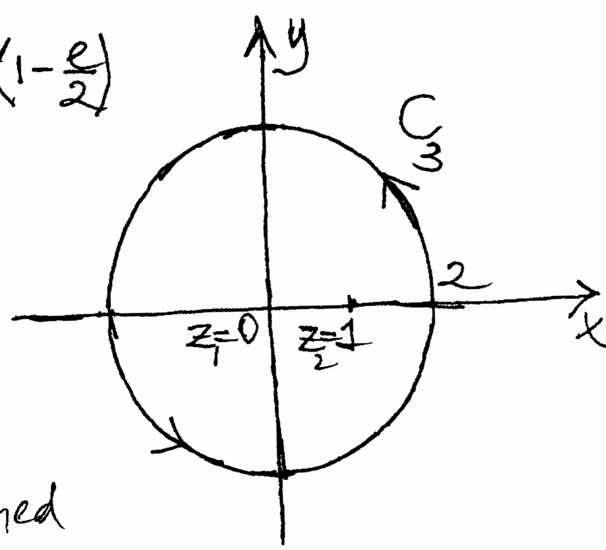


$C_2: |z-1|=\frac{1}{2}$

$f(z)=\dfrac{e^z}{z(1-z)^3}$ is analytic inside and on the simple closed contour C_2 except for the singular point $z=1$ inside C_2.

$B_2 = \operatorname*{Res}_{z=1} f(z) = -\frac{e}{2}$ was obtained in Question 4

(c) $\displaystyle\int_{C_3} \frac{e^z}{z(1-z)^3}\,dz = 2\pi i\,(B_1+B_2) = 2\pi i\left(1-\frac{e}{2}\right)$



$C_3: |z|=2$

$f(z)=\dfrac{e^z}{z(1-z)^3}$ is analytic inside and on C_3 except for the singular points $z_1=0$ and $z_2=1$ inside C_3.

$B_1+B_2 = \operatorname*{Res}_{z_1=0} f(z) + \operatorname*{Res}_{z_2=1} f(z) = 1-\frac{e}{2}$, obtained in Question 4.

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 "Complex Analysis II"

Spring Semester Second Midterm (Answers)
22.05.2001

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION :100 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :

1) Write your name and department.
2) Check that there are 5 questions.
3) Show all your work. Correct answers without the intermediate steps may not get credit.

1. Let $f(z)=\dfrac{z^{-1/2}}{z^2+1}=\dfrac{e^{-\frac{1}{2}\log z}}{z^2+1}$ $\quad(|z|>0,\ 0<\arg z<2\pi)$

(a) Show that $z_1=i$ and $z_2=-i$ are simple poles of $f(z)=\dfrac{p(z)}{q(z)}$ where $p(z)=e^{-\frac{1}{2}\log z}$ and $q(z)=z^2+1$.

Evaluate $B_1=\operatorname*{Res}_{z=i} f(z)$ and $B_2=\operatorname*{Res}_{z=-i} f(z)$.

[Answer: $B_1=\frac{1}{2\sqrt{2}}(-1-i)$, $B_2=\frac{1}{2\sqrt{2}}(1-i)$]

$$B_1=\operatorname*{Res}_{z=i} f(z)=\frac{p(i)}{q'(i)}=\frac{e^{-\frac{1}{2}\log i}}{2i}=\frac{e^{-\frac{1}{2}(\ln 1+i\frac{\pi}{2})}}{2i}=\frac{e^{-i\frac{\pi}{4}}}{2i}$$

$q'(z)=2z$, $q'(i)=2i$. $\quad e^{-i\frac{\pi}{4}}=\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}(1-i)$

$$B_1=\frac{1}{2i}\cdot\frac{1}{\sqrt{2}}(1-i)=-\frac{i}{2\sqrt{2}}(1-i)=\frac{1}{2\sqrt{2}}(-1-i)$$

$$B_2=\operatorname*{Res}_{z=-i} f(z)=\frac{p(-i)}{q'(-i)}=\frac{e^{-\frac{1}{2}\log(-i)}}{-2i}=\frac{e^{-\frac{1}{2}(\ln 1+\frac{3\pi}{2}i)}}{-2i}=\frac{e^{-i\frac{3\pi}{4}}}{-2i}$$

$$B_2=-\frac{1}{2i}\cdot\frac{1}{\sqrt{2}}(-1-i)=\frac{i}{2\sqrt{2}}(-1-i)=\frac{1}{2\sqrt{2}}(1-i)$$

(b) Let C_ρ and C_R denote the circles $|z|=\rho$ and $|z|=R$, respectively, where $0<\rho<1<R$.

Show that
$$|f(z)| \leq \frac{1}{(R^2-1)\sqrt{R}} \quad \text{for } z\in C_R$$
$$|f(z)| \leq \frac{1}{(1-\rho^2)\sqrt{\rho}} \quad \text{for } z\in C_\rho$$

Use these inequalities to show that
$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0, \text{ and } \lim_{\rho\to 0} \int_{C_\rho} f(z)\,dz = 0,$$

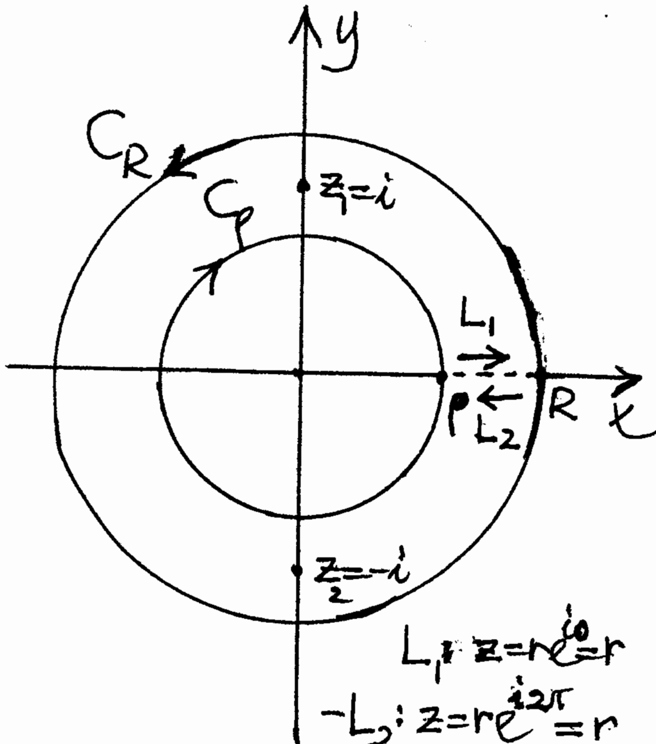
- $$|f(z)| = \frac{\left|e^{-\frac{1}{2}(\ln|z|+i\theta)}\right|}{|z^2+1|} \leq \frac{1/\sqrt{R}}{R^2-1} = \frac{1}{(R^2-1)\sqrt{R}} \quad \text{for } z\in C_\rho$$
$$|z^2+1| \geq |z|^2-1 = R^2-1 \quad \text{for } z\in C_R$$
$$\left|e^{-\frac{1}{2}(\ln|z|+i\theta)}\right| = \left|e^{-\frac{1}{2}\ln R}\cdot e^{-i\frac{\theta}{2}}\right| = e^{-\frac{1}{2}\ln R}\cdot 1 = e^{\ln R^{-1/2}} = R^{-1/2} = \frac{1}{\sqrt{R}} \quad \text{for } z\in C_R$$
$$|f(z)| = \frac{\left|e^{-\frac{1}{2}(\ln|z|+i\theta)}\right|}{|z^2+1|} \leq \frac{1/\sqrt{\rho}}{1-\rho^2} \quad \text{for } z\in C_\rho$$

- $$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{(R^2-1)\sqrt{R}}\,2\pi R \to 0 \text{ as } R\to\infty.$$
$$\left|\int_{C_\rho} f(z)\,dz\right| \leq \frac{1}{(1-\rho^2)\sqrt{\rho}}\,2\pi\rho = \frac{2\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho\to 0.$$

2. Use the results obtained in Question 1 to show that

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the branch of the multiple-valued function



$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{-\frac{1}{2}\log z}}{z^2+1}$$

$$(|z|>0,\ 0<\arg z<2\pi)$$

$L_1: z = re^{i0} = r$

$-L_2: z = re^{i2\pi} = r$

$\rho \le r \le R$

over the closed contour $\Gamma = L_1 + C_R + L_2 + C_\rho$ shown in the figure.

$$\int_\Gamma f(z)dz = \int_{L_1} f(z)dz + \int_{C_R} f(z)dz - \int_{-L_2} f(z)dz + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2)$$

$$\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} + \int_{C_R} f(z)dz + \int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} + \int_{C_\rho} f(z)dz = 2\pi i\left(-\frac{1}{\sqrt{2}}i\right) = \frac{2\pi}{\sqrt{2}}$$

$$2\int_\rho^R \frac{dr}{\sqrt{r}(r^2+1)} + \int_{C_R} f(z)dz + \int_{C_\rho} f(z)dz = \frac{2\pi}{\sqrt{2}}$$

If we take the limit of both sides as $\rho \to 0$ and $R \to \infty$ we obtain

$$\cancel{2}\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\cancel{2}\pi}{\sqrt{2}}$$

3.- Use Rouché's theorem to show that all the roots of the equation

$$h(z) = z^7 - 5z^3 + 12 = 0$$

lie in the annulus $1 \leq |z| < 2$.

Hint: For the circle $C_1 : |z| = 1$ let $f(z) = 12$, and $g(z) = z^7 - 5z^3$.
For the circle $C_2 : |z| = 2$ let $f(z) = z^7$, $g(z) = -5z^3 + 12$.

$f(z) = 12$ and $g(z) = z^7 - 5z^3$ are analytic inside and on the simple closed contour $C_1 : |z| = 1$, and that

$$|g(z)| \leq |z|^7 + 5|z|^3 = 1 + 5 = 6 < 12 = |f(z)| \text{ for } z \in C_1$$

Then $Z_f = 0 = Z_{f+g} = Z_h$ by Rouché's Thm. So $h(z)$ has no zeros inside C_1.

$f(z) = z^7$ and $g(z) = -5z^3 + 12$ are analytic inside an on the simple closed contour $C_2 : |z| = 2$, and that

$$|g(z)| \leq 5|z|^3 + 12 = 5 \cdot 8 + 12 = 52 < 128 = |f(z)| = |z|^7 = |2|^7, \; z \in C_2$$

Then $Z_f = 7 = Z_{f+g} = Z_h$ inside C_2 by Rouché's Thm.

All the ~~roots~~ zeros of $h(z)$ are inside C_2 and none of them lies inside C_1. Therefore all the zeros of $h(z)$ lie in the annulus $1 \leq |z| < 2$.

4.- Let $F(s) = \dfrac{\coth(\pi s/2)}{s^2+1} = \dfrac{\cosh(\pi s/2)}{(s-i)(s+i)\sinh(\pi s/2)}$.

(a) Obtain the following equations:

$$F(s) = \frac{\frac{\pi i}{2} + \frac{\pi^3 i}{2^3 \cdot 3!}(s-i)^2 + \frac{\pi^5 i}{2^5 \cdot 5!}(s-i)^4 + \cdots}{(s+i)\sinh(\pi s/2)} \qquad (0<|s-i|<1)$$

$$F(s) = -\frac{\frac{\pi i}{2} + \frac{\pi^3 i}{2^3 \cdot 3!}(s+i)^2 + \frac{\pi^5 i}{2^5 \cdot 5!}(s+i)^4 + \cdots}{(s-i)\sinh(\pi s/2)} \qquad (0<|s+i|<1)$$

- Explain why $s=\pm i$ are removable singular points of $F(s)$.

Let $g(z) = \cosh(\pi s/2) = \sum_{n=0}^{\infty} \frac{g^{(n)}(i)}{n!}(z-i)^n = \frac{\pi i}{2}(s-i) + \frac{\pi^3 i}{2^3 \cdot 3!}(s-i)^3 + \frac{\pi^5 i}{2^5 \cdot 5!}(s-i)^5 + \cdots$

$g(i) = \cosh\left(\frac{\pi i}{2}\right) = 0$, $g'(z) = \frac{\pi}{2}\sinh\left(\frac{\pi s}{2}\right)$

$g'(i) = \frac{\pi}{2}\sinh\left(\frac{\pi i}{2}\right) = \frac{\pi}{2} \cdot \frac{1}{2}\left(e^{i\frac{\pi}{2}} - e^{-i\frac{\pi}{2}}\right) = \frac{\pi}{2} \cdot \frac{2i}{2} = \frac{\pi i}{2}$

$g''(s) = \left(\frac{\pi}{2}\right)^2 \cosh\left(\frac{\pi s}{2}\right)$, $g''(i) = 0$.

- $g'''(s) = \left(\frac{\pi}{2}\right)^3 \sinh\left(\frac{\pi s}{2}\right)$, $g'''(i) = \frac{\pi^3}{2^3}\sinh\left(\frac{\pi i}{2}\right) = \frac{\pi^3}{2^3} i$.

$$\frac{\cosh(\pi s/2)}{s-i} = \frac{\pi i}{2} + \frac{\pi^3 i}{2^3 \cdot 3!}(s-i)^2 + \frac{\pi^5 i}{2^5 \cdot 5!}(s-i)^4 + \cdots \quad 0<|s-i|<1$$

The series on the right hand side converges also for $s=i$ to $\frac{\pi i}{2}$. Therefore if we define $F(i) = \frac{\pi i}{2} \cdot \frac{1}{2i \cdot i} = \frac{-\pi i}{4}$ F becomes analytic at $s=i$, and therefore $s=i$ is a removable singular point of F

Similarly, $z=-i$ is a removable singular point of $F(s)$.

(b) Let $F(s)=\frac{p(s)}{q(s)}$ where $p(s)=\frac{\cosh(\pi s/2)}{s^2+1}$ and $q(s)=\sinh(\pi s/2)$.

Show that $s_n=2ni$, and $\bar{s}_n=-2ni$ $(n=0,1,2,3,\dots)$ are simple poles of $F(s)$ with $\operatorname{Res}_{s=s_n} F(s)=\frac{2}{\pi}\cdot\frac{1}{1-4n^2}$.

The zeros of $q(s)=\sinh(\pi s/2)$ are the isolated singularities of $F(s)$. $\sinh(\pi s/2)=0$ iff $\frac{\pi s}{2}=n\pi i$, $n=0,\pm1,\pm2,\dots$ So $s_n=2ni$ and $\bar{s}_n=-2ni$ $(n=0,1,2,3,\dots)$ are isolated singular points of $F(s)$.

$q(s)=\sinh(\pi s/2)$ and $p(s)=\frac{\cosh(\pi s/2)}{s^2+1}$ are both analytic at $s_n=2ni$, $p(s_n)=\frac{\cosh(n\pi i)}{1-4n^2}=\frac{(-1)^n}{1-4n^2}\neq 0$

$q(s_n)=0$, and $q'(s_n)=\frac{\pi}{2}\cosh(n\pi i)=\frac{\pi}{2}(-1)^n$. Therefore,

$$\operatorname{Res}_{s=s_n} F(s)=\frac{p(s_n)}{q'(s_n)}=\frac{\cancel{(-1)^n}}{1-4n^2}\cdot\frac{1}{\frac{\pi}{2}\cancel{(-1)^n}}=\frac{2}{\pi}\cdot\frac{1}{1-4n^2}.$$

Thus we have

$$\operatorname{Res}_{s=0} F(s)=\frac{2}{\pi}, \text{ and } \operatorname{Res}_{s=s_n} F(s)=\frac{2}{\pi}\cdot\frac{1}{1-4n^2} \quad (n=1,2,3,\dots)$$

5. Let $F(s)$ be the function given in Question 4.

(a) Use the results in Question 4 to show that

$$\operatorname*{Res}_{s=0}[e^{st}F(s)] = \frac{2}{\pi}, \quad \operatorname*{Res}_{s=s_n}[e^{st}F(s)] + \operatorname*{Res}_{s=\bar{s}_n}[e^{st}F(s)] = -\frac{4}{\pi}\cdot\frac{\cos 2nt}{4n^2-1}$$

where $s_n = 2ni$ $(n = 1, 2, 3, \ldots)$

<u>Hint:</u> If s_0 is a simple pole then $\operatorname*{Res}_{s=s_0}[e^{st}F(s)] = e^{s_0 t}\operatorname*{Res}_{s=s_0}F(s)$

When the <u>simple</u> pole s_0 is of the form $s_0 = \alpha + i\beta$ $(\beta \neq 0)$ and $\overline{F(s)} = F(\bar{s})$ at points points of analyticity of $F(s)$ then

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)] + \operatorname*{Res}_{s=\bar{s}_0}[e^{st}F(s)] = 2e^{\alpha t}\operatorname{Re}[e^{i\beta t}\operatorname*{Res}_{s=s_0}F(s)].$$

Since $s=0$ is a simple pole of $F(s)$ we have

$$\operatorname*{Res}_{s=0}[e^{st}F(s)] = e^{0t}\operatorname*{Res}_{s=0}F(s) = \operatorname*{Res}_{s=0}F(s) = \frac{2}{\pi}.$$

$s_n = \alpha + i\beta = 2ni$ $(\alpha = 0, \beta = 2n \neq 0)$ is a simple pole and $\overline{F(s)} = F(\bar{s})$ at points of analyticity of $F(s)$ then

$$\operatorname*{Res}_{s=s_n}[e^{st}F(s)] + \operatorname*{Res}_{s=\bar{s}_n}[e^{st}F(s)] = 2e^{\alpha t}\operatorname{Re}[e^{i\beta t}\operatorname*{Res}_{s=s_n}F(s)]$$

$$= 2\operatorname{Re}\left[e^{i2nt}\cdot\frac{2}{\pi}\cdot\frac{1}{1-4n^2}\right]$$

$$= \frac{4}{\pi}\cdot\frac{1}{1-4n^2}\cos 2nt \quad (n = 1, 2, 3, \ldots)$$

(b) Find the inverse Laplace transform $f(t)$ of $F(s)$,

<u>Hint:</u> $f(t) = \sum_{n=0}^{\infty} \mathrm{Res}[e^{st}F(s)] \quad (t>0)$

$$f(t) = \underset{s=0}{\mathrm{Res}}[e^{st}F(s)] + \sum_{n=1}^{\infty}\left\{\underset{s=s_n}{\mathrm{Res}}[e^{st}F(s)] + \underset{s=\bar{s}_n}{\mathrm{Res}}[e^{st}F(s)]\right\}$$

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2-1} \quad (t>0).$$

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 "Complex Analysis II"

Spring Semester Final Examination (Answers)
08.06.2001

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION :120 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :
1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

1. Let $f(z) = \dfrac{z^a}{(z^2+1)^2} = \dfrac{e^{a\log z}}{(z^2+1)^2}$, where $-1 < a < 3$ and $|z| > 0$, $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

(a) Determine all singular points of $f(z)$, and tell which ones are isolated. Show that $z = i$ is a pole of order $m = 2$ and $\operatorname{Res}_{z=i} f(z) = \frac{i}{4}(a-1)e^{i\frac{a\pi}{2}}$.

The negative y-axis, $\arg z = -\frac{\pi}{2}$, is the branch cut for $\log z$. Therefore all the points on the negative y-axis together with the origin are singular points of $f(z)$ which are not isolated. The zeros of z^2+1 which are $+i$ and $-i$ are also singular points of $f(z)$. Only $+i$ is isolated.

To show that $z = i$ is a pole of order $2 = m$, we write $f(z) = \dfrac{\phi(z)}{(z-i)^2}$, where $\phi(z) = \dfrac{e^{a\log z}}{(z+i)^2}$ is analytic and nonzero at $z_0 = i$. $\phi(i) = \dfrac{e^{a(\ln 1 + i\frac{\pi}{2})}}{(2i)^2} = -\dfrac{e^{i\frac{a\pi}{2}}}{4}$,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} = \frac{\phi'(z_0)}{1!} = \phi'(i)$$

$$\phi'(z) = \frac{\frac{a}{z}e^{a\log z}(z+i)^2 - 2(z+i)e^{a\log z}}{(z+i)^4} = e^{a\log z}\cdot\frac{a\left(1+\frac{i}{z}\right)-2}{(z+i)^3}$$

$$\phi'(i) = e^{a\log i}\cdot\frac{a(1+1)-2}{-8i} = e^{a(\ln 1 + i\frac{\pi}{2})}\cdot\frac{a-1}{-4i} = e^{i\frac{a\pi}{2}}\cdot\frac{i(a-1)}{4}$$

Thus we have obtained $\operatorname*{Res}_{z=i} f(z) = \frac{i}{4}(a-1)e^{i\frac{a\pi}{2}}$.

(b) Let C_ρ and C_R denote the upper halves of the circles $|z|=\rho$ and $|z|=R$, where $\rho<1<R$.

Obtain the following inequalities:

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^a}{(1-\rho^2)^2}\cdot\pi\rho$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^a}{(R^2-1)^2}\cdot\pi R$$

- Show that $\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0 = \lim_{R\to\infty}\int_{C_R} f(z)\,dz$.

$|f(z)| = \dfrac{|e^{a\log z}|}{|z^2+1|^2} \le \dfrac{\rho^a}{(1-\rho^2)^2}$ for $z\in C_\rho$ since $|z^2+1| \ge 1-|z|^2 = 1-\rho^2$, and $|e^{a\log z}| = |e^{a(\ln\rho+i\theta)}| = \rho^a$ for $z\in C_\rho$.

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^a}{(1-\rho^2)^2}\pi\rho = \frac{\pi\rho^{1+a}}{(1-\rho^2)^2} \to 0 \text{ since } 1+a>0$$

- $|f(z)| = \dfrac{|e^{a\log z}|}{|z^2+1|^2} \le \dfrac{R^a}{(R^2-1)^2}$ for $z\in C_R$ since we have $|z^2+1| \ge |z|^2-1 = R^2-1$, and $|e^{a\log z}| = |e^{a(\ln R+i\theta)}| = R^a$ for $z\in C_R$.

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^a}{(R^2-1)^2}\cdot\pi R = \frac{\pi R^{a+1}}{(R^2-1)^2} \to 0 \text{ since } a+1<4$$

2: Take the indented contour

$\Gamma = L_1 + C_R + L_2 + C_\rho$

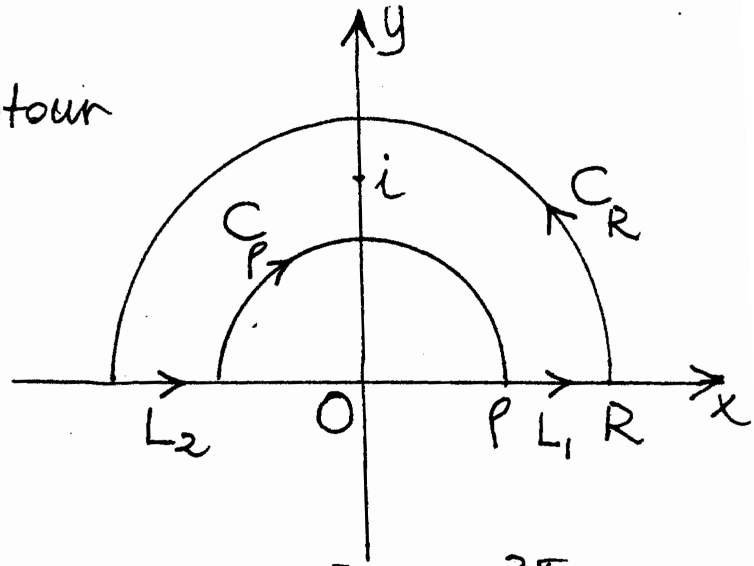
shown in the figure, and use the function

$$f(z) = \frac{z^a}{(z^2+1)^2} = \frac{e^{a \log z}}{(z^2+1)^2},$$

where $-1 < a < 3$,

to show that

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{(1-a)\pi}{4\cos\left(\frac{a\pi}{2}\right)}.$$



$|z|>0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$L_1 :\ z = re^{i0} = r,\ \rho \le r \le R$

$-L_2 :\ z = re^{i\pi} = -r,\ \rho \le r \le R$

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz - \int_{-L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2\pi i \cdot \frac{i}{4}(a-1)e^{i\frac{a\pi}{2}}.$$

$$\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_R} f(z)\,dz + e^{ia\pi} \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_\rho} f(z)\,dz = \frac{\pi(1-a)}{2} e^{i\frac{a\pi}{2}}$$

$$(1+e^{ia\pi}) \int_\rho^R \frac{r^a}{(1+r^2)^2}\,dr + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz = \frac{\pi(1-a)}{2} \cdot e^{i\frac{a\pi}{2}}.$$

Letting $\rho \to 0$ and $R \to \infty$ we get

$$(1+e^{ia\pi}) \int_0^\infty \frac{r^a}{(1+r^2)^2}\,dr = \frac{\pi(1-a)}{2} e^{i\frac{a\pi}{2}},$$

$$\int_0^\infty \frac{r^a}{(1+r^2)^2}\,dr = \frac{\pi(1-a)}{2} \cdot \frac{e^{i\frac{a\pi}{2}}}{1+e^{ia\pi}} = \frac{(1-a)\pi}{2} \cdot \frac{1}{e^{-i\frac{a\pi}{2}} + e^{i\frac{a\pi}{2}}}$$

$$\int_0^\infty \frac{r^a}{(1+r^2)^2}\,dr = \frac{(1-a)\pi}{2} \cdot \frac{1}{2\cos\left(\frac{a\pi}{2}\right)} = \frac{(1-a)\pi}{4\cos\left(\frac{a\pi}{2}\right)}.$$

3.- Let $F(s)=\dfrac{s^2-a^2}{(s^2+a^2)^2}=\dfrac{\phi(s)}{(s-ai)^2}$, where $\phi(s)=\dfrac{s^2-a^2}{(s+ai)^2}$ is analytic at $s_0=ai$ and hence has the Taylor representation

$$\phi(s)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(ai)}{n!}(s-ai)^n=\phi(ai)+\frac{\phi'(ai)}{1!}(s-ai)+\cdots$$

(a) Obtain the principal part $\dfrac{b_1}{(s-ai)}+\dfrac{b_2}{(s-ai)^2}$ in the Laurent series representation of $F(s)$ in the punctured disk $0<|s-ai|<2a$.

$$\phi'(s)=\frac{2s(s+ai)^2-2(s+ai)(s^2-a^2)}{(s+ai)^4}=\frac{2s(s+ai)-2(s^2-a^2)}{(s+ai)^3}=\frac{2ais+2a^2}{(s+ai)^3}$$

$$b_1=\phi'(ai)=\frac{2ai(ai)+2a^2}{(2ai)^3}=\frac{-2a^2+2a^2}{-8a^3i}=0.$$

$$b_2=\phi(ai)=\frac{-2a^2}{(2ai)^2}=\frac{-2a^2}{-4a^2}=\frac{1}{2}.$$

$$F(s)=\frac{1}{(s-ai)^2}\left[\phi(ai)+\frac{\phi'(ai)}{1!}(s-ai)+\frac{\phi''(ai)}{2!}(s-ai)^2+\cdots\right]$$

$$F(s)=\frac{1}{(s-ai)^2}\left[\frac{1}{2}+0\cdot(s-ai)+\frac{\phi''(ai)}{2!}(s-ai)^2+\cdots\right]$$

$$F(s)=\underbrace{\frac{0}{s-ai}+\frac{1/2}{(s-ai)^2}}_{\text{Principal Part}}+\frac{\phi''(ai)}{2!}+\frac{\phi'''(ai)}{3!}(s-ai)+\cdots$$

$0<|s-ai|<2a$

$b_1=0,\ b_2=\frac{1}{2}$

(b) Show that $\operatorname*{Res}_{s=ai}[e^{st}F(s)] + \operatorname*{Res}_{s=-ai}[e^{st}F(s)] = t\cos at$

<u>Hint:</u> You may use the following technique:

Suppose $F(s)$ has a pole of order m at a point $s_0=\alpha+i\beta$ $(\beta\neq 0)$ and that its Laurent series representation in a punctured disk $0<|s-s_0|<R_2$ has principal part

$$\frac{b_1}{(s-s_0)}+\frac{b_2}{(s-s_0)^2}+\cdots+\frac{b_m}{(s-s_0)^m} \qquad (b_m\neq 0)$$

Then

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)]+\operatorname*{Res}_{s=\bar{s}_0}[e^{st}F(s)]=2e^{\alpha t}\operatorname{Re}\left\{e^{i\beta t}\left[b_1+\frac{b_2}{1!}t+\cdots+\frac{b_m}{(m-1)!}t^{m-1}\right]\right\}$$

provided that $\overline{F(s)}=F(\bar{s})$ at points of analyticity of $F(s)$.

We use the formula with $s_0=ia$ $(\alpha=0, \beta=a)$, $b_1=0$, $b_2=\frac{1}{2}$:

$$\operatorname*{Res}_{s=ai}[e^{st}F(s)]+\operatorname*{Res}_{s=-ai}[e^{st}F(s)]=2\operatorname{Re}\left[e^{iat}\cdot\frac{1}{2}t\right]=t\cos at.$$

An alternative method:

$$\operatorname*{Res}_{s=ai}[e^{st}F(s)]=\operatorname*{Res}_{s=ai}\frac{\psi(s)}{(s-ai)^2}=\psi'(ai), \text{ where } \psi(s)=\frac{e^{st}(s^2-a^2)}{(s+ai)^2}.$$

$$\psi'(s)=\frac{[te^{st}(s^2-a^2)+2se^{st}](s+ai)^2-2(s+ai)e^{st}(s^2-a^2)}{(s+ai)^4}.$$

$$\psi'(s)=e^{st}\frac{[t(s^2-a^2)+2s](s+ai)-2(s^2-a^2)}{(s+ai)^3}.$$

$$\psi'(ai)=e^{iat}\frac{[t(-2a^2)+2ai](2ai)+4a^2}{-8a^3i}=e^{iat}\frac{-4a^3it}{-8a^3i}=\frac{t}{2}e^{iat}$$

Similarly, $\operatorname*{Res}_{s=-ai}[e^{st}F(s)]=\frac{t}{2}e^{-ait}$.

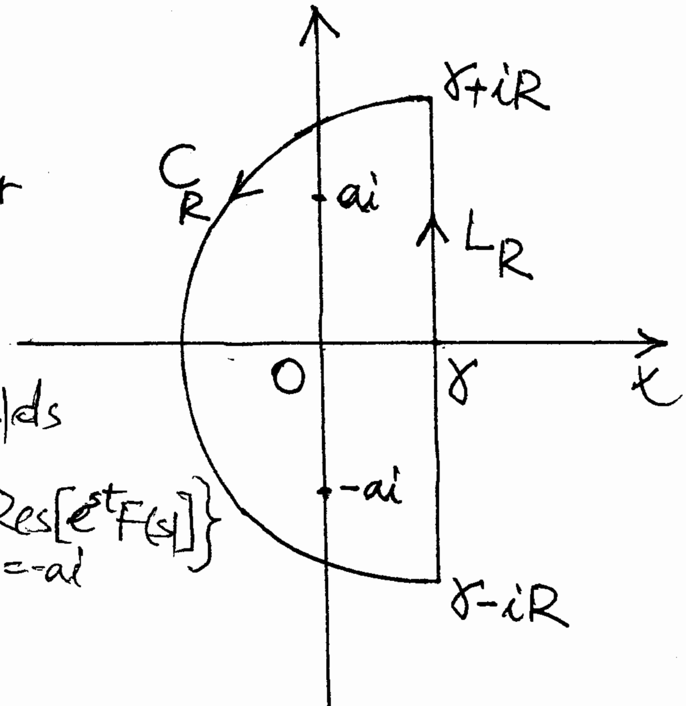
$$\operatorname*{Res}_{s=ai}[e^{st}F(s)]+\operatorname*{Res}_{s=-ai}[e^{st}F(s)]=\frac{t}{2}\left[e^{iat}+e^{-iat}\right]=t\cos at.$$

4. Let $F(s)=\dfrac{s^2-a^2}{(s^2+a^2)^2}$, the function in Question 3.

a) Use Cauchy's residue theorem to show that

$$\int_{L_R} e^{st}F(s)\,ds+\int_{C_R} e^{st}F(s)\,ds = 2\pi i\, t\cos at,$$ where

$\Gamma = L_R + C_R$ is the positively oriented simple closed contour shown in the figure.



$$\int_{L_R} e^{st}F(s)\,ds+\int_{C_R} e^{st}F(s)\,ds=\int_{\Gamma} e^{st}F(s)\,ds$$
$$=2\pi i\left\{\operatorname*{Res}_{s=ai}\left[e^{st}F(s)\right]+\operatorname*{Res}_{s=-ai}\left[e^{st}F(s)\right]\right\}$$
$$=2\pi i\, t\cos at.$$

$C_R:\ s=\gamma+Re^{i\theta},\ R>\gamma+a$

$\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}.$

b) Show that $|F(s)|\le \dfrac{(R+\gamma)^2+a^2}{[(R-\gamma)^2-a^2]^2}=M_R$ for $s\in C_R$, and that $M_R\to 0$ as $R\to\infty$.

<u>Hint</u>: Note that $a<R-\gamma\le|s|\le R+\gamma$ for $s\in C_R$.

$|s^2-a^2|\le|s|^2+a^2\le(R+\gamma)^2+a^2$ for $s\in C_R$.

$|s^2+a^2|\ge|s|^2-a^2\ge(R-\gamma)^2-a^2>0$ for $s\in C_R$.

$|F(s)|=\dfrac{|s^2-a^2|}{|s^2+a^2|^2}\le\dfrac{(R+\gamma)^2+a^2}{[(R-\gamma)^2-a^2]^2}=M_R$ for $s\in C_R$.

$\deg[(R+\gamma)^2+a^2]=2<4=\deg\{[(R-\gamma)^2-a^2]^2\}\Rightarrow M_R\to 0$ as $R\to\infty$

(c) Use the parametric representation of the circle
$C_R: s=\gamma+Re^{i\theta}, \frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$
to obtain the following inequalities:

$$\left|\int_{C_R} e^{st}F(s)ds\right| \le e^{\gamma t}M_R R\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta = e^{\gamma t}M_R R\int_0^{\pi} e^{-Rt\sin\theta}d\theta$$

$$< e^{\gamma t}M_R R\cdot\frac{\pi}{Rt}$$

$$\left|\int_{C_R} e^{st}F(s)ds\right| = \left|\int_{\pi/2}^{3\pi/2} e^{\gamma t}e^{Rt\cos\theta}\cdot e^{iRt\sin\theta}Rie^{i\theta}F(\gamma+Re^{i\theta})d\theta\right|$$

$$\le M_R e^{\gamma t}R\int_{\pi/2}^{3\pi/2} e^{Rt\cos\theta}d\theta \underset{(*)}{=} e^{\gamma t}M_R R\int_0^{\pi} e^{-Rt\sin\theta}d\theta$$

$$< e^{\gamma t}M_R R\cdot\frac{\pi}{Rt}$$

To obtain the eq. (*) we make the substitution
$\phi=\theta-\frac{\pi}{2}$ or $\theta=\phi+\frac{\pi}{2}$. Then $\cos\theta=\cos\phi\cos\frac{\pi}{2}-\sin\phi\sin\frac{\pi}{2}$.
The last inequality follows from Jordan's inequality.

(d) Use parts (a) and (c) to show that

$$\mathcal{L}^{-1}\{F(s)\}=f(t)=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{L_R} e^{st}F(s)ds = t\cos at.$$

From Part (a) we have $\frac{1}{2\pi i}\int_{L_R} e^{st}F(s)ds = t\cos at - \frac{1}{2\pi i}\int_{C_R} e^{st}F(s)ds$

$$f(t)=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{L_R} e^{st}F(s)ds = t\cos at - \frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds$$

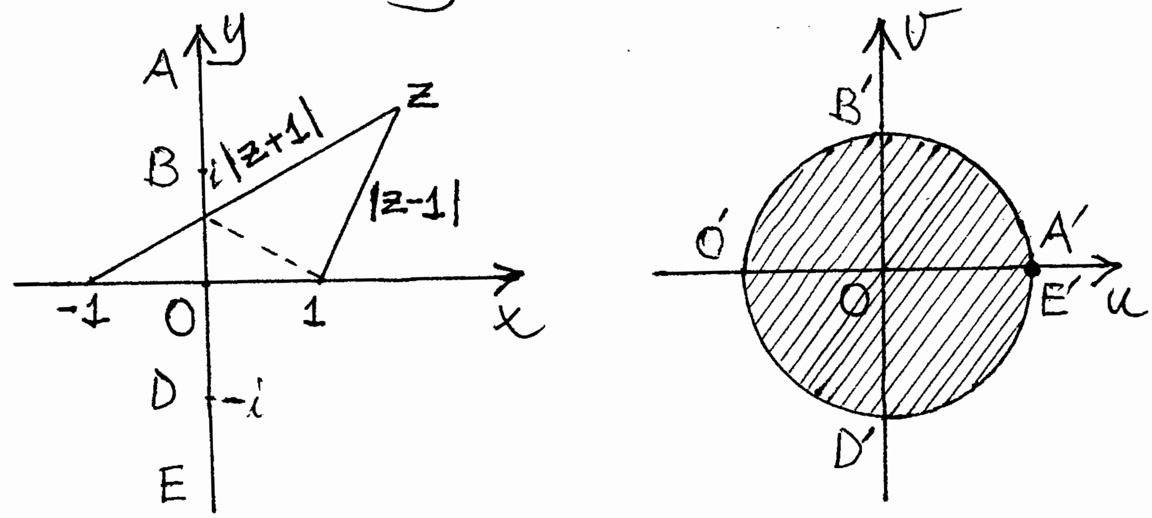
$$= t\cos at, \text{ since } \lim_{R\to\infty}\int_{C_R} e^{st}F(s)ds=0$$

from Part (c).

5.- Show that the linear fractional transformation

$$T(z) = w = \frac{z-1}{z+1}$$

maps the right half plane $\operatorname{Re} z > 0$ onto the open disk $|w| < 1$ and the boundary $\operatorname{Re} z = x = 0$ onto the boundary $|w| = 1$ in a one-to-one manner.



(a) If $\operatorname{Re} z > 0$ then $|z+1| > |z-1|$, therefore $|w| = \frac{|z-1|}{|z+1|} < 1$. So the ~~upper~~ right half plane $\operatorname{Re} z > 0$ is mapped onto the open disk $|w| < 1$.

(b) If $\operatorname{Re} z = 0$ then $|z+1| = |z-1|$, hence $|w| = 1$. Therefore the imaginary axis is mapped onto the circle $|w| = 1$.

(c) If $\operatorname{Re} z < 0$ then $|z+1| < |z-1|$, so $|w| = \frac{|z-1|}{|z+1|} > 1$. This shows that if $\operatorname{Im} z < 0$ then z is mapped to a point outside the circle $|w| = 1$.

A linear fractional transformation maps the extended z plane onto the extended w plane in a one-to-one manner. From the facts (a), (b) and (c) it follows that $w = T(z)$ has the desired mapping property.

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 "Complex Analysis II"

First Midterm Examination (Answers)
02.04.2002

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION :90 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :
1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

1: With the aid of series, prove that the function f defined by means of equations

$$f(z) = \begin{cases} \dfrac{1-\cos z}{z^2} & \text{if } z \neq 0 \\ 1/2 & \text{if } z = 0 \end{cases}$$

is entire.

$$f(z) = \frac{1}{z^2}\left[\cancel{1} - \left(\cancel{1} - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right] \quad 0<|z|<\infty$$

$$f(z) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \frac{z^6}{8!} + \cdots + (-1)^n \frac{z^{2n-2}}{(2n)!} + \cdots \quad 0<|z|<\infty$$

$$f(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^{2n-2}}{(2n)!}, \quad 0<|z|<\infty$$

The series on the right side converges also at $z=0$, and the sum of the power series for $z=0$ is $f(0)=\frac{1}{2}$. Thus $f(z)$ is the sum of the power series for all z. The sum of a power series is analytic inside its circle of convergence. Thus $f(z)$ is analytic throughout the entire plane, that is it is entire.

2.- Let $f(z)=\dfrac{1}{1-z-z^2}=\sum\limits_{n=0}^{\infty} c_n z^n = c_0+c_1 z+c_2 z^2+\cdots+c_n z^n+\cdots$

(a) Use the identity

$$1\equiv(1-z-z^2)\left(c_0+c_1z+c_2z^2+\cdots+c_{n-2}z^{n-2}+c_{n-1}z^{n-1}+c_nz^n+\cdots\right)$$

to show that the coefficients c_n of the expansion satisfy the relation $c_n=c_{n-1}+c_{n-2}\quad(n\geq 2)$

Note: The numbers c_n are called Fibonacci numbers

$$1\equiv c_0+(c_1-c_0)z+(c_2-c_1-c_0)z^2+\cdots+(c_n-c_{n-1}-c_{n-2})z^n+\cdots$$

We conclude that $c_0=1$, $c_1=1$; $c_2=2$, and $c_n=c_{n-1}+c_{n-2}$, $n\geq 2$.

(b) Show that $c_n=\dfrac{1}{\sqrt5}\left[\left(\dfrac{1+\sqrt5}{2}\right)^{n+1}-\left(\dfrac{1-\sqrt5}{2}\right)^{n+1}\right]$, and the radius of convergence of the **series** is $\frac{1}{2}(\sqrt5-1)$.

Hint: Use the identity

$$f(z)=\frac{-1}{\left(z+\frac{1+\sqrt5}{2}\right)\left(z+\frac{1-\sqrt5}{2}\right)}=\frac{1}{\sqrt5}\left[\frac{1}{\frac{1+\sqrt5}{2}\left(1+\frac{z}{\frac{1+\sqrt5}{2}}\right)}-\frac{1}{\frac{1-\sqrt5}{2}\left(1+\frac{z}{\frac{1-\sqrt5}{2}}\right)}\right]$$

$$f(z)=\frac{1}{\sqrt5}\left[\frac{1}{\frac{1+\sqrt5}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{\left(\frac{1+\sqrt5}{2}\right)^n}-\frac{1}{\frac{1-\sqrt5}{2}}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{\frac{(1-\sqrt5)^n}{2^n}}\right]$$

$$|z|<\tfrac{1}{2}(\sqrt5+1)\qquad\qquad |z|<\tfrac{1}{2}(\sqrt5-1)$$

$$f(z)=\frac{1}{\sqrt5}\sum_{n=0}^{\infty}\left[\frac{(-1)^n2^{n+1}}{(1+\sqrt5)^{n+1}}-\frac{(-1)^n2^{n+1}}{(1-\sqrt5)^{n+1}}\right]z^n\qquad |z|<\tfrac{1}{2}(\sqrt5-1)$$

$$\frac{1}{(1+\sqrt5)^{n+1}}-\frac{1}{(1-\sqrt5)^{n+1}}=\frac{(1-\sqrt5)^{n+1}-(1+\sqrt5)^{n+1}}{\left[(1+\sqrt5)(1-\sqrt5)\right]^{n+1}}=\frac{(1-\sqrt5)^{n+1}-(1+\sqrt5)^{n+1}}{(-2^2)^{n+1}}$$

$$f(z)=\frac{1}{\sqrt5}\sum_{n=0}^{\infty}\cancel{(-1)^n}2^{n+1}\frac{(1-\sqrt5)^{n+1}-(1+\sqrt5)^{n+1}}{\cancel{(-1)^n}(-1)2^{2n+2}}z^n=\frac{1}{\sqrt5}\sum_{n=1}^{\infty}\left[\left(\frac{1+\sqrt5}{2}\right)^n-\left(\frac{1-\sqrt5}{2}\right)^n\right]z^{n-1}$$

$$f(z)=\frac{1}{\sqrt5}\sum_{n=0}^{\infty}\left[\left(\frac{1+\sqrt5}{2}\right)^{n+1}-\left(\frac{1-\sqrt5}{2}\right)^{n+1}\right]z^n=\sum_{n=0}^{\infty}c_nz^n=\sum_{n=1}^{\infty}c_{n-1}z^{n-1}$$

3.- Let $f(z)=\frac{z+1}{z^3(z^2+1)}$

(a) Derive the Laurent series representation

$$f(z)=\frac{-1}{z}+\frac{1}{z^2}+\frac{1}{z^3}-1+z+z^2-z^3-z^4+\cdots \qquad (0<|z|<1)$$

$$f(z)=\frac{z+1}{z^3}\cdot\sum_{n=0}^{\infty}(-1)^n z^{2n}=(z+1)\sum_{n=0}^{\infty}(-1)^n z^{2n-3} \qquad (0<|z|^2<1)$$

$$f(z)=\sum_{n=0}^{\infty}(-1)^n z^{2n-2}+\sum_{n=0}^{\infty}(-1)^n z^{2n-3} \qquad 0<|z|<1$$

$$f(z)=\left(\frac{1}{z^2}-1+z^2-z^4+\cdots\right)+\left(\frac{1}{z^3}-\frac{1}{z}+z-z^3+\cdots\right)$$

$$f(z)=-\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}-1+z+z^2-z^3-z^4+\cdots \qquad (0<|z|<1)$$

(b) Represent $f(z)$ by its Laurent series for the domain $1<|z|<\infty$.

Answer. $f(z)=\frac{1}{z^4}+\frac{1}{z^5}-\frac{1}{z^6}-\frac{1}{z^7}+\frac{1}{z^8}+\frac{1}{z^9}-\cdots \qquad (1<|z|<\infty)$

$$f(z)=\frac{z+1}{z^5\left(1+\frac{1}{z^2}\right)}=\frac{z+1}{z^5}\cdot\sum_{n=0}^{\infty}(-1)^n\frac{1}{z^{2n}}=(z+1)\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+5}}$$

$$\left(\frac{1}{|z|^2}<1\right) \qquad (1<|z|<\infty)$$

$$f(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+4}}+\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+5}} \qquad (1<|z|<\infty)$$

$$f(z)=\frac{1}{z^4}+\frac{1}{z^5}-\frac{1}{z^6}-\frac{1}{z^7}+\frac{1}{z^8}+\frac{1}{z^9}-\cdots \qquad (1<|z|<\infty)$$

4.- Show that the singular points of the function
$$f(z) = \frac{z+1}{z^3(z^2+1)}$$
are poles. Determine the order m of each pole, and find the corresponding residue B.

$f(z) = \frac{\phi(z)}{z^3}$ where $\phi(z) = \frac{z+1}{z^2+1}$ is analytic and nonzero at $z_0 = 0$. Therefore $z_0 = 0$ is a pole of order $m=3$, and

$$\operatorname*{Res}_{z=0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} = \frac{\phi''(0)}{2!} = \frac{-2}{2} = -1$$

$$\phi'(z) = \frac{z^2+1-2z(z+1)}{(z^2+1)^2} = \frac{-z^2-2z+1}{(z^2+1)^2}$$

$$\phi''(z) = \frac{(-2z-2)(z^2+1)^2 - 4z(z^2+1)(-z^2-2z+1)}{(z^2+1)^4}. \quad \phi''(0) = -2$$

$f(z) = \frac{p(z)}{q(z)}$, where $p(z) = z+1$, and $q(z) = z^3(z^2+1) = z^5+z^3$ are entire fcts,
$q'(z) = 5z^4+3z^2$

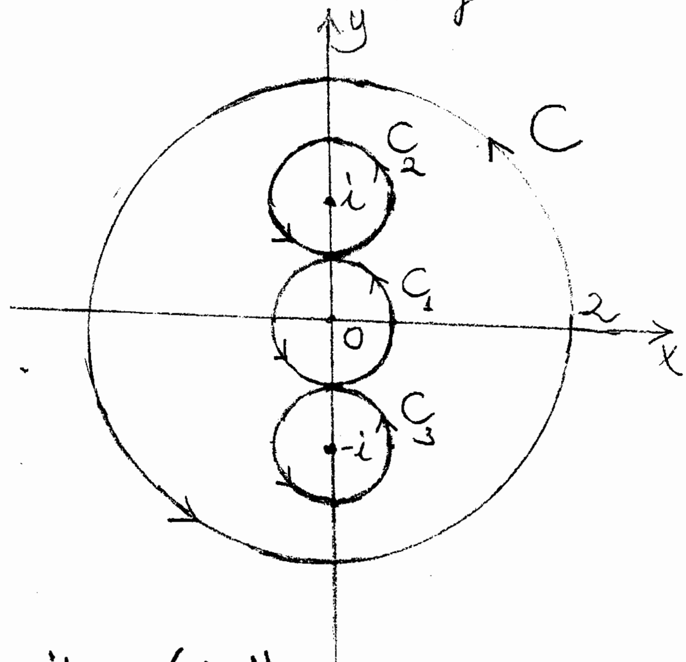
$p(i) = 1+i \neq 0$, $q(i) = 0$, $q'(i) = 5-3 = 2 \neq 0$, so $z_1 = i$ is a simple pole

$$\operatorname*{Res}_{z=i} f(z) = \frac{p(i)}{q'(i)} = \frac{1+i}{2} = \frac{1}{2}(1+i)$$

$p(-i) = 1-i \neq 0$, $q(-i) = 0$, $q'(-i) = 2 \neq 0$, so $z_2 = -i$ is a simple pole

$$\operatorname*{Res}_{z=-i} f(z) = \frac{p(-i)}{q'(-i)} = \frac{1-i}{2} = \frac{1}{2}(1-i)$$

5. Let $f(z)=\frac{z+1}{z^3(z^2+1)}$, C_1, C_2, C_3, and C be the circles $|z|=\frac{1}{2}$, $|z-i|=\frac{1}{2}$, $|z+i|=\frac{1}{2}$ and $|z|=2$ respectively, all described in the positive sense. Evaluate the integrals;



(a) $\int_{C_1} f(z)\,dz$

$$\int_{C_1} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = -2\pi i$$

(b) $\int_{C_2} f(z)\,dz$

$$\int_{C_2} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = \cancel{2}\pi i \cdot \frac{1}{\cancel{2}}(1+i) = \pi(-1+i)$$

(c) $\int_{C_3} f(z)\,dz$

$$\int_{C_3} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-i} f(z) = \cancel{2}\pi i \cdot \frac{1}{\cancel{2}}(1-i) = \pi(1+i)$$

(d) $\int_{C} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z)\right]$

$$= 2\pi i \left[-1 + \frac{1}{2}(1+i) + \frac{1}{2}(1-i)\right] = 2\pi i \cdot 0 = 0.$$

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 "Complex Analysis II"

Second Midterm Examination (Answers)
07.05.2002

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION :100 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :
1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

1. Let $f(z)=\frac{z^a}{(z^2+1)^2}$, where $-1<a<3$, and $z^a=e^{a\log z}$

$\left(|z|>0,\ -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$

(a) Show that $z_0=i$ is a pole of $f(z)$ of order $m=2$, and that $\operatorname*{Res}_{z=i} f(z)=\frac{i(a-1)}{4}e^{i\frac{a\pi}{2}}$.

$f(z)=\frac{\phi(z)}{(z-i)^2}$, where $\phi(z)=\frac{e^{a\log z}}{(z+i)^2}$ is analytic and nonzero at $z_0=i$, so $z_0=i$ is a pole of $f(z)$ of order $m=2$.

$$\operatorname*{Res}_{z=i} f(z)=\frac{\phi^{(m-1)}(z_0)}{(m-1)!}=\phi'(i)=i\,\frac{a-1}{4}e^{i\frac{a\pi}{2}}.$$

$$\phi'(z)=\frac{\frac{a}{z}e^{a\log z}(z+i)^2-2(z+i)e^{a\log z}}{(z+i)^4}$$

$$\phi'(i)=\frac{\frac{a}{i}e^{a\log i}(2i)^2-2(2i)e^{a\log i}}{(2i)^4}=\frac{-aie^{i\frac{a\pi}{2}}(-4)-4ie^{i\frac{a\pi}{2}}}{16},$$

$$\phi'(i)=\frac{(ai-i)e^{i\frac{a\pi}{2}}}{4}=i\,\frac{a-1}{4}e^{i\frac{a\pi}{2}}.$$

(b) Let C_R and C_ρ denote any semicircles $z=Re^{i\theta}$ $(0\le\theta\le\pi)$ and $z=\rho e^{i\theta}$ $(0\le\theta\le\pi)$ respectively. Show that $\lim\limits_{R\to\infty}\int_{C_R} f(z)\,dz = 0 = \lim\limits_{\rho\to 0}\int_{C_\rho} f(z)\,dz$.

$|z^2+1| \ge |z|^2-1 = R^2-1 > 0$ for each $z\in C_R$.

$|z^2+1|^2 \ge (R^2-1)^2 \Rightarrow \dfrac{1}{|z^2+1|^2} \le \dfrac{1}{(R^2-1)^2}$.

$\log z = \ln|z| + i\arg z = \ln R + i\theta$, $-\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$.

$|e^{a\log z}| = |e^{a\ln R}\cdot e^{ia\theta}| = e^{a\ln R} = R^a$ for $z\in C_R$.

$$|f(z)| = \frac{|e^{a\log z}|}{|z^2+1|^2} \le \frac{R^a}{(R^2-1)^2} = M_R \quad \text{for } z\in C_R$$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^a}{(R^2-1)^2}\pi R = \frac{\pi R^{a+1}}{(R^2-1)^2} \to 0 \text{ as } R\to\infty$$

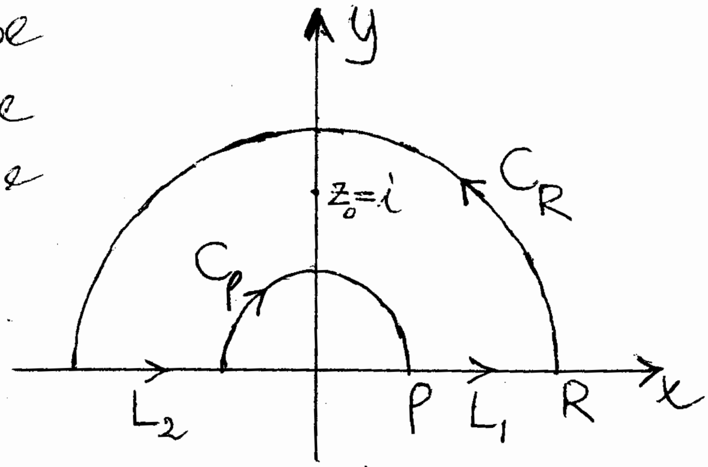
since $a+1<4$.

Similarly, $|f(z)| \le \dfrac{\rho^a}{(1-\rho^2)^2} = M_\rho$ for $z\in C_\rho$

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^a}{(1-\rho^2)^2}\pi\rho = \frac{\pi\rho^{a+1}}{(1-\rho^2)^2} \to 0 \text{ as } \rho\to 0^+$$

since $0<a+1$

2. Let $C = L_1 + C_R + L_2 + C_\rho$ be the positively oriented simple closed contour shown in the figure.



$L_1: z = re^{i0} = r, \ \rho \le r \le R$

$-L_2: z = re^{i\pi} = -r, \ \rho \le r \le R$

(a) Integrating the function $f(z)$ in Question 1 around the positively oriented simple closed contour C (by using the Cauchy's Residue Theorem) obtain the equation

$$(*) \quad \left(1+e^{ia\pi}\right)\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz = \frac{\pi(1-a)}{2} e^{i\frac{a\pi}{2}}$$

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz - \int_{-L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

$$\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_R} f(z)\,dz + \int_\rho^R \frac{(-r)^a}{(r^2+1)}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i \cdot i\,\frac{a-1}{4} e^{\frac{ia\pi}{2}}.$$

$$(-r)^a = e^{a\log(-r)} = e^{a(\ln r + i\pi)} = r^a e^{ia\pi},$$

$$\left(1+e^{ia\pi}\right)\int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr + \int_{C_R} f(z)\,dz + \int_{C_\rho} f(z)\,dz = \frac{\pi(1-a)}{2} e^{\frac{ia\pi}{2}},$$

(b) Use the equation (*) in part (a) to establish the integration formula

$$\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{(1-a)\pi}{4\cos\left(\frac{a\pi}{2}\right)}, \text{ where } -1<a<3 \text{ (Why?)}$$

By letting $R\to\infty$, $\rho\to 0$ and using the facts that $\lim_{R\to\infty}\int_{C_R} f(z)dz = 0 = \lim_{\rho\to 0}\int_{C_\rho} f(z)dz$ we obtain the eq.

$$(1+e^{ia\pi})\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{\pi(1-a)}{2}e^{i\frac{a\pi}{2}}.$$

Solving this eq. for the improper integral we get

$$\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{(1-a)\pi}{2}\cdot\frac{e^{i\frac{a\pi}{2}}}{1+e^{ia\pi}} = \frac{(1-a)\pi}{2}\cdot\frac{1}{e^{-i\frac{a\pi}{2}}+e^{i\frac{a\pi}{2}}}$$

$$e^{-i\frac{a\pi}{2}}+e^{i\frac{a\pi}{2}} = \left(\cos\frac{a\pi}{2} - i\sin\frac{a\pi}{2}\right)+\left(\cos\frac{a\pi}{2}+i\sin\frac{a\pi}{2}\right)$$
$$= 2\cos\frac{a\pi}{2}.$$

Therefore we have established the integration formula

$$\int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr = \frac{(1-a)\pi}{4\cos\frac{a\pi}{2}} \quad (-1<a<3)$$

3- Apply Rouché's Theorem (giving full details) to determine the number of roots of the equation

$$z^4 - 8z + 10 = 0$$

(a) inside the circle $|z|=1$, (b) in the annulus $1<|z|<3$.

(a) Let $f(z)=10$, and $g(z)=z^4-8z$. $f(z)$ and $g(z)$ are analytic inside and on the simple closed contour C_1 which is the circle $|z|=1$. $|f(z)|=10$, $|g(z)| \le |z|^4+8|z|=9$ at each point z on C_1. Thus $|f(z)|>|g(z)|$ at each point z on C_1. By the Rouché's Theorem $f(z)$ and $f(z)+g(z)$ have the same number of zeros, counting multiplicities, inside C_1. $Z_f=0=Z_{f+g}$ inside C_1. So the given equation has no zeros inside C_1.

(b) Let $f(z)=z^4$, and $g(z)=-8z+10$. $f(z)$ and $g(z)$ are analytic inside and on the simple closed contour C_2 which is the circle $|z|=3$. $|f(z)|=|z|^4=81>34 \ge 8|z|+10=34=|g(z)|$ at each point z on C_2. By the Rouché's Theorem

$$4=Z_f=Z_{f+g} \quad \text{inside } C_2$$

$f+g$ has no zeros inside C_1 and four zeros inside C_2 therefore $f(z)+g(z)=z^4-8z+10$ has four zeros in the annulus $1<|z|<3$

4. Let $F(s)=\dfrac{1}{s\cosh(s^{1/2})}$

where $s^{1/2}$ denotes any branch of this double-valued function. We agree, however, that the branch cut of $s^{1/2}$ does not lie along the negative real axis, so that $\cosh(s^{1/2})$ is well defined along that axis.

(a) Show that $F(s)=\dfrac{1}{s}\left(1-\dfrac{s}{2}+\dfrac{5}{24}s^2-\cdots\right) \qquad (0<|s|<R_2)$
and that $s_0=0$ is a simple pole of $F(s)$ with $\operatorname*{Res}_{s=0}F(s)=1$.

$$F(s)=\frac{1}{s}\cdot\frac{1}{1+\frac{s}{2!}+\frac{s^2}{4!}+\frac{s^3}{6!}+\cdots}=\frac{1}{s}\left(1-\frac{s}{2}+\frac{5}{24}s^2-\cdots\right)$$

$$\begin{array}{l|l}
1 & 1+\frac{s}{2}+\frac{s^2}{24}+\cdots \\ \hline
\mp 1 \mp \frac{s}{2} \mp \frac{s^2}{24} \mp \cdots & 1-\frac{s}{2}+\frac{5}{24}s^2-\cdots \\ \hline
-\frac{s}{2}-\frac{s^2}{24}-\cdots & \\
\pm\frac{s}{2}\pm\frac{s^2}{4}-\cdots & \\ \hline
+\frac{5}{24}s^2+\cdots &
\end{array}$$

Thus we have obtained a Laurent series representation of $F(s)$ in a punctured nbd. of zero

$$F(s)=\frac{1}{s}-\frac{1}{2}+\frac{5}{24}s-\cdots \qquad 0<|s|<R_2$$

The principal part of $F(s)$ consists of $\frac{1}{s}$, therefore $s_0=0$ is a simple pole of $F(s)$ with $\operatorname*{Res}_{s=0}F(s)=1$.

(b) Show that $s_n = -\frac{\pi^2}{4}(2n-1)^2$ $(n=1,2,3,\cdots)$ are the other singular points of $F(s)$ which are also simple poles of $F(s)$ with $\operatorname*{Res}_{s=s_n} F(s) = (-1)^n \frac{4}{\pi}(2n-1)^{-1} = \frac{4}{\pi}\cdot\frac{(-1)^n}{2n-1}$
[Hint: $\cosh z = \cos(iz)$, and $\sinh(iz) = i\sin z$.]

The other singularities of $F(s)$ are the zeros of $\cosh(s^{1/2})$.
$\cosh(s^{1/2}) = 0$ if and only if $s^{1/2} = \pm i\frac{\pi}{2}(2n-1)$, $n = 1, 2, \cdots$
Therefore $s_n = -\frac{\pi^2}{4}(2n-1)^2$ $(n=1,2,3,\cdots)$ are the other singular points of $F(s)$.

$F(s) = \frac{p(s)}{q(s)}$, where $p(s)=1$, $q(s) = s\cosh(s^{1/2})$

$p(s_n)=1$, $q(s_n)=0$; $p(s)$ and $q(s)$ are analytic at s_n.

$q'(s) = \cosh(s^{1/2}) + s\left(\frac{1}{2}s^{-1/2}\right)\sinh(s^{1/2})$

$q'(s_n) = \frac{1}{2}s_n^{1/2}\sinh(s_n^{1/2}) = \frac{1}{2}\left[\pm i\frac{\pi}{2}(2n-1)\right]\sinh\left[\pm i\frac{\pi}{2}(2n-1)\right]$

$= (2n-1)\frac{i\pi}{4}i\sin(2n-1)\frac{\pi}{2} = -\frac{\pi}{4}(2n-1)(-1)^{n-1} = \frac{\pi}{4}(2n-1)(-1)^n \neq 0$

Therefore s_n $(n=1,2,3,\cdots)$ is a simple of $F(s)$ with

$$\operatorname*{Res}_{s=s_n} F(s) = \frac{p(s_n)}{q'(s_n)} = \frac{1}{\frac{\pi}{4}(2n-1)(-1)^n} = \frac{4}{\pi}\cdot\frac{(-1)^n}{2n-1} \qquad (n=1,2,3,\cdots)$$

5. Let $F(s)$ be the function in Question 4.

(a) Evaluate $\operatorname*{Res}_{s=s_0}[e^{st}F(s)]$ and $\operatorname*{Res}_{s=s_n}[e^{st}F(s)]$, where $s_0=0$, and $s_n=-\frac{\pi^2}{4}(2n-1)^2$ $(n=1,2,3,\dots)$ are simple poles of $F(s)$ shown in Question 4.

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)]=e^{s_0t}\operatorname*{Res}_{s=s_0}F(s)=1\cdot 1=1,$$

$$\operatorname*{Res}_{s=s_n}[e^{st}F(s)]=e^{s_nt}\operatorname*{Res}_{s=s_n}F(s)=e^{-\frac{\pi^2}{4}(2n-1)^2}\cdot\frac{4}{\pi}\cdot\frac{(-1)^n}{(2n-1)}$$

(b) Find $\mathcal{L}^{-1}\{F(s)\}=f(t)$.

[Ans. $f(t)=1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-\frac{\pi^2}{4}(2n-1)^2t}$ $(t>0)$]

We write the formal series for $f(t)$:

$$f(t)=\operatorname*{Res}_{s=s_0}[e^{st}F(s)]+\sum_{n=1}^{\infty}\operatorname*{Res}_{s=s_n}[e^{st}F(s)] \quad (t>0)$$

$$f(t)=1+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-\frac{\pi^2}{4}(2n-1)t} \quad (t>0)$$

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 "Complex Analysis II"

Final Examination (Answers)

07.06.2002

NAME, SURNAME :

NUMBER :

DEPARTMENT :

SECTION :

SIGNATURE :

INSTRUCTOR'S NAME :

DURATION :120 minutes

TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :

1) Write your name and department.

2) Check that there are 5 questions.

3)Show all your work. Correct answers without the intermediate steps may not get credit.

1- Let $g(z) = \frac{1}{z^3}\left(3e^{iz} - e^{i3z}\right)$

(a) Show that $\lim\limits_{R\to\infty} \int_{C_R} g(z)\,dz = 0$, where C_R denotes any semicircle $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$.

Hint: <u>Use Jordan's Lemma</u> which is stated below:

Suppose $f(z)$ is analytic at all points in the upper half plane $y \ge 0$ except possibly the points that lie below a semicircle $z = R_0 e^{i\theta}$ $(0 \le \theta \le \pi)$; and let C_R denote any semicircle $z = Re^{i\theta}$ $(0 \le \theta \le \pi)$, where $R > R_0$. If for all points z on C_R there is a positive constant M_R such that $|f(z)| \le M_R$, where $M_R \to 0$ as $R \to \infty$, then
$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iaz}\,dz = 0. \qquad (a>0)$$

Let $f(z) = \frac{1}{z^3}$. $f(z)$ is analytic everywhere except $z=0$.

$|f(z)| = \frac{1}{|z|^3} = \frac{1}{R^3} = M_R$ for z on C_R, where $M_R \to 0$ as $R \to \infty$.

By Jordan's Lemma, by choosing $a=1$, and $a=3$ we get
$$\lim_{R\to\infty} \int_{C_R} f(z) e^{iz}\,dz = 0 = \lim_{R\to\infty} \int_{C_R} f(z) e^{i3z}\,dz.$$

Therefore
$$\lim_{R\to\infty} \int_{C_R} g(z)\,dz = 3\lim_{R\to\infty} \int_{C_R} f(z) e^{iz}\,dz - \lim_{R\to\infty} \int_{C_R} f(z) e^{i3z}\,dz$$
$$= 3\cdot 0 - 0 = 0.$$

(b) Show that the Laurent series representation of $g(z)$ in the domain $0<|z|$ can be written as

$$g(z)=\frac{1}{z^3}\left(3e^{iz}-e^{i3z}\right)=\sum_{n=0}^{\infty}a_n z^n+\frac{3}{z}+\frac{2}{z^3}=h(z)+\frac{3}{z}+\frac{2}{z^3}.$$

Point out why $h(z)$ is continuous and thus bounded, that is, $|h(z)|\leq M$, in the closed disk $|z|\leq \rho_0$. Use the equation $g(z)=h(z)+\frac{3}{z}+\frac{2}{z^3}$ and a parametric representation for $-C_\rho$: $z=\rho e^{i\theta}$ $(0\leq\theta\leq\pi)$ to show that

$$\lim_{\rho\to 0}\int_{C_\rho} g(z)\,dz=-3\pi i,$$

where C_ρ denotes the upper half of a circle and the direction is clockwise.

$$g(z)=\frac{1}{z^3}\left\{3\left[1+\frac{iz}{1!}+\frac{(iz)^2}{2!}+\frac{(iz)^3}{3!}+\cdots\right]-\left[1+\frac{i3z}{1!}+\frac{(i3z)^2}{2!}+\frac{(i3z)^3}{3!}+\cdots\right]\right\}$$

$$g(z)=\frac{1}{z^3}\left(2+0z+3z^2+a_0z^3+a_1z^4+\cdots\right)\qquad(0<|z|<\infty)$$

$$g(z)=\sum_{n=0}^{\infty}a_n z^n+\frac{3}{z}+\frac{2}{z^3}=h(z)+\frac{3}{z}+\frac{2}{z^3}\qquad(0<|z|<\infty)$$

$h(z)$ is, being the sum of a power series, continuous in a closed disk $|z|\leq\rho$. Therefore there is a constant $M>0$ such that $|h(z)|\leq M$ in the closed disk $|z|\leq\rho\leq\rho_0$.

$$\int_{C_\rho} g(z)\,dz=\int_{C_\rho} h(z)\,dz+3\int_{C_\rho}\frac{1}{z}\,dz+2\int_{C_\rho}\frac{1}{z^3}\,dz$$

$$\left|\int_{C_\rho} h(z)\,dz\right|\leq M\pi\rho\to 0 \text{ as } \rho\to 0.$$

$$\int_{C_\rho}\frac{1}{z}\,dz=-\int_{-C_\rho}\frac{1}{z}\,dz=-\int_0^{\pi}\frac{1}{\rho}e^{-i\theta}\rho i e^{i\theta}\,d\theta=-i\int_0^{\pi}d\theta=-\pi i.$$

$$\int_{C_\rho}\frac{1}{z^3}\,dz=-\int_{-C_\rho}\frac{1}{z^3}\,dz=-\int_0^{\pi}\frac{1}{\rho^3}e^{-i3\theta}\rho i e^{i\theta}\,d\theta=-\frac{i}{\rho^2}\left(-\frac{1}{2i}\right)e^{-i2\theta}\Big|_0^{\pi}=0.$$

$$\lim_{\rho\to 0}\int_{C_\rho} g(z)\,dz=\lim_{\rho\to 0}\int_{C_\rho} h(z)\,dz-3\pi i=-3\pi i.$$

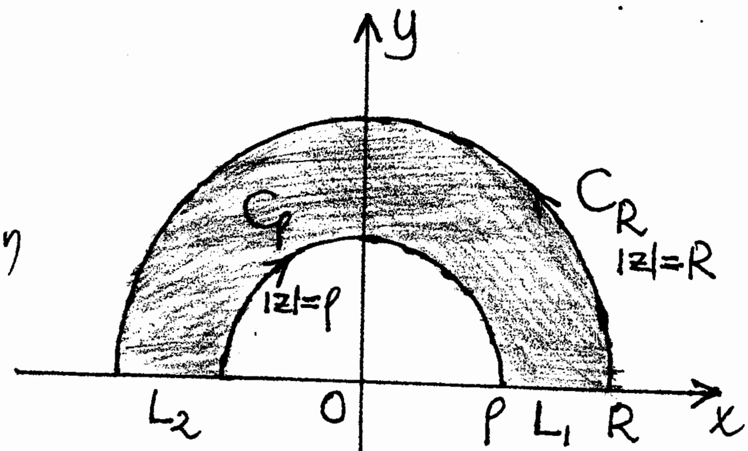
2.- Show that

$$I=\int_0^{\infty}\frac{3\sin r-\sin 3r}{r^3}\,dr=\frac{3\pi}{2}$$

by integrating the function

$$g(z)=\frac{3e^{iz}-e^{i3z}}{z^3}$$

over the indented path



$C=L_1+C_R+L_2+C_\rho$, shown in the figure.

- $g(z)$ is analytic at all points interior to and on the simple closed contour C, hence by the Cauchy-Goursat Theorem

$$(*)\qquad \int_C g(z)dz=\int_{L_1} g(z)dz+\int_{C_R} g(z)dz-\int_{-L_2} g(z)dz+\int_{C_\rho} g(z)dz=0,$$

L_1: $z=re^{i0}=r$, $\rho\le r\le R$; L_2: $z=re^{i\pi}=-r$, $\rho\le r\le R$.

$$\int_{L_1} g(z)dz-\int_{-L_2} g(z)dz=\int_\rho^R\frac{3e^{ir}-e^{i3r}}{r^3}dr-\int_\rho^R\frac{3e^{-ir}-e^{-i3r}}{(-r)^3}(-dr)$$

$$=\int_\rho^R\frac{3(e^{ir}-e^{-ir})-(e^{i3r}-e^{-i3r})}{r^3}dr$$

$$=2i\int_\rho^R\frac{3\sin r-\sin 3r}{r^3}dr$$

- Eq. (*) can be written as

$$2i\int_\rho^R\frac{3\sin r-\sin 3r}{r^3}dr=-\int_{C_R} g(z)dz-\int_{C_\rho} g(z)dz$$

By taking the limit as $\rho\to 0$ and $R\to\infty$ we get

$$2i\int_0^\infty\frac{3\sin r-\sin 3r}{r^3}dr=0-(-3\pi i)=3\pi i$$

Thus we have established the integration formula

$$\int_0^\infty\frac{3\sin r-\sin 3r}{r^3}dr=\frac{3\pi}{2}.$$ [Since $\sin^3 r=\frac{1}{4}(3\sin r-\sin 3r)$ we can write the integration formula

$$\int_0^\infty\frac{\sin^3 r}{r^3}dr=\frac{3\pi}{8}.$$]

3. Write the function $F(s)=\dfrac{8a^3s^2}{(s^2+a^2)^3}$ $(a>0)$ as

$$F(s)=\frac{\phi(s)}{(s-ai)^3} \quad \text{where} \quad \phi(s)=8a^3\frac{s^2}{(s+ai)^3}.$$

Point out why $\phi(s)$ has a Taylor series representation about $s=ai$, and then use it to show that the principal part of F at that point is

$$\frac{\phi''(ai)/2}{s-ai}+\frac{\phi'(ai)}{(s-ai)^2}+\frac{\phi(ai)}{(s-ai)^3}=-\frac{i/2}{s-ai}-\frac{a/2}{(s-ai)^2}-\frac{a^2i}{(s-ai)^3}.$$

- $\phi(s)$ is a rational function of s, therefore it is analytic at each point $s=s_0$ in its domain. $s=ai$ is in the domain of $\phi(s)$, and an analytic function is the sum of its Taylor series in a neighborhood $|s-s_0|<R_2$.

$$\phi(s)=\sum_{n=0}^{\infty}\frac{\phi^{(n)}(ai)}{n!}(s-ai)^n=\phi(ai)+\frac{\phi'(ai)}{1!}(s-ai)+\frac{\phi''(ai)}{2!}(s-ai)^2+\cdots$$

$$\phi(ai)=8a^3\cdot\frac{(ai)^2}{(2ai)^3}=8a^3\cdot\frac{1}{8ai}=-a^2i.$$

- $$\phi'(s)=8a^3\cdot\frac{2s(s+ai)^3-3(s+ai)^2s^2}{(s+ai)^6}=\frac{2s(s+ai)-3s^2}{(s+ai)^4}\cdot 8a^3=\frac{-s^2+2ais}{(s+ai)^4}\cdot 8a^3$$

$$\phi'(ai)=\frac{-(ai)^2+2(ai)^2}{(2ai)^4}\cdot 8a^3=8a^3\cdot\frac{-1}{16a^2}=-\frac{a}{2}.$$

$$\phi''(s)=\frac{(-2s+2ai)(s+ai)^4-4(s+ai)^3(-s^2+2ais)}{(s+ai)^8}\cdot 8a^3=8a^3\frac{(-2s+2ai)(s+ai)-4(-s^2+2ais)}{(s+ai)^5}$$

$$\phi''(ai)=\frac{-4(a^2-2a^2)}{(2ai)^5}\cdot 8a^3=\frac{4a^2}{32a^5i}\cdot 8a^3=\frac{1}{i}=-i$$

$$F(s)=\frac{1}{(s-ai)^3}\left[-a^2i-\frac{a/2}{1!}(s-ai)-\frac{i}{2!}(s-ai)^2+\frac{\phi'''(ai)}{3!}(s-ai)^3+\cdots\right]$$

$$F(s)=\underbrace{\frac{-a^2i}{(s-ai)^3}-\frac{a/2}{(s-ai)^2}-\frac{i/2}{(s-ai)}}_{\text{The principal part of } F}+\frac{\phi'''(ai)}{3!}+\frac{\phi^{(4)}(ai)}{4!}(s-ai)+\cdots$$

$$(0<|s-ai|<2a)$$

4- Use residues to find the inverse Laplace transform $f(t)$ corresponding to $F(s)=\dfrac{8a^3s^2}{(s^2+a^2)^3}$ $(a>0)$

Hint: You may use the following fact:
If $F(s)$ has a pole of order m at $s=s_0$, with a Laurent series expansion

$$F(s)=\sum_{n=0}^{\infty}a_n(s-s_0)^n+\frac{b_1}{s-s_0}+\frac{b_2}{(s-s_0)^2}+\cdots+\frac{b_m}{(s-s_0)^m} \quad (b_m\neq 0)$$

in the punctured disk $0<|s-s_0|<R_2$, then $\bar{s}_0=\alpha-i\beta$ $(\beta\neq 0)$ is also a pole of $F(s)$ of order m provided that $\overline{F(s)}=F(\bar{s})$ at points of analyticity of $F(s)$. In that case

$$\operatorname*{Res}_{s=s_0}[e^{st}F(s)]+\operatorname*{Res}_{s=\bar{s}_0}[e^{st}F(s)]=2e^{\alpha t}\operatorname{Re}\left\{e^{i\beta t}\left[b_1+\frac{b_2}{1!}t+\cdots+\frac{b_m}{(m-1)!}t^{m-1}\right]\right\}.$$

In Question 3 we have obtained the Laurent series of F

$$F(s)=\sum_{n=0}^{\infty}\frac{\phi^{(n+3)}(ai)}{n+3}(s-ai)^n-\frac{i/2}{s-ai}-\frac{a/2}{(s-ai)^2}-\frac{a^2i}{(s-ai)^3} \quad (0<|s-ai|<2a)$$

$s_0=ai$ is a pole of $F(s)$ of order 3, and $\overline{F(s)}=F(\bar{s})$ at points of analyticity of $F(s)$. So $\bar{s}_0=-ai$ is also a pole of $F(s)$ of order 3. By the rule given as hint above we have

$$f(t)=\operatorname*{Res}_{s=s_0}[e^{st}F(s)]+\operatorname*{Res}_{s=\bar{s}_0}[e^{st}F(s)]=2\operatorname{Re}\left\{e^{iat}\left[-\frac{i}{2}-\frac{a/2}{1!}t-\frac{a^2i}{2!}t^2\right]\right\}$$

$$f(t)=-2\operatorname{Re}\left[(\cos at+i\sin at)\left(\frac{i}{2}+\frac{a}{2}t+\frac{a^2i}{2}t^2\right)\right]$$

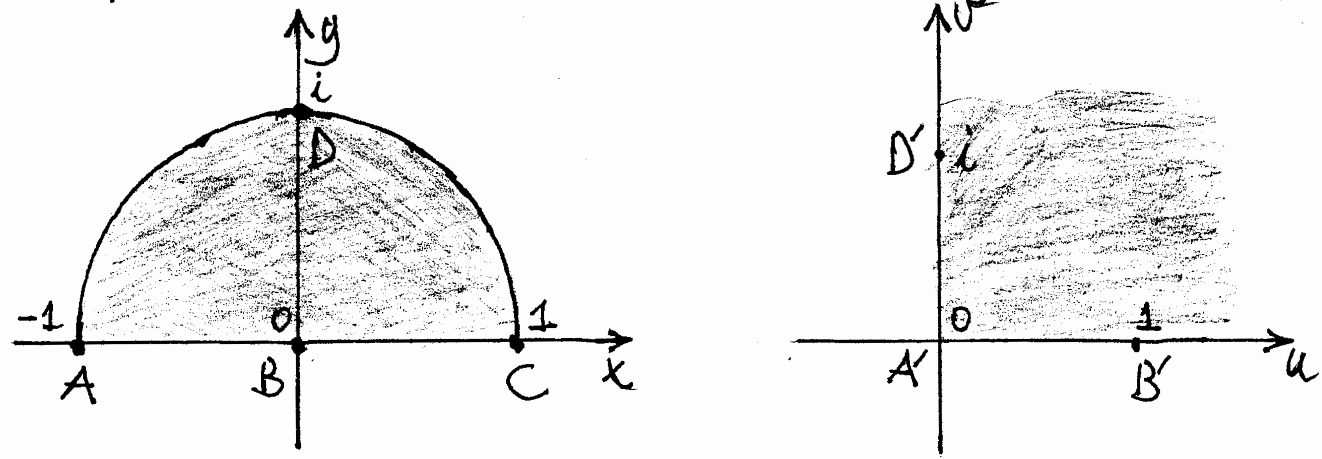
$$f(t)=-2\left[\frac{1}{2}at\cos at-\left(\frac{1}{2}+\frac{a^2}{2}t^2\right)\sin at\right]$$

$$f(t)=(1+a^2t^2)\sin at-at\cos at. \qquad (t>0)$$

5. Show that the linear fractional transformation

$$W = u + iv = T(z) = \frac{1+z}{1-z}$$

transforms the semicircular region $S: |z| \le 1$, $\mathrm{Im}\, z \ge 0$ onto the closed first quadrant region $u \ge 0, v \ge 0$ in the w plane in a one-to-one manner.



$$W = T(z) = \frac{(1+x)+iy}{(1-x)-iy} = \frac{[(1+x)+iy][(1-x)+iy]}{(1-x)^2+y^2} = \frac{(1-x^2)-y^2}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2} = u+iv$$

If $z = x$ then $W = u = \frac{1-x^2}{(1-x)^2}$ provided that $x \ne 1$.

(a) So T transforms the real axis onto the real axis. In particular $T(-1)=0$, $T(0)=1$, and $T(1)=\infty$. Hence T transforms the line segment $[AC]$ onto nonnegative real axis $u \ge 0$.

If $|z|=1$ and $y = \mathrm{Im}\, z > 0$ then $W = T(z) = i\frac{2y}{(1-x)^2+y^2} = iv$. Thus

(b) T transforms the semicircle $x^2+y^2=1$, $y \ge 0$ onto the nonnegative imaginary axis $v \ge 0$. In particular we note that $T(i) = i$ and $T(1) = \infty$.

(c) If $|z|^2 = x^2+y^2 < 1$, and $\mathrm{Im}\, z = y > 0$ then

$$W = T(z) = u + iv = \frac{1-x^2-y^2}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2} \text{ implies that}$$

$u > 0$ and $v > 0$. Therefore $T(z)$ lies in the first quadrant.

(d) If $|z| > 1$ then $u < 0$, and if $\mathrm{Im}\, z = y < 0$ then $v < 0$. So if z isn't in the semicircular region then $T(z)$ isn't in the 1st quadrant.

T maps the extended z plane onto the extended w plane in a one-to-one manner. So we conclude from the facts we have shown as (a), (b), (c), and (d) that T maps the semicircular region onto the first quadrant region in a one-to-one manner.

ANSWERS

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352
Complex Analysis II (Summer)

First Midterm
18.07.2002

NAME, SURNAME :

ID NUMBER :

DEPARTMENT :

SIGNATURE :

INSTRUCTOR :

TIME :10:00-11:30

NUMBER OF QUESTIONS: 5

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :

1) Write your name and department.
2) Check that there are 5 questions.
3) Show all your work. Correct answers without the intermediate steps may not get credit.

1) Find two Laurent series expansions in powers of z for the function $f(z) = \dfrac{1}{z^2(4+z^2)}$ and specifiy the two domains.

We can make separate expansions on $0 < |z| < 2$ and $2 < |z|$

On $0 < |z| < 2$:

- $$\frac{1}{z^2(4+z^2)} = \frac{1}{4z^2} \cdot \frac{1}{\left(1+\left(\frac{z}{2}\right)^2\right)}, \qquad \left|\frac{z}{2}\right| < 1 \Rightarrow \frac{1}{1+\frac{z^2}{4}} = 1 - \frac{z^2}{4} + \frac{z^4}{16} - \dots$$

$$f(z) = \frac{1}{4z^2} - \frac{1}{16} + \frac{z^2}{64} - \dots$$

$$= \sum_{n=-1}^{\infty} 4^{-n-2} z^{2n} (-1)^{n+1} = \sum_{n=0}^{\infty} 4^{-n-1} z^{2n-2} (-1)^n$$

- On $2 < |z|$

$$\frac{1}{z^2(4+z^2)} = \frac{1}{z^4\left(1+\frac{4}{z^2}\right)} = \frac{1}{z^4}\left(1 - \frac{4}{z^2} + \left(\frac{4}{z^2}\right)^2 + \dots\right)$$

$$f(z) = \frac{1}{z^4} - \frac{4}{z^6} + \frac{16}{z^8} - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n 4^n z^{-2n-4}$$

2) Find the residue of $f(z) = \dfrac{1}{z^2 \sinh z}$ at $z = 0$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$$

$$f(z) = \frac{1}{z^3\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots\right)}$$

$$\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots$$

$$1 = a_0 + a_1 z + \left(a_2 + \frac{a_0}{3!}\right) z^2 + \left(a_3 + \frac{a_1}{3!}\right) z^3 + \left(a_4 + \frac{a_2}{3!} + \frac{a_0}{5!}\right) z^4 + \cdots$$

$\Rightarrow$ $a_0 = 1$

$a_1 = 0$

$a_2 + \frac{a_0}{3!} = 0 \Rightarrow a_2 = -\frac{1}{3!}$

$a_3 + \frac{a_1}{3!} = 0 \Rightarrow a_3 = 0$

$a_4 + \frac{a_2}{3!} + \frac{a_0}{5!} = 0 \Rightarrow a_4 = \frac{1}{36} - \frac{1}{120} = \frac{7}{360}$

$$f(z) = \frac{1}{z^3}\left(1 - \frac{1}{6} z^2 + \frac{7}{360} z^4 - \cdots\right)$$

$$b_1 = -\frac{1}{6} = \operatorname*{Res}_{z=0} f(z)$$

3) Find the residue of $f(z) = \dfrac{z + (2z^2+1)^2}{1+z^3}$ at $z = -1$

$$1 + z^3 = 0 \Rightarrow z^3 = -1$$

$$z = e^{i(\pi + 2k\pi)/3}$$

$$z_0 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_1 = -1$$

$$z_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$f(z) = \frac{z + (2z^2+1)^2}{(z+1)\left(z - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)} = \frac{\phi(z)}{z+1}$$

$z = -1$ is simple pole so $\operatorname{Res}_{z=-1} = \phi(-1)$

$$\operatorname{Res}_{z=-1} = \frac{-1 + (2+1)^2}{\left(-\frac{3}{2} - i\frac{\sqrt{3}}{2}\right)\left(-\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)} = \frac{8}{\frac{9}{4} + \frac{3}{4}} = \frac{8}{3}$$

4) Evaluate the integral $\int_0^\infty \frac{dx}{1+x^4}$

Let $f(z) = \frac{1}{1+z^4}$. Singularities are: $z^4 = -1$

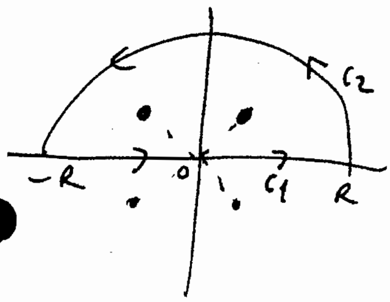
$z = e^{i(\pi + 2k\pi)/4}$

$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$z_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

$z_3 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$



z_0 and z_1 are inside the contour.

$$f(z) = \frac{1}{\left(z - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)\left(z + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i\right)\left(z + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)\left(z - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)}$$

$$\operatorname*{Res}_{z=z_0} = \frac{1}{\sqrt{2}\,(\sqrt{2}+\sqrt{2}i)\sqrt{2}i} = \frac{1}{2i}\,\frac{1}{\sqrt{2}+\sqrt{2}i}$$

$$\operatorname*{Res}_{z=z_1} = \frac{1}{(-\sqrt{2})\,\sqrt{2}i\,(-\sqrt{2}+\sqrt{2}i)} = -\frac{1}{2i}\,\frac{1}{(-\sqrt{2}+\sqrt{2}i)}$$

$$\int_C f(z)\,dz = 2\pi i\,\frac{1}{2i}\left(\frac{1}{\sqrt{2}+\sqrt{2}i} - \frac{1}{-\sqrt{2}+\sqrt{2}i}\right)$$

$$= \pi\left(\frac{-\sqrt{2}+\sqrt{2}i-\sqrt{2}-\sqrt{2}i}{-2-2}\right) = \frac{\pi\sqrt{2}}{2}$$

$$\left|\int_{C_2} f(z)dz\right| \leq \frac{1}{R^4-1}\left|\int_{C_2} dz\right| \leq \frac{1}{R^4-1}\pi R \Rightarrow \lim_{R\to\infty}\left|\int_{C_2} f(z)dz\right| = 0$$

So, in the limit $R \to \infty$, we have $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{2}$

$$\int_0^\infty \frac{dx}{1+x^4} = \frac{\pi\sqrt{2}}{4}$$

5) Evaluate the integral $\int_0^{\pi} \frac{d\theta}{2+\cos\theta}$

$$\int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{1}{4}\int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\cos\theta}$$

if $|z|=1$ then $z = \cos\theta + i\sin\theta$

$$z^{-1} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$$\frac{1}{4}\int_0^{2\pi} \frac{d\theta}{1+\frac{1}{2}\cos\theta} = \frac{1}{4}\int_C \frac{\frac{dz}{iz}}{1+\frac{1}{4}\left(z+\frac{1}{z}\right)} \quad \text{where } C: |z|=1$$

$$= \frac{1}{8i}\int_C \frac{dz}{z+\frac{z^2}{4}+\frac{1}{4}} = \frac{1}{2i}\int_C \frac{dz}{z^2+4z+1}$$

$$z^2+4z+1=0 \Rightarrow z = -2 \pm \sqrt{3}$$

$z=-2+\sqrt{3}$ is inside the circle.

$$\operatorname*{Res}_{z=-2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}-(-2-\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

$$\frac{1}{2i}\int_C \frac{dz}{z^2+4z+1} = \frac{1}{2i}\cdot 2\pi i \cdot \frac{1}{2\sqrt{3}} = \frac{\pi}{2\sqrt{3}}$$

ANSWERS

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352

Complex Analysis II (Summer)

Second Midterm
01.08.2002

NAME, SURNAME :

ID NUMBER :

DEPARTMENT :

SIGNATURE :

INSTRUCTOR :

TIME :10:00-11:30

NUMBER OF QUESTIONS: 6

Question	Grade	Out of
1		20
2		20
3		15
4		20
5		20
6		15
Total		110

IMPORTANT :

1) Write your name and department.
2) Check that there are 6 questions.
3) Show all your work. Correct answers without the intermediate steps may not get credit.

1) Determine the number of roots of the equation $z^4 + z^3 - 5z^2 + z + 1 = 0$ on $1 \leq |z| < 3$.

on $|z| = 3$:

$|z^4| = 81$

$|z^3 - 5z^2 + z + 1| \leq |z^3| + |5z^2| + |z| + 1 = 27 + 45 + 3 + 1 = 76$

$|z^4| > |z^3 - 5z^2 + z + 1|$ so by Rouche's Theorem, number of zeros of

$z^4 + z^3 - 5z^2 + z + 1$ are equal to number of zeros of $|z^4|$ inside $|z| = 3$.

$\Rightarrow$ There are 4 zeros inside $|z| = 3$

on $|z| = 1$:

$|5z^2| = 5$

$|z^4 + z^3 + z + 1| \leq |z^4| + |z^3| + |z| + 1 = 4$

$|5z^2| > |z^4 + z^3 + z + 1|$

$5z^2$ has 2 zeros inside $|z| = 1$, so does the polynomial.

$\Rightarrow$ $z^4 + z^3 - 5z^2 + z + 1 = 0$ has $4 - 2 = 2$ zeros on $1 \leq |z| < 3$

2) Find and graph the image of the line segment $x = 2,\ -2 \le y \le 2$ under the mapping $w = \dfrac{1}{z}$.

$$x = \frac{u}{u^2+v^2} \qquad y = \frac{-v}{u^2+v^2}$$

Let's find the image of the line $x=2$ first.

$$2 = \frac{u}{u^2+v^2}$$

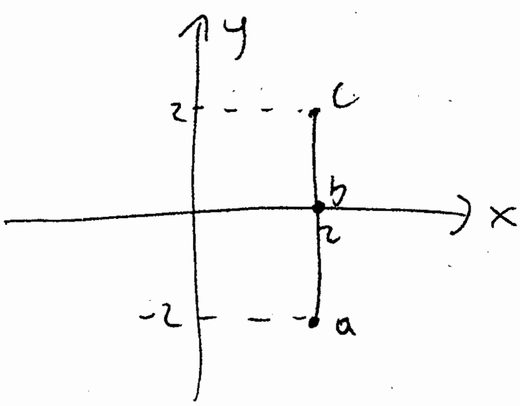
$$u^2+v^2-\frac{u}{2}=0$$

$$\left(u-\frac{1}{4}\right)^2+v^2=\frac{1}{16}$$

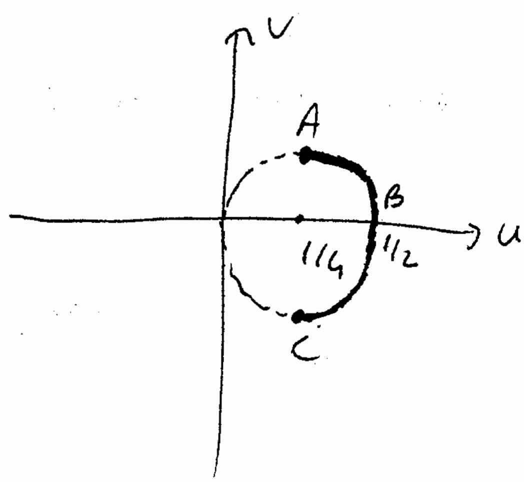
This is a circle with orgin $u=\frac{1}{4}, v=0$ and radius $=\frac{1}{4}$

$x=2,\ y=-2 \Rightarrow \quad u=\dfrac{2}{2^2+2^2}=\dfrac{1}{4}, \quad v=\dfrac{2}{2^2+2^2}=\dfrac{1}{4}$

$x=2,\ y=2 \Rightarrow \quad u=\dfrac{1}{4}, \quad v=-\dfrac{1}{4}$



Z



W

3) Find the Linear Fractional Transformation that maps $1+i, 2+2i, 4+4i$ onto $0, 1, \infty$

$$\frac{z-1-i}{z-4-4i} \cdot \frac{z+2i-4-4i}{z+2i-1-i} = \frac{w-0}{w-\infty} \cdot \frac{1-\infty}{1-0}$$

$$\frac{z-1-i}{z-4-4i} \cdot \frac{-2(1+i)}{1+i} = \frac{w}{1}$$

$$w = -2 \cdot \frac{z-1-i}{z-4-4i}$$

4) Find and graph the image of the square $x = \pm\frac{\pi}{4}$, $y = \pm\frac{\pi}{4}$ under the mapping $w = e^z$

$$W = e^{x+iy} = e^x \cdot e^{iy}$$

$x = \frac{\pi}{4} \Rightarrow w = e^{\pi/4} e^{iy}$: Circle with radius $e^{\pi/4}$, $-\frac{\pi}{4} < \phi < \frac{\pi}{4}$

$-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$

$x = -\frac{\pi}{4}$

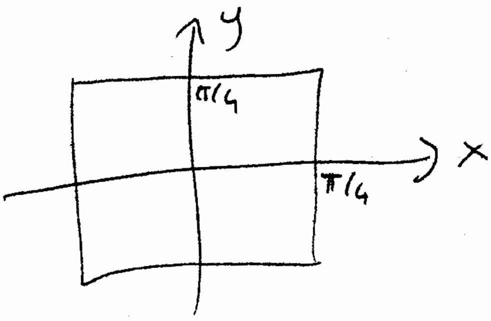
$-\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$ $\Rightarrow w = e^{-\pi/4} e^{iy}$: Circle with radius $e^{-\pi/4}$, $-\frac{\pi}{4} < \phi < \frac{\pi}{4}$

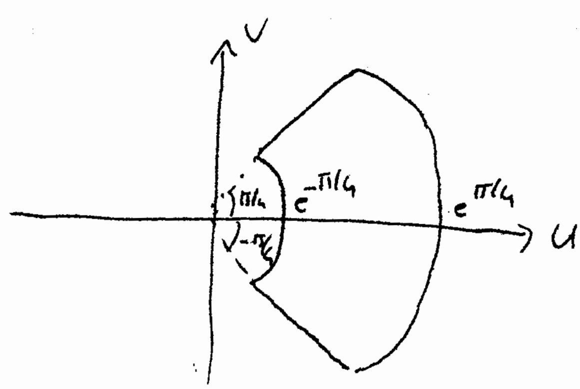
$y = \frac{\pi}{4}$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \Rightarrow w = e^x e^{i\frac{\pi}{4}}$

$u = e^x \cos\frac{\pi}{4}$, $v = e^x \sin\frac{\pi}{4} \Rightarrow u = v$, $e^{-\pi/4} \leq u \leq e^{\pi/4}$

$y = -\frac{\pi}{4}$, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \Rightarrow w = e^x e^{-i\pi/4}$

$\Rightarrow u = -v$, $e^{-\pi/4} \leq u \leq e^{\pi/4}$





Z

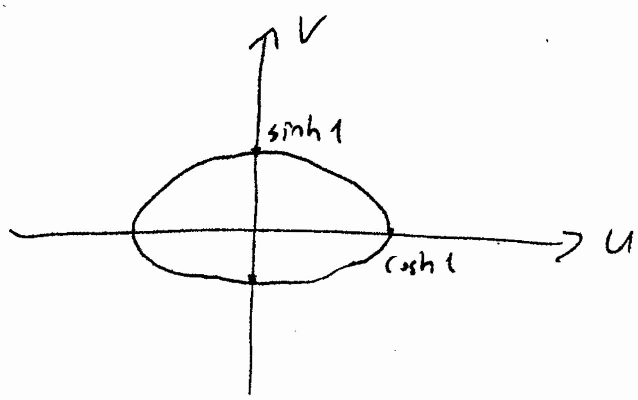
5) Find and graph the image of the line $y = -1$ under the mapping $w = \cos z$

$$W = \cos z = \cos x \cosh y - i \sin x \sinh y$$

$$U = \cos x \cosh y = \cos x \cosh(-1)$$

$$V = -\sin x \sinh y = -\sin x \sinh(-1)$$

$$\frac{u^2}{\cosh^2(-1)} + \frac{v^2}{\sinh^2(-1)} = 1 \Rightarrow \text{ellipse}$$



6) Find and graph the image of the line $x = 5$ under the mapping $w = z^2$.

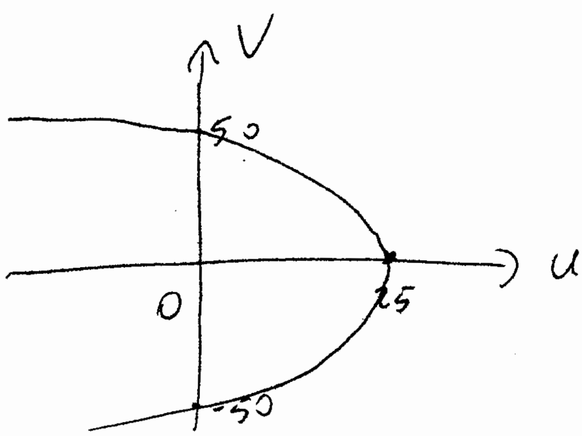
$W = (x+iy)^2 = x^2 - y^2 + 2ixy$

$u = x^2 - y^2 = 25 - y^2$

$v = 2xy = 10y$

$\Rightarrow \quad u = 25 - \left(\frac{v}{10}\right)^2 = 25 - \frac{v^2}{100}$

Parabola



ANSWERS

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

MATH 352
Complex Analysis II (Summer)

FINAL
09.08.2002

NAME, SURNAME :
ID NUMBER :
DEPARTMENT :
SIGNATURE :
INSTRUCTOR :
TIME :10:00-11:30
NUMBER OF QUESTIONS: 6

Question	Grade	Out of
1		15
2		20
3		20
4		15
5		20
6		20
Total		110

IMPORTANT :

1) Write your name and department.
2) Check that there are 6 questions.
3) Show all your work. Correct answers without the intermediate steps may not get credit.

1) Find the Laurent series expansion of $f(z) = e^{2z}\cosh(z^2)$.

$$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = 1 + 2z + \frac{4z^2}{2} + \frac{8z^3}{6} + \cdots$$

$$\cosh z^2 = \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = 1 + \frac{z^4}{2!} + \frac{z^8}{4!} + \frac{z^{12}}{6!} + \cdots$$

$$e^{2z}\cosh z^2 = \left(1 + 2z + \frac{4z^2}{2} + \frac{8z^3}{6} + \cdots\right)\left(1 + \frac{z^4}{2} + \frac{z^8}{24} + \cdots\right)$$

$$= 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \left(\frac{16}{24} + \frac{1}{2}\right)z^4 + \cdots$$

$$= 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{7}{6}z^4 + \cdots$$

2) Find the residue of $f(z) = \dfrac{1-e^z}{z^3}$ at $z = 0$.

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

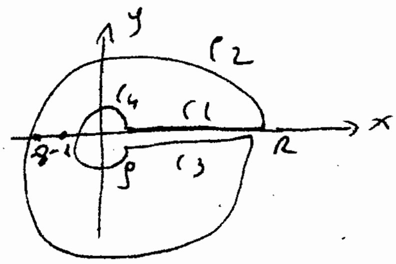
$$1 - e^z = -z - \frac{z^2}{2} - \frac{z^3}{6} - \cdots$$

$$\frac{1-e^z}{z^3} = -\frac{1}{z^2} - \frac{1}{2z} - \frac{1}{6} - \cdots$$

$$\operatorname*{Res}_{z=0} = -\frac{1}{2}$$

3) Evaluate the integral $\int_0^\infty \frac{\sqrt[3]{x}\,dx}{(x+1)(x+8)}$.

$$f(z) = \frac{z^{1/3}}{(z+1)(z+8)}$$



$$\operatorname*{Res}_{z=-1} f = \frac{(-1)^{1/3}}{-1+8} = \frac{e^{i\pi/3}}{7}$$

$$\operatorname*{Res}_{z=-8} f = \frac{(-8)^{1/3}}{-8+1} = \frac{(8e^{i\pi})^{1/3}}{-7} = -\frac{2}{7}e^{i\pi/3}$$

$$\int_\rho^R f(z)\,dz + \int_{C_2} f(z)\,dz + \int_R^\rho f(z)\,dz + \int_{C_4} f(z)\,dz = 2\pi i\left(\frac{1}{7}-\frac{2}{7}\right)e^{i\pi/3} = \frac{-2\pi i}{7}e^{i\pi/3}$$

$$\left|\int_{C_2} f(z)\,dz\right| \leq \frac{\sqrt[3]{R}\;2\pi R}{(R-1)(R-8)} \qquad \left|\int_{C_4} f(z)\,dz\right| \leq \frac{\sqrt[3]{\rho}\;2\pi\rho}{1-\rho\;\;8-\rho}$$

In the limit $R\to\infty$, $\rho\to 0$, these two integrals are zero

$$\int_0^\infty \frac{x^{1/3}dx}{(x+1)(x+8)} + \int_\infty^0 \frac{x^{1/3}e^{i2\pi/3}\,dx}{(x+1)\;(x+8)} = \frac{-2\pi i}{7}e^{i\pi/3}$$

$$(1-e^{i2\pi/3})\int_0^\infty \frac{x^{1/3}dx}{(x+1)(x+8)} = \frac{-2\pi i}{7}e^{i\pi/3}$$

$$\int_0^\infty \frac{x^{1/3}dx}{(x+1)\;(x+8)} = -\frac{2\pi i}{7}\,\frac{e^{i\pi/3}}{1-e^{i2\pi/3}} = -\frac{2\pi i}{7}\,\frac{1}{e^{-i\pi/3}-e^{i\pi/3}}$$

$$= \frac{-2\pi i}{7}\,\frac{1}{\left[\frac{1}{2}-\frac{i\sqrt{3}}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}\right]}$$

$$= \frac{-2\pi i}{7}\cdot\frac{1}{(-i\sqrt{3})}$$

$$= \frac{2\pi}{7\sqrt{3}}$$

4) Find the Linear Fractional Transformation that maps $\infty, 2i, i$ onto $0, 1, \infty$

$$\frac{z-\infty}{z-i} \cdot \frac{2i-i}{2i-\infty} = \frac{w-0}{w-\infty} \cdot \frac{1-\infty}{1-0}$$

$$\frac{i}{z-i} = w$$

$$w = \frac{i}{z-i}$$

5) Find and graph the image of the x−axis under the mapping $w = \dfrac{z-i}{z+i}$

$$z = x$$

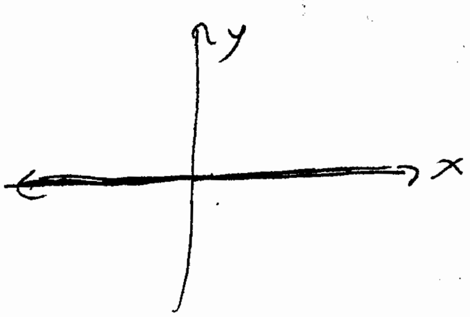
$$w = \frac{x-i}{x+i} = \frac{(x-i)^2}{x^2+1} = \frac{x^2-2xi+1}{x^2+1}$$

$w = u + iv$ where $u = \dfrac{x^2-1}{x^2+1}$

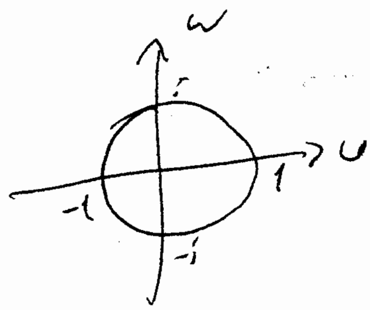
$$v = \frac{-2x}{x^2+1}$$

$$u^2+v^2 = \frac{x^4-2x^2+1+4x^2}{(x^2+1)^2} = \frac{x^4+2x^2+1}{(x^2+1)^2} = \frac{(x^2+1)^2}{(x^2+1)^2} = 1$$

The image is the unit circle



z



w

6) Find the images of the lines $y = x + 4$ and $y = 0$ under the transformation $w = \frac{1}{z}$. Verify conformality at the point $z = -4$

$w = \frac{1}{z}$ If $w = u + iv$, $z = x + iy$ Then $x = \frac{u}{u^2+v^2}$, $y = \frac{-v}{u^2+v^2}$

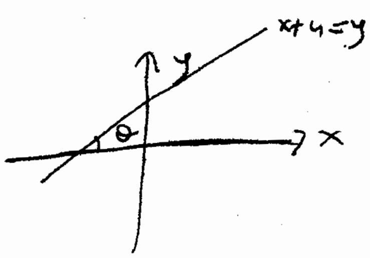
$y = 0 \Rightarrow v = 0$

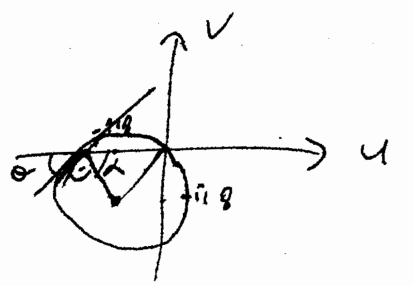
$y = x+4 \Rightarrow \frac{-v}{u^2+v^2} = \frac{u}{u^2+v^2} + 4$

$-v = u + 4(u^2+v^2)$

$u^2+v^2 + \frac{1}{4}u + \frac{1}{4}v = 0$

$(u+\frac{1}{8})^2 + (v+\frac{1}{8})^2 = \frac{1}{32}$





$\theta = 45° = \frac{\pi}{4}$

W

$\theta = 180 - 90 - \alpha$

$\alpha = 45$

$\theta = 45°$

ANSWERS

ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES

MATH 352
Complex Analysis II

FIRST MIDTERM
03.04.2003

NAME, SURNAME :

ID NUMBER :

DEPARTMENT :

SIGNATURE :

INSTRUCTOR :

TIME :15:00-16:30

NUMBER OF QUESTIONS: 5

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :

1) Write your name and department.

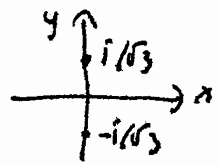
2) Check that there are 5 questions.

3) Show all your work. Correct answers without the intermediate steps may not get credit.

1) Give two Laurent Expansions in powers of z for $f(z) = \dfrac{1}{z^4(1+3z^2)}$ and specify the domains where these expansions are valid.

$$1+3z^2 = 0 \Rightarrow 3z^2 = -1 \Rightarrow z = \pm \frac{i}{\sqrt{3}}$$

$$|z| = \frac{1}{\sqrt{3}}$$



- on $0 < |z| < \frac{1}{\sqrt{3}}$

$$f(z) = \frac{1}{z^4(1+3z^2)} = \frac{1}{z^4} \cdot \frac{1}{1-(-3z^2)}$$

$$= \frac{1}{z^4}\left(1 - 3z^2 + 9z^4 - 27z^6 + \dots\right)$$

$$= \frac{1}{z^4} - \frac{3}{z^2} + 9 - 27z^2 + \dots$$

or $f(z) = \sum_{n=0}^{\infty} (-1)^n 3^n z^{2n-4}$

- on $|z| > \frac{1}{\sqrt{3}}$

$$f(z) = \frac{1}{z^4} \cdot \frac{1}{3z^2\left(1+\frac{1}{3z^2}\right)} = \frac{1}{3z^6} \cdot \frac{1}{1-\left(-\frac{1}{3z^2}\right)}$$

$$= \frac{1}{3z^6}\left(1 - \frac{1}{3z^2} + \frac{1}{9z^4} - \dots\right)$$

$$= \frac{1}{3z^6} - \frac{1}{9z^8} + \frac{1}{27z^{10}} - \dots$$

or $f(z) = \sum_{n=0}^{\infty} (-1)^n 3^{-n-1} z^{-2n-6}$

2) Find the Laurent expansion of $f(z) = \dfrac{1}{e^z - 1 - z}$ around $z = 0$. (Three terms are sufficient)

$$\frac{1}{e^z-1-z} = \frac{1}{\left(1+z+\frac{z^2}{2!}+\dots\right)-1-z} = \frac{1}{\frac{z^2}{2!}+\frac{z^3}{3!}+\dots}$$

$$= \frac{1}{z^2}\left(\frac{1}{\frac{1}{2!}+\frac{z}{3!}+\frac{z^2}{4!}+\dots}\right) = \frac{1}{z^2}\left(a_0+a_1 z+a_2 z^2+\dots\right)$$

$$\Rightarrow 1 = \left(\frac{1}{2}+\frac{z}{6}+\frac{z^2}{24}+\frac{z^3}{120}+\dots\right)\left(a_0+a_1 z+a_2 z^2+\dots\right)$$

$$1 = \frac{a_0}{2} + z\left(\frac{a_0}{6}+\frac{a_1}{2}\right) + z^2\left(\frac{a_2}{2}+\frac{a_1}{6}+\frac{a_0}{24}\right)+\dots$$

$$\Rightarrow \quad a_0 = 2$$

$$\frac{a_0}{6}+\frac{a_1}{2} = 0 \quad \Rightarrow a_1 = -\frac{2}{3}$$

$$\frac{a_2}{2}+\frac{a_1}{6}+\frac{a_0}{24} = 0 \Rightarrow \frac{a_2}{2}+\left(-\frac{2}{18}\right)+\frac{2}{24} = 0$$

$$\frac{a_2}{2} = \frac{1}{9}-\frac{1}{12} = \frac{1}{36}$$

$$a_2 = \frac{1}{18}$$

$$f(z) = \frac{1}{z^2}\left(2-\frac{2}{3}z+\frac{1}{18}z^2+\dots\right)$$

3) Let $f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{4n}}{16^n}$.

a) Find the domain of $f(z)$.

b) Find $g(z)$, which is the analytic continuation of $f(z)$. ($g = f$ over the domain of f)

c) Find the domain of $g(z)$

a) $f(z) = \sum_{n=0}^{\infty} \left(\frac{-z^4}{16}\right)^n$ converges when $\left|\frac{-z^4}{16}\right| < 1$

or $|z| < 2$

b) on $|z| < 2$,

$$f(z) = \frac{1}{1-\left(-\frac{z^4}{16}\right)} = \frac{1}{1+\frac{z^4}{16}} = \frac{16}{16+z^4}$$

So $g(z) = \dfrac{16}{16+z^4}$

c) $g(z)$ is defined on the whole complex plane except

$z^4 = -16$

or $z = 2e^{i\pi/4}$

$z = 2e^{i3\pi/4}$

$z = 2e^{-i3\pi/4}$

$z = 2e^{-i\pi/4}$

4) Let $f(z) = \dfrac{(1+3z-5z^2+7z^4+8z^8)\, e^{1/z}}{z^3}$. Find the residue of $f(z)$ at $z=0$

$$f(z) = \left(\frac{1}{z^3} + \frac{3}{z^2} - \frac{5}{z} + 7z + 8z^5\right)\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots\right)$$

The coefficient of the term $\frac{1}{z}$ in this expansion will be:

$\frac{1}{z^3}$

$$b_1 = -5 + \frac{7}{2!} + \frac{8}{6!}$$

$$= -5 + \frac{7}{2} + \frac{8}{720}$$

$$= \frac{-3}{2} + \frac{1}{90}$$

$$= \frac{-135+1}{90}$$

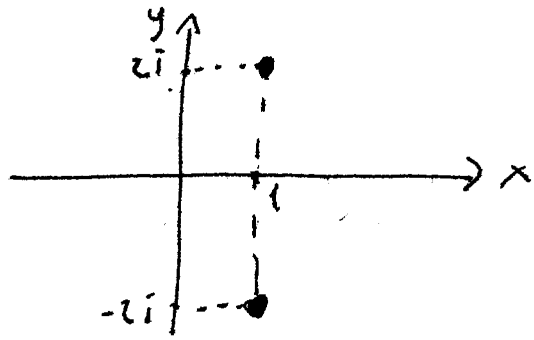
$$= \frac{-134}{90}$$

$$= -\frac{67}{45}$$

5) Evaluate the integral $\int_C \frac{1}{z^2 - 2z + 5}\, dz$ where C is the circle given by

a) $C : |z - 2i| = 2$

b) $C : |z - 2i| = 6$



$$z^2 - 2z + 5 = 0$$

$$z = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

$f(z) = \dfrac{1}{(z-1-2i)(z-1+2i)}$ $\Rightarrow$ both singularities are simple poles.

$$\operatorname*{Res}_{z=1+2i} f(z) = \frac{1}{1+2i-1+2i} = -\frac{i}{4}$$

$$\operatorname*{Res}_{z=1-2i} f(z) = \frac{1}{1-2i-1-2i} = \frac{i}{4}$$

a) The circle $|z-2i| = 2$ includes only the first singularity

$$\Rightarrow I = 2\pi i \left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

b) The circle $|z-2i| = 6$ contains both singularities

$$\Rightarrow I = 2\pi i \left(-\frac{i}{4} + \frac{i}{4}\right) = 0$$

ANSWERS

ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES

MATH 352
Complex Analysis II

SECOND MIDTERM
01.05.2003

NAME, SURNAME :

ID NUMBER :

DEPARTMENT :

SIGNATURE :

INSTRUCTOR :

TIME :15:00-16:30

NUMBER OF QUESTIONS: 6

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
6		20
Total		120

IMPORTANT :

1) Write your name and department.

2) Check that there are 6 questions.

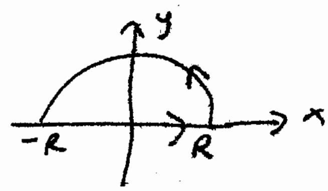
3) Show all your work. Correct answers without the intermediate steps may not get credit.

$$I = \frac{2\pi i \dfrac{e^{i\pi/3} + e^{-i\pi/3}}{2}}{i \cdot \dfrac{e^{i2\pi/3} - e^{-i2\pi/3}}{2i}} = 2\pi \frac{\cos(\pi/3)}{\sin(2\pi/3)} = 2\pi \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$$

1) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{3\,dx}{9x^2+6x+2}$

Consider $\int_C f(z)\,dz$ where $f(z) = \dfrac{1}{3\left(z^2+\frac{2}{3}z+\frac{2}{9}\right)}$

and C is the semicircle:



$z^2+\frac{2}{3}z+\frac{2}{9}=0 \Rightarrow z=-\frac{1}{3}\pm\frac{1}{3}i$ only $z=-\frac{1}{3}+\frac{1}{3}i$ is inside the contour

$$\operatorname*{Res}_{z=-\frac{1}{3}+\frac{1}{3}i} f(z) = \frac{1}{3\left(2z+\frac{2}{3}\right)}\bigg|_{z=-\frac{1}{3}+\frac{1}{3}i} = \frac{1}{-2+2i+2} = \frac{1}{2i}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \frac{1}{2i} = \pi$$

$$\lim_{R\to\infty}\left|\int_{C_R} f(z)\,dz\right| = 0 \quad \text{because} \quad \deg\left(9z^2+6z+2\right) = 2+\deg(3)$$

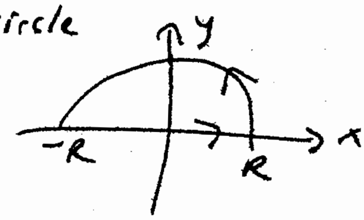
$$\Rightarrow \lim_{R\to\infty}\int_{-R}^{R} f(z)\,dz = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{3\,dx}{9x^2+6x+2} = \pi$$

2) Evaluate the integral $\int_{-\infty}^{\infty} \frac{x \sin x \, dx}{1+x^2}$

Consider $\int_C f(z)\,dz$ where $f(z) = \frac{z e^{iz}}{1+z^2}$

and C is the upper semicircle



$1+z^2=0 \Rightarrow z=\pm i$

only $z=i$ is inside the contour

$$\operatorname*{Res}_{z=i} f(z) = \left.\frac{z e^{iz}}{z+i}\right|_{z=i} = \frac{i e^{i^2}}{2i} = \frac{e^{-1}}{2} = \frac{1}{2e}$$

$$\int_C f(z)\,dz = 2\pi i \cdot \frac{1}{2e} = \frac{\pi i}{e}$$

$\lim_{R\to\infty} \left| \int_{C_R} f(z)\,dz \right| = 0$ by Jordan's lemma, because $\deg(1+z^2) = 1 + \deg z$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \frac{\pi i}{e}$$

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\,dx = \frac{\pi}{e} i$$

By taking the imaginary part of both sides, we obtain

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\,dx = \frac{\pi}{e}$$

3) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{(\sqrt{3}+\cos\theta)^2}$

Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{z+z^{-1}}{2}$$

$$\int_0^{2\pi} \frac{d\theta}{(\sqrt{3}+\cos\theta)^2} = \int_C \frac{\frac{dz}{iz}}{\left(\sqrt{3}+\frac{z+z^{-1}}{2}\right)^2} \quad \text{where } C: |z|=1$$

$$I = \int_C \frac{z\,dz}{i\left(\sqrt{3}z+\frac{z^2}{2}+\frac{1}{2}\right)^2} = \int_C \frac{4z\,dz}{i(z^2+2\sqrt{3}z+1)^2}$$

$$z^2+2\sqrt{3}z+1=0 \Rightarrow z = -\sqrt{3}\pm\sqrt{2}$$

Only $z=-\sqrt{3}+\sqrt{2}$ is inside $|z|=1$

Let $f(z) = \frac{4z}{i(z-z_1)^2(z-z_2)^2}$ where $z_1 = -\sqrt{3}+\sqrt{2}$, $z_2 = -\sqrt{3}-\sqrt{2}$

Then $f(z) = \frac{\phi(z)}{(z-z_1)^2}$ where $\phi(z) = \frac{4z}{i(z-z_2)^2}$

$$\operatorname*{Res}_{z=z_1} f(z) = \phi'(z_1)$$

$$\phi'(z) = \frac{4}{i}\left(\frac{(z-z_2)^2 - 2(z-z_2)z}{(z-z_2)^4}\right)$$

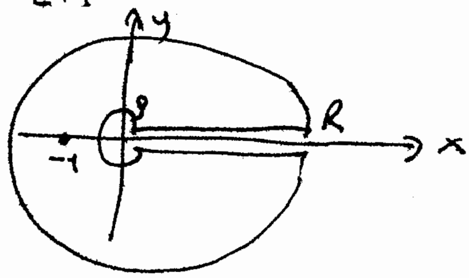
$$\phi'(z_1) = \frac{4}{i}\left(\frac{(z_1-z_2)^2 - 2z_1(z_1-z_2)}{(z_1-z_2)^4}\right) \quad \text{where } z_1-z_2 = 2\sqrt{2}$$

$$\phi'(z_1) = \frac{4}{i}\left(\frac{8-4\sqrt{2}(-\sqrt{3}+\sqrt{2})}{64}\right) = \frac{\sqrt{2}\sqrt{3}}{4i} = \text{Residue}$$

$$I = 2\pi i \cdot \text{Res} = \frac{2\pi i\sqrt{2}\sqrt{3}}{4i} = \frac{\pi\sqrt{2}\sqrt{3}}{2} = \pi\sqrt{\frac{3}{2}}$$

4) Evaluate the integral $\displaystyle\int_0^\infty \frac{x^{-1/3}}{x+1}\,dx$

Let $f(z) = \dfrac{z^{-1/3}}{z+1}$. Consider $\int_C f(z)\,dz$ where C is the contour



The only singularity inside the contour is $z=-1$.

$$\operatorname*{Res}_{z=-1} f(z) = (-1)^{-1/3} = e^{-i\pi/3}$$

$$\int_\rho^R \frac{z^{-1/3}}{z+1}\,dz + \int_{C_R} \frac{z^{-1/3}}{z+1}\,dz + \int_R^\rho \frac{z^{-1/3}}{z+1}\,dz + \int_{C_\rho} \frac{z^{-1/3}}{z+1}\,dz = 2\pi i\, e^{-i\pi/3}$$

$$z^{-1/3} = e^{(\log z)(-\frac{1}{3})} = e^{-\frac{1}{3}\ln r - \frac{1}{3}i\theta} = r^{-1/3}\, e^{-\frac{i\theta}{3}}$$

On the first integral, $\theta = 0$, on the third, $\theta = 2\pi$

We can prove that the second and fourth integrals are zero in the limit $\rho \to 0$, $R \to \infty$, because $\left|\int_{C_R} f(z)\,dz\right| \leq \dfrac{R^{-1/3}\, 2\pi R}{R-1}$

$\left|\int_{C_\rho} f(z)\,dz\right| \leq \dfrac{\rho^{-1/3}\, 2\pi\rho}{1-\rho}$

Therefore $\displaystyle\int_\rho^R \frac{x^{-1/3}}{x+1}\,dx + \int_R^\rho \frac{x^{-1/3}\, e^{-i2\pi/3}}{x+1}\,dx = 2\pi i\, e^{-i\pi/3}$

In the limit $\rho \to 0$, $R \to \infty$, we obtain

$$\left(\int_0^\infty \frac{x^{-1/3}}{x+1}\,dx\right)\left(1 - e^{-i2\pi/3}\right) = 2\pi i\, e^{-i\pi/3}$$

$$\int_0^\infty \frac{x^{-1/3}}{x+1}\,dx = \frac{2\pi i\, e^{-i\pi/3}}{1-e^{-i2\pi/3}}\,\frac{(1+e^{i2\pi/3})}{(1+e^{i2\pi/3})} = \frac{2\pi i\,(e^{-i\pi/3} + e^{i\pi/3})}{1+e^{i2\pi/3} - e^{-i2\pi/3} - 1}$$

Continues at cover

5) Find the number of roots of the equation $z^8 - z^3 + z + 18 = 0$ on

a) $|z| < 1$

b) $1 \leq |z| < 2$

c) $2 \leq |z|$

On $|z| = 1$, Let $f(z) = 18$, $g(z) = z^8 - z^3 + z$

$$|f(z)| > |g(z)|$$

$f(z) = 0$ has no roots $\Rightarrow$ $f(z) + g(z) = 0$ has no roots inside $|z| < 1$

On $|z| = 2$, $f(z) = z^8$, $g(z) = -z^3 + z + 18$

$$|f(z)| = z^8 = 256$$

$$|g(z)| \leq 8 + 2 + 18 = 28$$

$$|f(z)| > |g(z)|$$

$f(z) = 0$ has 8 roots $\Rightarrow$ $f(z) + g(z) = 0$ has 8 roots inside $|z| < 2$

a) 0 roots

b) 8 roots

c) 0 roots

6) Find the Inverse Laplace Transform of $F(s) = \dfrac{5s-7}{s^3-2s^2-s+2}$.

$$s^3-2s^2-s+2 = (s-1)(s+1)(s-2)$$

$$F(s) = \frac{5s-7}{(s-1)(s+1)(s-2)}$$

$$g(s) = F(s)\, e^{st} = \frac{(5s-7)\, e^{st}}{(s-1)\,(s+1)\,(s-2)}$$

$$f(t) = \sum \operatorname{Res} g(s)$$

$$\operatorname*{Res}_{s=1} g(s) = \frac{5-7}{2\,(-1)}\, e^{t} = e^{t}$$

$$\operatorname*{Res}_{s=-1} g(s) = \frac{-5-7}{(-2)(-3)}\, e^{-t} = \frac{-12}{6}\, e^{-t} = -2\, e^{-t}$$

$$\operatorname*{Res}_{s=2} g(s) = \frac{10-7}{1\cdot 3}\, e^{2t} = e^{2t}$$

$$f(t) = e^{t} + e^{2t} - 2e^{-t}$$

ANSWERS

ÇANKAYA UNIVERSITY
DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES

MATH 352
Complex Analysis II

FINAL
20.06.2003

NAME, SURNAME :
ID NUMBER :
DEPARTMENT :
SIGNATURE :
INSTRUCTOR :
TIME :15:00-16:30
NUMBER OF QUESTIONS: 6

Question	Grade	Out of
1		20
2		20
3		20
4		20
5		20
6		20
Total		120

IMPORTANT :

1) Write your name and department.

2) Check that there are 6 questions.

3) Show all your work. Correct answers without the intermediate steps may not get credit.

1) Find the Laurent expansion of $f(z) = \dfrac{1}{\sinh z}$ around $z = 0$ up to z^3.
In which region is this expansion valid?

$$\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots} = \frac{1}{z} \quad \frac{1}{\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)}$$

$$= \frac{1}{z}\left(a_0 + a_1 z + a_2 z^2 + \dots\right)$$

$$\Rightarrow \quad 1 = \left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)\left(a_0 + a_1 z + a_2 z^2 + \dots\right)$$

$a_0 = 1$

$a_1 = 0$

$a_2 + \frac{a_0}{3!} = 0 \quad \Rightarrow \quad a_2 = -\frac{1}{6}$

$a_3 + \frac{a_1}{3!} = 0 \quad \Rightarrow \quad a_3 = 0$

$a_4 + \frac{a_2}{3!} + \frac{a_0}{5!} = 0 \quad \Rightarrow \quad a_4 = -\frac{1}{120} + \frac{1}{36} = \frac{7}{360}$

$$\frac{1}{\sinh z} = \frac{1}{z}\left(1 - \frac{1}{6} z^2 + \frac{7}{360} z^4 + \dots\right)$$

$$= \frac{1}{z} - \frac{1}{6} z + \frac{7}{360} z^3 + \dots$$

$\sinh i\pi = 0 \quad \Rightarrow \quad$ Valid on $\quad 0 < |z| < \pi$

2) Find the residue of $f(z) = e^{-1/z^2} \cosh z$ at $z = 0$

$$f(z) = \left(1 - \frac{1}{z^2} + \frac{1}{z^4 \cdot 2!} - \frac{1}{z^6 3!} + \cdots\right)\left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\right)$$

The coefficient of $\frac{1}{z}$ in this product is zero

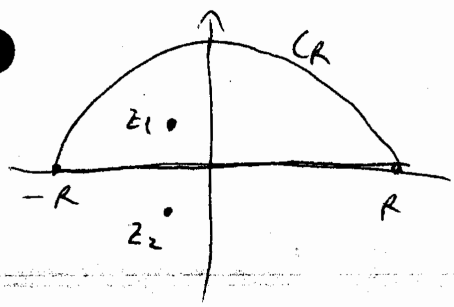
$\Rightarrow$ The residue is zero

3) Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{29x^2+4x+1}$

$$29z^2+4z+1=0 \Rightarrow z=\frac{-4\pm\sqrt{16-116}}{58} = -\frac{2}{29} \pm \frac{5i}{29}$$

Let $z_1 = -\frac{2}{29} + \frac{5i}{29}$, $\quad z_2 = \frac{-2}{29} - \frac{5i}{29}$

$$f(z) = \frac{1}{29z^2+4z+1} = \frac{1/29}{(z-z_1)(z-z_2)}$$



$$\operatorname*{Res}_{z=z_1} f(z) = \frac{1/29}{z_1-z_2} = \frac{1}{29}\cdot\frac{1}{\frac{5i}{29}+\frac{5i}{29}} = \frac{1}{10i}$$

$$\int_C f(z)\,dz = 2\pi i\,\frac{1}{10i} = \frac{\pi}{5}$$

$\left|\int_{C_R} f(z)\,dz\right| = 0$ in the limit $R\to\infty$ because $\deg p = \deg q - 2$

$$\Rightarrow \lim_{R\to\infty}\int_C f(z)\,dz = \int_{-\infty}^{\infty}\frac{dx}{29x^2+4x+1} = \frac{\pi}{5}$$

4) Evaluate the integral $\int_0^{2\pi} \frac{d\theta}{3+\sin\theta}$

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{z - z^{-1}}{2i}$$

$$I = \int_C \frac{dz/iz}{3 + \frac{z-z^{-1}}{2i}} = \int \frac{2\,dz}{6iz + z^2 - 1}$$ where $C: |z| = 1$

$$= \int \frac{2\,dz}{z^2 + 6iz - 1}$$

$$z^2 + 6iz - 1 = 0 \Rightarrow z = \frac{-6i \pm \sqrt{-36+4}}{2} = -3i \pm 2\sqrt{2}\,i$$

$z_1 = -3i + 2\sqrt{2}\,i$ is inside $|z| = 1$

$$I = \int \frac{2\,dz}{(z-z_1)(z-z_2)}$$

The residue of $f(z)$ at $z = z_1$ is $\frac{2}{z_1 - z_2} = \frac{2}{4\sqrt{2}\,i} = \frac{1}{2\sqrt{2}\,i}$

$$I = 2\pi i \frac{1}{2\sqrt{2}\,i} = \frac{\pi}{\sqrt{2}}$$

5) Find a linear fractional transformation that maps the points $z_1 = 0,\ z_2 = 1,\ z_3 = \infty$ onto the points $w_1 = 1,\ w_2 = \dfrac{1-i}{2},\ w_3 = 1+i$

$w = az+b / cz+d$

$z=0 \Rightarrow w=1$ $\qquad$ $z=1 \quad w=\dfrac{1-i}{2}$ $\qquad$ $z=\infty \quad w=1+i$

$\dfrac{b}{d}=1$ $\qquad$ $\dfrac{a+b}{c+d}=\dfrac{1-i}{2}$ $\qquad$ $\dfrac{a}{c}=1+i$

- $b=d$

$a=(1+i)c$

$a+b=(c+d)\left(\frac{1-i}{2}\right)$

$(1+i)c+d=(c+d)\left(\frac{1-i}{2}\right)$

$\frac{1+3i}{2}c=-\frac{1-i}{2}d$

$c=-\frac{(1+i)}{(1+3i)}d$

- Let $d=1+3i$

Then $c=-(1+i)$

$a=-(1+i)^2=-2i$

$b=1+3i$

$$w=\frac{-2iz+(1+3i)}{-(1+i)z+(1+3i)}$$

OR $\quad w=\dfrac{(-1-i)z+i+2}{-z+i+2}$

6) Find and plot the image of the line segment $x = \frac{\pi}{4}$, $-\ln 2 \leq y \leq \ln 2$ under the transformation $w = \sin z$

$$W = \sin z = \sin x \cosh y + i \cos x \sinh y$$

$$x = \frac{\pi}{4} \Rightarrow w = \frac{\sqrt{2}}{2} \cosh y + i \frac{\sqrt{2}}{2} \sinh y$$

$$u = \frac{\sqrt{2}}{2} \cosh y \qquad v = \frac{\sqrt{2}}{2} \sinh y$$

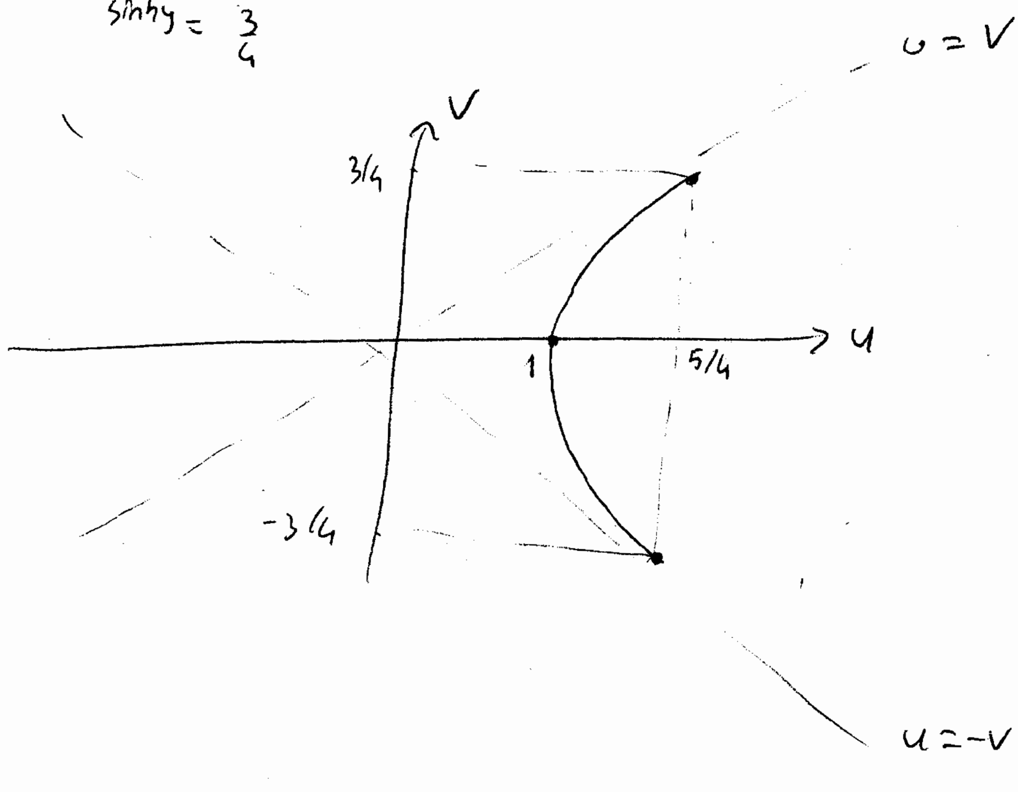
$$\left(\frac{u}{\frac{\sqrt{2}}{2}}\right)^2 - \left(\frac{v}{\frac{\sqrt{2}}{2}}\right)^2 = 1$$

$$y = -\ln 2 \Rightarrow \cosh y = \frac{e^{-\ln 2} + e^{\ln 2}}{2} = \frac{\frac{1}{2} + 2}{2} = \frac{5}{4}$$

$$\sinh y = \frac{e^{-\ln 2} - e^{\ln 2}}{2} = \frac{\frac{1}{2} - 2}{2} = -\frac{3}{4}$$

$$y = \ln 2 \Rightarrow \cosh y = \frac{5}{4}$$

$$\sinh y = \frac{3}{4}$$



Answer Key!

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 Complex Analysis II
First Midterm Examination

23.03.2005

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION : 90 minutes
TOTAL NUMBER OF QUESTIONS : 4

Questions	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

IMPORTANT :
1) Write your name and department.
2) Check that there are 4 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

(25 pts.)

a) Prove that if $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \to 0$. If $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly, then show that $g_k \to 0$ uniformly. Explain clearly

b) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^z}$ does not converge uniformly on $A = \{z \mid \operatorname{Re} z > 1\}$.

a) Suppose $a_k \not\to 0$. Then $\exists\, \epsilon > 0$ such that $\forall N > 0\ \exists k > N$ with $|a_k| > \epsilon$. But then $|S_k - S_{k-1}| = |a_k| > \epsilon$. The sequence of partial sums is not Cauchy and cannot converge. Thus the sum $\sum a_k$ does not converge.

b) If $\sum_{k=1}^{\infty} g_k(z)$ converge uniformly, the sequence of partial sums $S_n(z)$ converges uniformly. Thus it is uniformly Cauchy and given $\epsilon > 0$, $\exists\, N > 0$ so that

$|S_n(z) - S_{n+p}(z)| < \epsilon$, $\forall z$, $\forall n > N$, $p = 1, 2, 3 \ldots$

Take $p = 1$.

$|g_{n+1}(z)| = |S_n(z) - S_{n+1}(z)| < \epsilon$ $\forall z$, $\forall n > N$.

The sequence g_n is uniformly Cauchy and so uniformly convergent

1) (25 pts.)
Compute the Taylor series of the following:

a) $\sin(z^2)$, $z_0 = 0$

b) e^{2z} , $z_0 = 0$

a) $$\sin z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} z^{2n-1} , \quad z_0 = 0$$

Therefore

$$\sin(z^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} (z^2)^{2n-1}$$

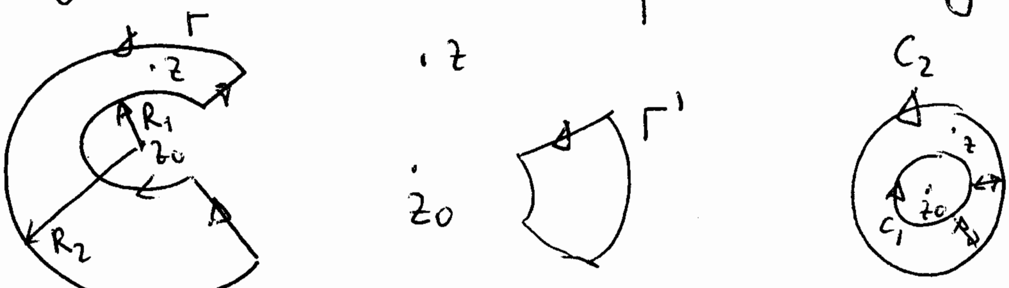
$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} z^{2n-1}$$

b) $$e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$$

III) (25 Pts.) Prove the following theorem:

Let f be analytic in the annulus $r < |z-z_0| < R$. The f can be expressed there as the sum of two series $f(z) = \sum_{j=0}^{\infty} a_j (z-z_0)^j + \sum_{j=1}^{\infty} a_{-j} (z-z_0)^{-j}$ both series converging in the annulus, and converging uniformly in any closed subannulus $r < \rho_1 \le |z-z_0| \le \rho_2 < R$. The coefficients a_j are given by $a_j = \frac{1}{2\pi i} \oint \frac{f(\xi)}{(\xi - z_0)^{j+1}} d\xi$

$(j = 0, \pm 1, \pm 2 \ldots)$. Explain clearly.



a b c

$\forall z$, $r < \rho_1 \le |z - z_0| < \rho_2 < R$ we have to prove that

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\xi)}{\xi - z} d\xi + \frac{1}{2\pi i} \oint_{C_2} \frac{f(\xi)}{\xi - z} d\xi, \quad R_1 = \frac{r + \rho_1}{2}$$

$R_2 = \frac{R + \rho_2}{2}$. Consider the contour Γ of Fig. a.

therefore $f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{\xi - z} d\xi$. Let Γ' the contour as in Fig. b, then $\cancel{R(z)}\, \frac{1}{2\pi i} \int_{\Gamma'} \frac{f(\xi)}{\xi - z} d\xi = 0$ because the integrand is analytic inside and on Γ'.

Consequently $f(z) = \frac{1}{2\pi i} \int \frac{f(\xi)}{\ldots} d\xi$.

Since z lies inside C_2 the integral over C_2 is

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{n} a_j (z-z_0)^j + T_n(z), \quad T_n(z) \to 0$$

uniformly as $n \to \infty$ for $|z - z_0| \leq \rho_2$ and

$$a_j = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}}\, d\zeta, \quad j = 0, 1, 2, \ldots$$ Hence

$$\frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=0}^{\infty} a_j (z - z_0)^j, \quad (|z - z_0| \leq \rho_2).$$ we have

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{z - z_0} \frac{1}{1 - \frac{\zeta - z_0}{z - z_0}} = -\frac{1}{z - z_0}\left[1 + \frac{\zeta - z_0}{z - z_0} + \frac{(\zeta - z_0)^2}{(z - z_0)^2} + \ldots\right]$$

we obtain

$$\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{m+1} a_{-j} (z - z_0)^{-j} + J_m(z)$$

where $a_{-j} = -\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z_0)^{-j+1}}\, d\zeta, \ (j = 1, 2, 3 \ldots)$

and $$J_m(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\left(\frac{(\zeta - z_0)^{m+1}}{(z - z_0)^{m+1}}\right) d\zeta$$

Now for ζ on C_1 we have $|\zeta - z| \geq \rho_1 - R_1$

$|\zeta - z_0| = R_1$ and $|z - z_0| \geq \rho_1$. Thus $|J_m(z)| \leq \frac{1}{2\pi} \cdot \max_{\zeta \text{ on } C_1} |f(\zeta)| \frac{1}{\rho_1 - R_1}$

$\times \left(\frac{R_1}{\rho_1}\right)^{m+1} 2\pi R_1$. Since $\frac{R_1}{\rho_1} < 1$, $J_m(z) \to 0$ uniformly for $|z - z_0| \geq \rho_1$

and $$\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \sum_{j=1}^{\infty} a_{-j} (z - z_0)^{-j}, \quad (|z - z_0| \geq \rho_1)$$

. Similarly, the integral over C_1 can be changed into an integral over C incorporating the minus sign to account for the change in orientation. Q.E.D.

1) (25 pts.) classify the zeros and singularities of the following functions

a)
$\sin(1-z^{-1})$

b) $\dfrac{\tan z}{z}$

Explain clearly

a) $1-z^{-1}=n\pi, \quad z=\dfrac{1}{1-n\pi}, \quad n=0,\pm1,\pm2\ldots$

The zeros are simple because

$$\frac{d}{dz}\sin(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}}=\frac{1}{z^2}\cos(1-z^{-1})\Big|_{z=(1-n\pi)^{-1}}$$

$$=(1-n\pi)^2\cos n\pi\neq0.$$

The only singularity of $\sin(1-z^{-1})$ arises when $z=0$. On the other hand we observe that when z approach 0 through positive value, then $\sin(1-z^{-1})$ oscillates between ±1. in conclusion we have an essential singularity.

b) $$\frac{\tan z}{z}=\frac{\sin z}{z\cos z}=\frac{1}{z\cos z}\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}+\ldots\right)=$$

$$=\frac{1}{\cos z}\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\ldots\right)$$

$\dfrac{\tan z}{z}\to1$ as $z\to0$. Conclusion: the origin is a removable singularity.

$\cos z$ has simple zeros at $z=(n+\frac{1}{2})\pi$ for $n=0,\pm1,\pm2,$

We observe that $f(z)$ has simple poles at these points.

Answer Key

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 Complex Analysis II

Second Midterm Examination

20.04.2005

NAME, SURNAME :

NUMBER :

DEPARTMENT :

SECTION :

SIGNATURE :

INSTRUCTOR'S NAME :

DURATION : 90 minutes

TOTAL NUMBER OF QUESTIONS : 4

Questions	Grade	Out of
1		25
2		25
3		25
4		25
Total		100

IMPORTANT :

1) Write your name and department.
2) Check that there are 4 questions.
3) Show all your work. Correct answers without the intermediate steps may not get credit.

)(25 Pts.) Prove the following theorem:
if f has a pole of order m at z_0, then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right].$$

Explain clearly.

Starting with the Laurent expansion for f around z_0,

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

we multiply by $(z-z_0)^m$,

$$(z-z_0)^m f(z) = a_{-m} + \cdots + a_{-2}(z-z_0)^{m-2} + a_{-1}(z-z_0)^{m-1}$$
$$+ a_0(z-z_0)^m + a_1(z-z_0)^{m+1} + \cdots$$

and differentiate $m-1$ times to derive

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] = (m-1)!\, a_{-1} + m!\, a_0 (z-z_0) + \cdots$$

Hence

$$\lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right] = (m-1)!\, a_{-1} \quad \text{Q.E.D.}$$

II) (25 Pts.)

a) Find $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{x^4+x^2+1}\, dx$

b) Evaluate the integral

$\displaystyle\int_{-\infty}^{+\infty} \frac{x e^{i\omega x}}{x^4+1}\, dx$ for $\omega > 0$.

Explain clearly.

- a) $\displaystyle\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)}\right]$, where $P(x)$ and $Q(x)$ are polynomials in x, and the degree of $Q(x)$ exceed that of $P(x)$ by two or more.

- $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2\,dx}{x^4+x^2+1} = 2\pi i \sum \text{Res} \frac{z^2}{z^4+z^2+1}$

$z^4+z^2+1=0$ gives $z^2 = \dfrac{-1\pm i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}}$

Taking square roots yields

$z = e^{i\frac{\pi}{3}}, e^{-i\frac{2\pi}{3}}, e^{-i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}$

Thus, $\dfrac{z^2}{z^4+z^2+1}$ has simple pole in u.h.p

at $e^{i\frac{\pi}{3}}$ and $e^{i\frac{2\pi}{3}}$

Conclusion: $2\pi i \sum \text{Res}\left[\dfrac{z^2}{z^4+z^2+1}\right] = \dfrac{\pi}{\sqrt{3}}$

b)

$$\int_{-\infty}^{\infty} \frac{x e^{i\omega x}}{x^4+1}\, dx = 2\pi i \sum \mathrm{Res}\left[e^{i\omega z} \frac{z}{z^4+1} \right]$$

$z^4+1=0$, $z_{1,2} = \frac{\pm 1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$ are the poles in the upper half- plane.

Therefore, $2\pi i \,\mathrm{Res}\left[e^{i\omega z} \frac{z}{z^4+1} \right]$ u.h.p =

$$= \frac{\pi i}{2} \left[\frac{e^{i\omega(1+i)/\sqrt{2}}}{i} + \frac{e^{i\omega(-1+i)/\sqrt{2}}}{-i} \right].$$

Conclusion:

$$\int_{-\infty}^{+\infty} \frac{x e^{i\omega x}}{x^4+1}\, dx = i\pi\, e^{-\frac{\omega}{\sqrt{2}}} \sin\left(\frac{\omega}{\sqrt{2}}\right)$$

VII) (25 pts.)

Locate the zeros of the polynomial $z^4 - z + 5$. Explain clearly.

We know that "if $f(z) = a_0 + a_1 z + \cdots + a_n z^n$, $n \geq 1$, $a_n \neq 0$, then f has exactly n roots."

Let $f(z) = z^4 - z + 5$. We know f has four zeros.

Let $g(z) = z^4$.

$$|f(z) - g(z)| = |z - 5| \leq |z| + 5$$

$$|g(z)| = |z|^4$$

So if $|z| = \sqrt{3}$, $|f(z) - g(z)| \leq |g(z)|$.

All four roots of f must lie in $\{ z \mid |z| < \sqrt{3} \}$.

(25 Pts.)

Deduce Cauchy's integral formula from Residue Theorem. Explain clearly

Let the closed curve γ and its interior lie in the region A of analyticity of f. Then, $\frac{f(z)}{(z-z_0)^{n+1}}$ for $z_0 \in A$, $z_0 \notin \gamma$, is analytic on the region less the point z_0.

writing $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ near z_0, one sees that

$$\operatorname{Res}\left(\frac{f(z)}{(z-z_0)^{n+1}}, z_0\right) = a_n = \frac{f^{(n)}(z_0)}{n!}$$

or

$$f^{(n)}(z_0)\, I(\gamma, z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

which is the Cauchy integral formula.

Answer Key!

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 Complex Analysis II
Final Examination

30.05.2005

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION : 110 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
Total		100

IMPORTANT :

1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.

pts.)

Find the image of the interior of the circle $C: |z-2| = 2$ under the Möbius transformation

$$w = f(z) = \frac{z}{2z-8}$$ Explain clearly.

Since f has a pole at $z=4$ and this point lies on C, the image has to be a straight line. To specify this line all we need is to determine two of its finite points. The points $z=0$ and $z=2+2i$ which lie on C have, as their images,

$w = f(0) = 0$ and $w = f(2+2i) = \frac{2+2i}{2(2+2i)-8} = -\frac{i}{2}$

Thus the image of C is the imaginary axis in the w-plane.

We know that the interior of C is mapped either onto the right half-plane $\operatorname{Re} w > 0$ or onto the left half-plane $\operatorname{Re} w < 0$.

Since $z=2$ lies inside C and

$w = f(2) = \frac{2}{4-8} = -\frac{1}{2}$ lies in the left half-plane, we conclude that the image of the exterior of C is the left half-plane.

we estimate the integrals along the circ

Γ_ϵ and C_ρ as usual.

For sufficiently small ϵ,

$$\left| \int_{\Gamma_\epsilon} \frac{\mathcal{L}_0(z)\, dz}{(z+1)(z^2+2z+2)} \right| \leq \frac{\sqrt{(\mathrm{Log}\,\epsilon)^2 + (2\pi)^2}\; 2\pi\epsilon}{(1-\epsilon)(2-2\epsilon-\epsilon^2)}$$

which goes to zero as $\epsilon \to 0^+$.

And for sufficiently large ρ

$$\left| \int_{C_\rho} \frac{\mathcal{L}_0(z)\, dz}{(z+1)(z^2+2z+2)} \right| \leq \frac{\sqrt{(\mathrm{Log}\,\rho)^2 + (2\pi)^2}\; 2\pi\rho}{(\rho-1)(\rho^2-2\rho-2)}$$

$$= \frac{\sqrt{(\mathrm{Log}\,\rho)^2 + (2\pi)^2}\; 2\pi\rho}{\rho^3 \left(1-\frac{1}{\rho}\right)\left(1-\frac{2}{\rho}-\frac{2}{\rho^2}\right)}$$

which goes to zero as $\rho \to \infty$.

Consequently, in the limit the contour integral $I_{\epsilon,\rho}$ approaches

$$-2\pi i \int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)} = -2\pi i\, I.$$

But, by the residue theorem, $I_{\epsilon,\rho}$ equals $2\pi i$ times the sum of the residues inside the contour.

$$I = 2\pi i\left(-\mathrm{Log}\sqrt{2}\right)/(+2\pi)i = \mathrm{Log}\sqrt{2}.$$

)(20 Pts.)

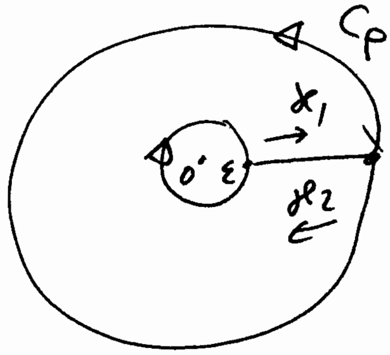
Calculate the integral

$$I = \int_0^\infty \frac{dx}{(x+1)(x^2+2x+2)}$$

by using the residue theorem.

$$\mathcal{L}_0(z) := \mathrm{Log}|z| + i \arg_0 z$$

$$I_{\epsilon,\rho} = \int \frac{\mathcal{L}_0(z)\,dz}{(z+1)(z^2+2z+2)}$$



On the upper side of the cut we have $\mathcal{L}_0(z) = \mathrm{Log}\,x + i \arg_0 x = \mathrm{Log}\,x$

On the lower side we have

$\mathcal{L}_0(z) = \mathrm{Log}\,x + i\,2\pi$.

integrating back and forth along the cut, then, we observe the cancellation of the real parts containing the unwanted $\mathrm{Log}\,x$ factors, and we obtain

$$\int_\epsilon^\rho \frac{\mathrm{Log}\,x\,dx}{(x+1)(x^2+2x+2)} + \int_\rho^\epsilon \frac{(\mathrm{Log}\,x + i\,2\pi)\,dx}{(x+1)(x^2+2x+2)} = -2\pi i \int_\epsilon^\rho \frac{dx}{(x+1)(x^2+2x+}$$

Thus

$$\tan z = z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \cdots$$

b) $\cos z$ has zeros of order 1 at

$$z = \frac{2n+1}{2}\pi \quad , n = 0, \pm 1, \pm 2, \ldots$$

So $\frac{1}{\cos z}$ has poles of order 1 at these points.

) (20 Pts.)

a) (10 Pts.) Find the first few terms of the Taylor expansion of $\tan z = \dfrac{\sin z}{\cos z}$ about $z = 0$.

b) (10 Pts.) Where are the poles of $\dfrac{1}{\cos z}$ and what are the orders?

a) Let $\dfrac{\sin z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \dots$

We know $\cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots$

and $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots$

So that

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$= \left\{1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots\right\}\left\{a_0 + a_1 z + a_2 z^2 + \dots\right\}$$

$$= a_0 + a_1 z + \left[a_2 - \frac{1}{2!}a_0\right]z^2 + \left[a_3 - \frac{1}{2!}a_1\right]z^3$$

$$+ \left[a_4 - \frac{1}{2!}a_2 + \frac{1}{4!}a_0\right]z^4 + \left(a_5 - \frac{1}{2!}a_3 + \frac{1}{4!}a_1\right]z^5$$

$$+ \left[a_6 - \frac{1}{2!}a_4 + \frac{1}{4!}a_2 - \frac{1}{6!}a_0\right]z^6 + \dots$$

Comparing coefficients gives:

$a_0 = 1$, $a_1 = 1$, $a_2 - \frac{1}{2}a_0 = 0$, $a_3 - \frac{1}{2}a_1 = -\frac{1}{6}$ ...

Hence $a_0 = a_2 = a_4 = \dots = a_{2n} = 0$, and $a_1 = 1$, $a_3 = \frac{1}{3}$, $a_5 = \frac{2}{15}$.

(20 Pts).

a) (10 Pts.) Find the Laurent series for the function $\dfrac{z^2-2z+3}{z-2}$ in the region $|z-1|>1$.

b) (10 Pts.) Expand $e^{\frac{1}{z}}$ in a Laurent.

a) $$\frac{1}{z-2} = \frac{1}{(z-1)-1} = \frac{1}{z-1} \, \frac{1}{1-\frac{1}{z-1}}$$

- For $|z-1|>1$

$$\frac{1}{z-2} = \frac{1}{z-1} \sum_{j=0}^{\infty} \frac{1}{(z-1)^j}$$

$$= \frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots$$

$$z^2-2z+3 = (z-1)^2 + 0\cdot(z-1) + 2 = (z-1)^2+2$$

- conclusion

$$\frac{z^2-2z+3}{z-2} = \left[(z-1)^2+2\right]\left[\frac{1}{z-1} + \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} + \cdots\right]$$

$$= (z-1) + 1 + \sum_{j=1}^{\infty} \frac{3}{(z-1)^j}$$

b) $$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

if $z \neq 0$, $w = \frac{1}{z}$, then

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$$

V) (20 Pts.)

a) (10 Pts.)
Evaluate the following integral

$$\int_{|z|=\frac{1}{2}} \frac{dz}{(1-z)^3}$$

b) Calculate

$$\int_{|z-1|=\frac{1}{2}} \frac{dz}{(1-z)^3}.$$

a) $\frac{1}{(1-z)^3}$ is analytic inside the path $|z|=\frac{1}{2}$, so the integral is 0.

b) The pole at 1 has residue 0. Therefore, the integral is zero.

KEY

CANKAYA UNIVERSITY
Department of Mathematics and Computer Science

Math 352 Complex Analysis
Midterm 1
30 March 2006, 12:40

STUDENT NUMBER:

NAME SURNAME:

SECTION:

SIGNATURE:

INSTRUCTOR:

DURATION: 110 min.

TOTAL NUMBER OF QUESTIONS: 6

Question	Grade	Out Of
1		18
2		14
3		15
4		18
5		15
6		20
TOTAL		100

IMPORTANT NOTES:

1) Please make sure that you have written your student number and name above.
2) Check that the exam paper contains 6 (six) problems.
3) Show all your work. No points will be given to correct answers without reasonable work.

(18 pts) 1. In each of the following functions defined on a punctured neighborhood of $0 \in \mathbb{C}$, decide what kind of an isolated singularity $z = 0$ is and compute the residue at $z = 0$.

$$a)\ \frac{\cos z}{z^5}, \qquad b)\ z^{2006} \exp\left(\frac{2}{z^2}\right)$$

(a) $$\frac{\cos z}{z^5} = \frac{1}{z^5}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right) \qquad |z| > 0$$

$$= \underbrace{\frac{1}{z^5} - \frac{1}{2!z^3} + \frac{1}{4!z}}_{\text{principal part}} \underbrace{- \frac{z}{6!} + \frac{z^3}{8!} - \frac{z^5}{10!} + \cdots}_{\text{analytic part}} \qquad 0 < |z|$$

$\Rightarrow$ $z = 0$ is a pole of order $m = 5$ with residue $B_1 = \operatorname{Res}_{z=0} \frac{\cos z}{z^5} = \frac{1}{4!}$

(b) $$z^{2006} \exp\left(\frac{2}{z^2}\right) = z^{2006}\left(1 + \frac{2}{z^2} + \frac{1}{2!}\left(\frac{2}{z^2}\right)^2 + \frac{1}{3!}\left(\frac{2}{z^2}\right)^3 + \cdots\right)$$

$$= z^{2006} + 2z^{2004} + 2z^{2002} + \frac{8}{6} z^{2000} + \frac{16}{24} \cdot z^{1998} + \cdots \frac{2^{1003}}{(1003)!} + \frac{2^{1004}}{1004!} \frac{1}{z^2} + \cdots$$

$\Rightarrow$ principal part contains infinitely many nonzero terms, so $z = 0$ is an essential singular point of $z^{2006} \exp\left(\frac{2}{z^2}\right)$

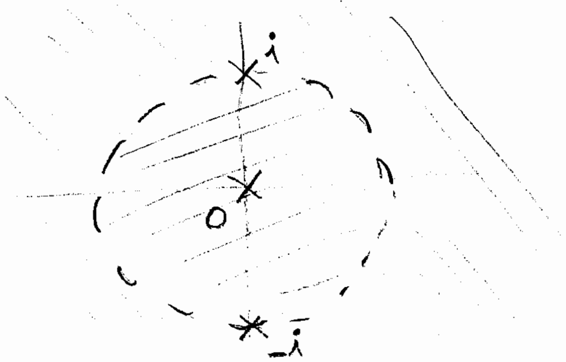
$\Rightarrow$ $\operatorname{Res}_{z=0} z^{2006} \exp\left(\frac{2}{z^2}\right) = 0.$

(14pts) 2. Write the two Laurent series in powers of z that represents the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

This function has isolated singularities at $z=0$ and $z=\pm i$, as indicated in the figure below.



Hence there is a Laurent series representation for the domain $<|z|<1$ and also one for the domain $1<|z|<\infty$, which is exterior to the circle $|z|=1$.

To find each of these Laurent series, we recall the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

For the domain $0<|z|<1$, we have

$$f(z) = \frac{1}{z}\cdot\frac{1}{1+z^2} = \frac{1}{z}\sum_{n=0}^{\infty}(-z^2)^n = \sum_{n=0}^{\infty}(-1)^n z^{2n-1} = \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1} + \frac{1}{z}.$$

On the other hand, when $1<|z|<\infty$,

7)

$$f(z) = \frac{1}{z^3}\,\frac{1}{1+\frac{1}{z^2}} = \frac{1}{z^3}\sum_{n=0}^{\infty}\left(-\frac{1}{z^2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+3}} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$$

In this second expansion, we have used the fact that

$$(-1)^{n-1} = (-1)^{n-1}(-1)^2 = (-1)^{n+1}.$$

(15 pts) 3. Use the residue theorem to show that

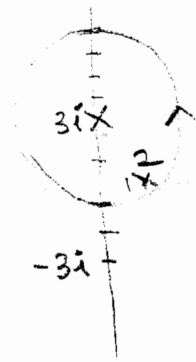
$$\int_C \frac{1+z}{(z-2)(z^2+9)}dz = \frac{\pi}{39}(7+9i)$$

where C is the circle $|z-3i| = 4$ in the positive sense.

$f(z) = \dfrac{1+z}{(z-2)(z^2+9)}$ has singularities at

$z=2$, $z = \pm 3i$

C:



$|2-3i| = \sqrt{2^2+(-3)^2} = \sqrt{13} < 4 \Rightarrow z=2$ is interior

o the contour C.

y the Cauchy's Residue Theorem:

$$\int_C f(z)dz = 2\pi i\left(\operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=3i} f(z)\right) = 2\pi i\left(\frac{3}{13} + \frac{1}{6}\,\frac{-9-7i}{13}\right)$$

$$= \frac{2\pi i}{6}\left(\frac{18-9-7i}{13}\right) = \frac{\pi i}{3}\,\frac{9-7i}{13}$$

$$\boxed{= \frac{\pi}{39}(7+9i)}$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{1+z}{z^2+9}\Big|_{z=2} = \frac{1+2}{2^2+9} = \frac{3}{13}$$

$$\operatorname*{Res}_{z=3i} f(z) = \frac{1+z}{(z-2)(z+3i)}\Big|_{z=3i} = \frac{1+3i}{(3i-2)(3i+3i)} = \frac{1+3i}{(3i-2)6i}$$

$$= \frac{1}{6}\,\frac{1+3i}{-3-2i}$$

$$= \frac{1}{6}\,\frac{1+3i}{-3-2i}\cdot\frac{-3+2i}{-3+2i}$$

$$= \frac{1}{6}\,\frac{-3+2i-9i-6}{9+4}$$

$$= \frac{1}{6}\,\frac{-9-7i}{13}$$

(18 pts) 4. Find the residues of the following functions at the indicated points

(9) (a) $\dfrac{z^4}{z - \frac{1}{6}z^3 - \sin z}$ at $z = 0$

(9) (b) $\dfrac{z^2+1}{z^4-1}$ at $z = 1$ and $z = i$

(9) (a) $$\frac{z^4}{z - \frac{1}{6}z^3 - \sin z} = \frac{z^4}{z - \frac{1}{6}z^3 - \left(z - \frac{z^3}{6} + \frac{z^5}{120} - \frac{z^7}{7!} + \cdots\right)}$$

$$= \frac{z^4}{-\frac{z^5}{5!} + \frac{z^7}{7!} - \frac{z^9}{9!} + \cdots}$$

$$= \frac{1}{-\frac{z}{5!} + \frac{z^3}{7!} - \frac{z^5}{9!} + \cdots}$$

$$f(z) = \frac{\phi(z)}{z}$$

where $\phi(z) = \dfrac{1}{-\frac{1}{5!} + \frac{z^2}{7!} - \frac{z^4}{3!} + \cdots}$ is analytic at $z=0$

and $\phi(0) = -\dfrac{1}{\frac{1}{5!}} = -5! \neq 0 \Rightarrow f(z)$ ~~has~~ a pole at $z=0$ of order $m=1$ with residue $B_1 = -120$.

(9) (b) $$f(z) := \frac{z^2+1}{z^4-1} = \frac{\cancel{(z+i)}\cancel{(z-i)}}{(z-1)(z+1)\cancel{(z-i)}\cancel{(z+i)}}$$

removable singularity at $z=i \Rightarrow \operatorname{Res}_{z=i} f(z) = 0$

pole at $z=1$ of order $m=1$. $\Rightarrow \operatorname{Res}_{z=1} f(z) = \left.\dfrac{z^2+1}{(z^2+1)(z+1)}\right|_{z=1} = \dfrac{1}{2}$

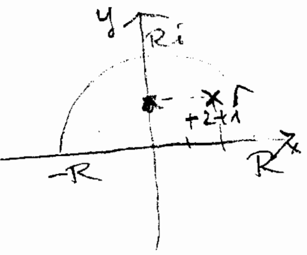
(15) 5. Show that

$$\int_{|z|=1} \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3}$$

We know the Laurent series representation

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360} z + \cdots$$

$$\Rightarrow \operatorname{Res}_{z=0} \frac{1}{z^2 \sinh z} = -\frac{1}{6} = \frac{1}{2\pi i} \int_{|z|=1} \frac{dz}{z^2 \sinh z} = 2\pi i\left(-\frac{1}{6}\right) = -\frac{\pi i}{3}.$$



6. Let

$$f(z) = \frac{1}{(z^2 - 4z + 5)^2}$$

5 (a) Prove that $f(z)$ has a pole of order 2 at the points $2 \pm i$.

5 (b) Find the residue of f at $2 + i$.

10 (c) Prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 4x + 5)^2} = \frac{\pi}{2}.$$

(a) $f(z) = \dfrac{1}{(z^2-4z+5)^2} = \dfrac{1}{(z-2-i)^2(z-2+i)^2} = \dfrac{\phi_1(z)}{(z-2-i)^2} = \dfrac{\phi_2(z)}{(z-2+i)^2}$

where $\phi_1(z) = \dfrac{1}{(z-2+i)^2}$ and $\phi_2(z) = \dfrac{1}{(z-2-i)^2}$ are analytic

at $z = 2+i$ and $z = 2-i$ respectively, and $\phi_1(2+i) = \dfrac{1}{(i+i)^2} = -\dfrac{1}{4} \neq 0$

and $\phi_2(2-i) = \dfrac{1}{(-2i)^2} = -\dfrac{1}{4} \neq 0$

$\Rightarrow f(z)$ has a pole at $2 \pm i$ of order $m_1 = m_2 = 2$.

(b) $\operatorname*{Res}_{z=2+i} f(z) = \dfrac{\phi_1'(2+i)}{1!} = -2(z-2+i)^{-3}\Big|_{z=2+i} = -2(2+i-2+i)^{-3} = -2(2i)^{-3} = -\dfrac{2}{-8i} = -\dfrac{i}{4}$

(c) $\displaystyle\int_C f(z)\,dz = 2\pi i \operatorname{Res} f(z) = 2\pi i\left(-\frac{i}{4}\right) = \frac{\pi}{2}$

$\displaystyle\int_C f(z)\,dz = \int_{-R}^{R} \frac{1}{(x^2-4x+5)^2}\,dx + \int_{C_R} \frac{1}{(z^2-4z+5)^2}\,dz$ $\qquad |z_1| = \sqrt{2^2+1^2} = \sqrt{5}$

Now if z is a point on C_R, then $|z| = R$ and

$|f(z)| \leq M_R$ where $M_R = \dfrac{1}{(R-|z_1|)^2(R-|\bar{z}_1|)^2} = \dfrac{1}{(R-\sqrt{5})^4}$

Hence

$\displaystyle\left|\int_{C_R} f(z)\,dz\right| \leq M_R \pi R = \frac{\pi R}{(R-\sqrt{5})^4} \longrightarrow 0$ as $R \to \infty$

$\displaystyle\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2-4x+5} = \frac{\pi}{2}$

KEY

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 Complex Analysis II
Second Midterm Examination

04.05.2006

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION : 100 minutes
TOTAL NUMBER OF QUESTIONS : 5

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
6		20
7 (Bonus)		15
Total		100

IMPORTANT :

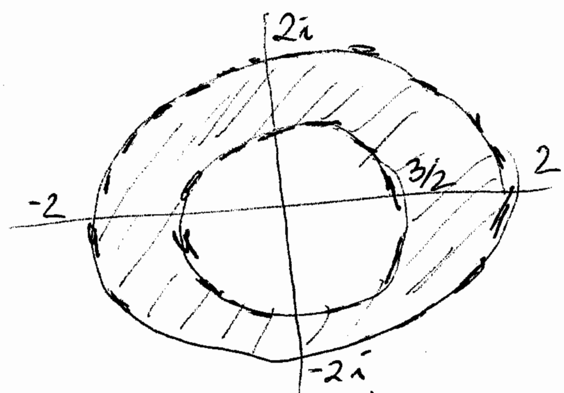
1) Write your name and department.
2) Check that there are 5 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.
4) It is forbidden to use calculaters
5) Choose five problems from the first six problems.

KEY

Solve 5 Problems from 1 to 6

1. Show that the function $P(z) = z^5 + 15z + 1$ has precisely four zeros inside the annulus $3/2 < |z| < 2$.

Apply Rouché's with $f(z) = z^5$ on outer circle, and use the same tactic with $f(z) = 15z+1$ on inner circle.



Can you sharpen this fact? Yes: Look for circles with larger (and smaller

2. Find the linear fractional transformation (LFT) which maps the points -1, ∞, i respectively to i, 1, $1+i$.

Let's instead find g mapping $-1, \infty, i$, respectively to $0, 1-i, 1$; then we'll just define $f(z) = g(z) + i$.

- To achieve $g(-1) = 0$ we look for g of the form

$\frac{z+1}{cz+d}$.

The equation $g(\infty) = 1-i$ means that $\frac{1}{c} = 1-i$, whence

$$c = \frac{1}{1-i} = \frac{1+i}{2}.$$

Thus $g(z) = \dfrac{z+1}{\frac{1+i}{2}z + d}$.

- The last equation $1 = g(i)$ thus becomes $1 = \dfrac{i+1}{i\,\frac{i+1}{2} + d}$

and thus $d + i\,\frac{i+1}{2} = i+1$, so $d = \frac{3+i}{2}$.

Therefore a solution is

$$\boxed{w = \frac{z+i+2}{z-i+2}}$$

$$f(z) = g(z) + i$$
$$= i + \frac{z+1}{\frac{i+1}{2}z + \frac{3+i}{2}} = i + \frac{2z+2}{(1+i)z + (3+i)}$$
$$= \frac{(1+i)z + (1+3i)}{(1+i)z + (3+i)}$$

3. Show that the function

$$f(z) = e^z - 3z$$

has one zero in the unit disk $|z| < 1$.

Apply Rouché's theorem with

$g(z) = e^z$, ~~$h(z) = e^z$~~ $h(z) = -3z$

On $|z| = 1$, $|h(z)| = 3$ and

$|g(z)| = |e^{\cos t + i\sin t}| = e^{\cos t} \leq e < 3$

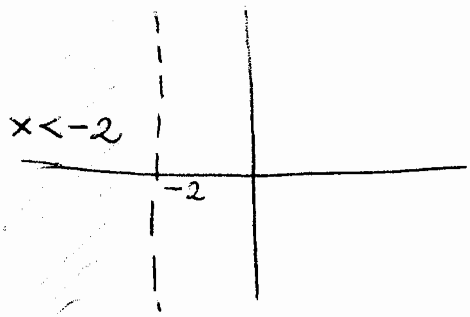
Since $|h| > |g|$ on $|z| = 1$, we conclude

that $f(z) = e^z - 3z$

has one zero in the unit disk.

4. Find the image of the left half plane $x < -2$ under the transformation

$$w = \frac{1}{z}.$$



$$w = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

$$x < -2 \longrightarrow \frac{u}{u^2+v^2} < -\cancel{2}4$$

$$\longrightarrow \quad u < -4(u^2+v^2) \longrightarrow \frac{-\cancel{4}u}{4} > u^2+v^2$$

$$\longrightarrow \quad -4u^2-u-4v^2 > 0 \qquad u^2+\frac{u}{4}+v^2<0$$

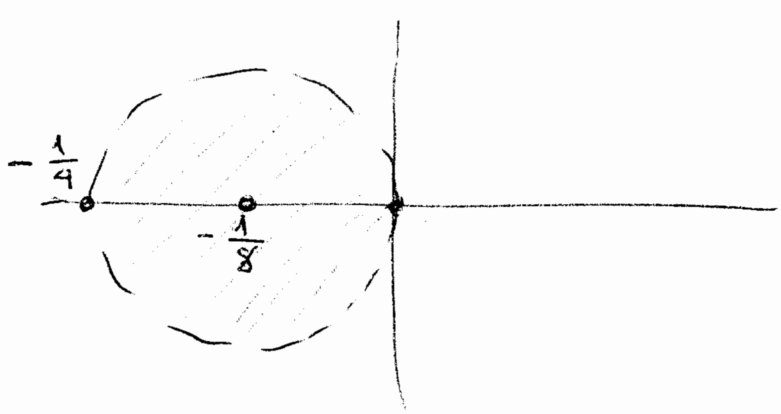
$$\longrightarrow \quad u^2+v^2+\frac{u}{4}<0 \qquad u^2+\frac{u}{4}+\frac{1}{64}+v^2<\frac{1}{64}$$

$$\longrightarrow \quad u^2-\frac{u}{4}+\frac{1}{8^2}-\frac{1}{8^2}+\frac{u}{4}\ \cancel{0} +v^2<0$$

$$\longrightarrow \quad \left(u-\frac{1}{8}\right)^2+v^2$$

$$\left(u+\frac{1}{8}\right)^2+v^2<\frac{1}{8^2}$$

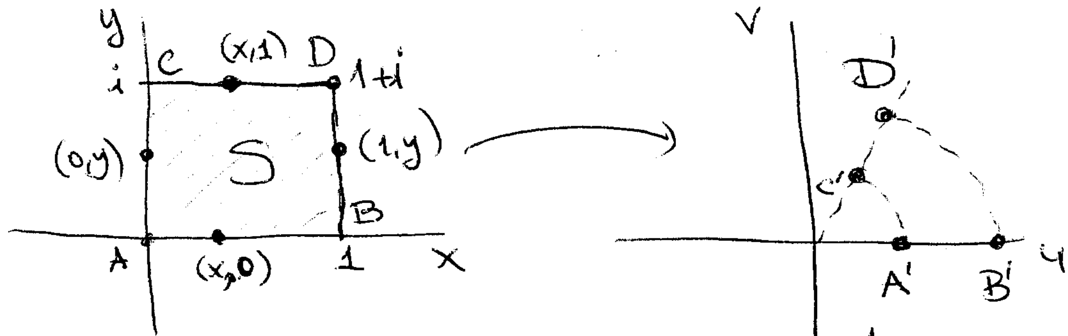
$\Rightarrow$ the image is the interior of the disk below



5. To what region does the function

$$f(z) = e^z$$

map the unit square S whose corners are at $0,\ 1,\ i$, and $1+i$? Sketch the regions S and $f(S)$.



$A' = f(A) = f(0) = e^0 = 1$, $B' = f(B) = f(1) = e^1 = e$

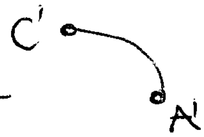
$C' = f(C) = f(i) = e^i$, $D' = f(D) = f(1+i) = e^{1+i} = e\,e^i$

(1) In $[AC]$, take a point $(0,y)$

$f(0,y) = f(iy) = e^{iy} \Rightarrow |r| = 1,\ \theta = y$, when $y = 0 \Rightarrow \theta = 0$

and as $y \nearrow$, we have $\theta \nearrow$

we have such an image like



2) In $[AB]$, take a point $(x,0)$

$f(x,0) = f(x) = e^x$, $\theta = 0$, $r = e \Rightarrow$ as $x \nearrow \Rightarrow e^x \nearrow$

$\Rightarrow f([AB]) = A'B'$ (in figure)

3) In $[BD]$, take a point, $(1,y)$

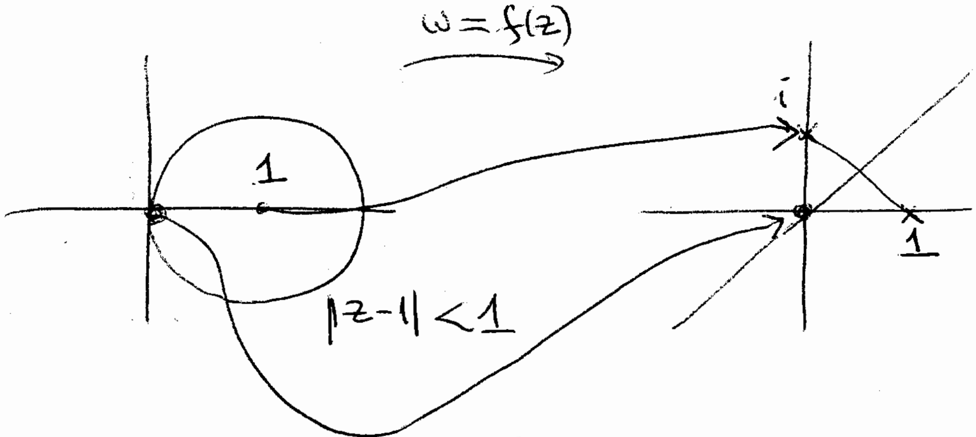
$e^{1+yi} = e\,e^{yi} \Rightarrow r = e,\ \theta = y \Rightarrow$ as $y \nearrow \Rightarrow \theta \nearrow$

$\Rightarrow f([BD]) = [B'D']$.

4) In $[CD]$, take a point $(x,1)$

$f(x,1) = e^{x+i} = e^x e^i \Rightarrow r = e^x$, $\theta = 1 \Rightarrow f([CD])$

6. Find the LFT that maps the open disk $|z-1|<1$ onto the half plane $v>u$.



$$w(0)=0 \qquad z_1=0,\ w_1=0$$
$$w(1)=i \qquad z_2=1,\ w_2=i$$
$$w(\infty)=1 \qquad z_3=\infty,\ w_3=1$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)}=\frac{(z-z_1)(z_1-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-0)(i-1)}{(w-1)(i-0)}=\cancel{\frac{(z-0)(0-\infty)}{(z-\infty)(1-0)}}$$

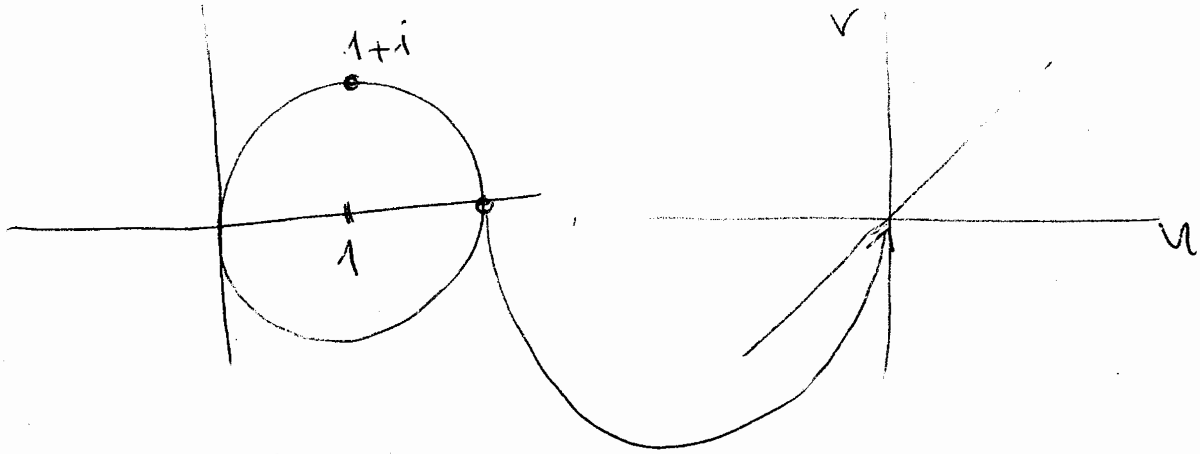
$$\frac{(w-0)(i-1)}{(w-1)\ i}=z \Rightarrow \frac{w}{w-1}=\frac{i}{-1+i}z$$
$$=\frac{i(-1-i)}{1+1}z$$

6

$$\frac{w}{w-1}=\frac{-i+1}{2}z$$

$$2w=(1-i)(w-1)z$$
$$=(1-i)zw-(1-i)$$

$$2w-(1-i)zw=-1+i \Rightarrow w\big(2-(1-i)z\big)=-1+$$
$$\Rightarrow \boxed{w=\frac{-1+i}{(-1+i)z+2}}$$



$$w(1+i) = 1+i$$
$$w(1) = 0$$
$$w(0) = \infty$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-(1+i))(0-\infty)}{(w-\infty)(0-1-i)} = \frac{(z-(1+i))(1-0)}{(z-0)(-i)}$$

$$(w-1-i) = \frac{(z-1-i)}{z} \; \frac{-1-i}{-i}$$
$$= \frac{(z-1-i)}{z} \; \frac{-i+1}{+1}$$

$$\Rightarrow w = \frac{z-1-i}{z} \cdot (1-i) + 1 + i$$

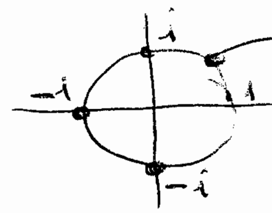
$$w = \frac{(z-1-i)(1-i) + (1+i)z}{z}$$

7. (Bonus) Describe and sketch the images of the circles in $|z| = 1$ and $|z| = 2$ under the mapping

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

We are mapping from the z-space to the w-space, where

$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) = u + iv.$



$z = 1 \cdot e^{i\theta} = e^{i\theta}$

For the circle $|z| = 1$, we write

$z = e^{i\theta}$, $\theta \in [0, 2\pi]$

$\longrightarrow w = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$= \cos\theta.$

So, the mapping takes the circle to the segment

$\{w = u + iv : u \in [-1, 1], v = 0\}$

For the circle $|z| = 2$, we write

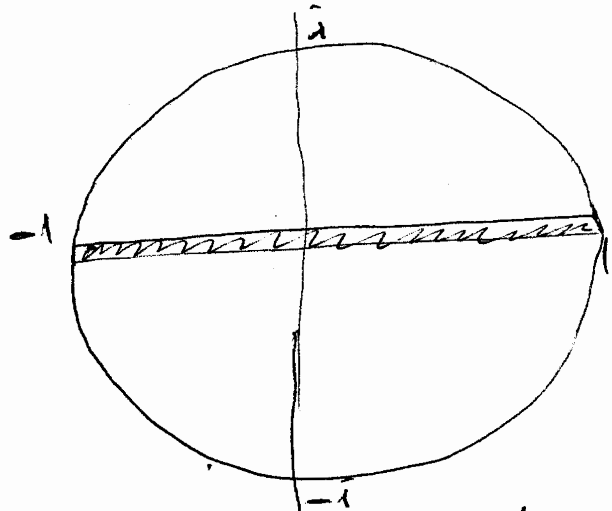
$z = 2e^{i\theta}$, $\theta \in [0, 2\pi]$

$\longrightarrow w = \frac{1}{2}\left(2e^{i\theta} + \frac{1}{2}e^{-i\theta}\right)$

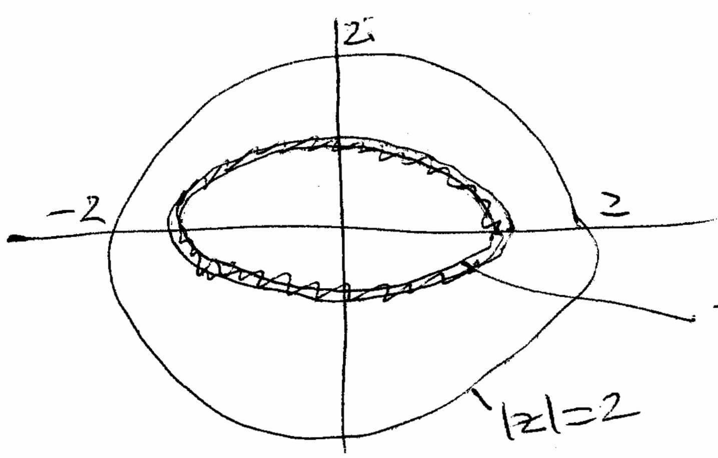
$= \frac{5}{4}\cos\theta + i\,\frac{3}{4}\sin\theta$

Thus, the image is the ellipse

$\left\{w = u + iv : \left(\frac{u}{5/4}\right)^2 + \left(\frac{v}{3/4}\right)^2 = 1\right\}$



The circle $|z| = 1$ and its image $\{w = u + iv : u \in [-1, 1], v = 0\}$



its image $\left\{w = u + iv : \left(\frac{u}{5/4}\right)^2 + \left(\frac{v}{3/4}\right)^2 = 1\right\}$,

$|z| = 2$

KEY

ÇANKAYA UNIVERSITY
MATHEMATICS AND COMPUTER SCIENCE DEPARTMENT

Math 352 Complex Analysis II
Final Examination

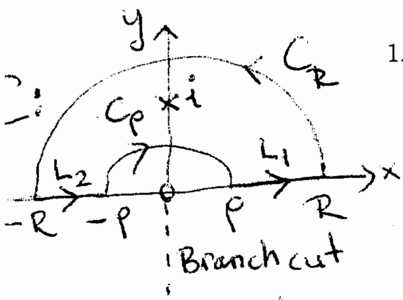
31.05.2006

NAME, SURNAME :
NUMBER :
DEPARTMENT :
SECTION :
SIGNATURE :
INSTRUCTOR'S NAME :
DURATION : 110 minutes
TOTAL NUMBER OF QUESTIONS : 7

Questions	Grade	Out of
1		20
2		20
3		20
4		20
5		20
6		20
7		20
Total		100

IMPORTANT :
1) Write your name and department.
2) Check that there are 7 questions.
3)Show all your work. Correct answers without the intermediate steps may not get credit.
4) It is forbidden to use calculaters
5) Solve five problems.

Solve 5 Problems



1. (20 pts.) Use the function

$$f(z) = \frac{z^{1/3}\log z}{z^2+1} = \frac{e^{1/3\log z}\log z}{z^2+1} \qquad \left(|z|>0,\quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

to find the value of the improper integral

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx$$

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i \operatorname*{Res}_{z=i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Since

$$f(z) = \frac{\phi(z)}{z-i} \quad \text{where} \quad \phi(z) = \frac{e^{(1/3)\log z}\log z}{z+i},$$

the point $z=i$ is a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{\pi}{4}e^{i\pi/6}.$$

The parametric representations

L_1: $z = re^{i0} = r$ $(\rho \le r \le R)$ and $-L_2$: $z = re^{i\pi} = -r$ $(\rho \le r \le R)$

can be used to write

$$\int_{L_1} f(z)dz = \int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr \quad \text{and} \quad \int_{L_2} f(z)dz = e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}dr$$

Thus

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + e^{i\pi/3}\int_\rho^R \frac{\sqrt[3]{r}\ln r + i\pi\sqrt[3]{r}}{r^2+1}dr = \frac{\pi^2}{2}ie^{i\pi/6} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz$$

By equating real parts on each side of this equation, we have

$$\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr + \cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}dr - \pi\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}dr = -\frac{\pi^2}{2}\sin(\pi/6)$$

$$-\operatorname{Re}\int_{C_\rho} f(z)dz - \operatorname{Re}\int_{C_R} f(z)dz;$$

and equating imaginary parts yields

$$\sin(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + \pi\cos(\pi/3)\int_\rho^R \frac{\sqrt[3]{r}}{r^2+1}\,dr = \frac{\pi^2}{2}\cos(\pi/6)$$
$$- \operatorname{Im}\int_{C_\rho} f(z)\,dz - \operatorname{Im}\int_{C_R} f(z)\,dz$$

Now $\sin(\pi/3) = \frac{\sqrt{3}}{2}$, $\cos(\pi/3) = \frac{1}{2}$, $\sin(\pi/6) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

and it is routine to show that

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)\,dz = 0 \quad \text{and} \quad \lim_{R\to\infty}\int_{C_R} f(z)\,dz = 0. \quad (*)$$

Thus

$$\frac{3}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr - \frac{\pi\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}\,dr = -\frac{\pi^2}{4},$$

$$\frac{\sqrt{3}}{2}\int_0^\infty \frac{\sqrt[3]{r}\ln r}{r^2+1}\,dr + \frac{\pi}{2}\int_0^\infty \frac{\sqrt[3]{r}}{r^2+1}\,dr = \frac{\pi^2\sqrt{3}}{4}.$$

That is,

$$\frac{3}{2}I_1 - \frac{\pi\sqrt{3}}{2}I_2 = -\frac{\pi^2}{4}$$

$$\frac{\sqrt{3}}{2}I_1 + \frac{\pi}{2}I_2 = \frac{\pi^2\sqrt{3}}{4}.$$

Solving these simultaneous equations for I_1 and I_2, we arrive at the desired integration formula

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}\,dx = \frac{\pi^2}{6}.$$

The first of the above limits in $(*)$ is shown by writing

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^{1/3}\ln\rho\,\pi\rho}{(1-\rho^2)} \longrightarrow 0 \text{ as } \rho\to 0, \quad \text{since } \frac{1}{3}+1>0.$$

As for the second limit,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{1/3}\ln R\,\pi R}{(R^2-1)} = \frac{\pi R^{4/3}}{R^2-1}\ln R \longrightarrow 0 \text{ as } R\to\infty.$$

2. (20 pts.) Use residues to evaluate the definite integral

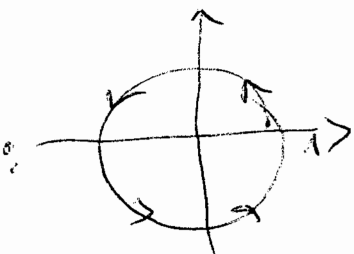
$$\int_0^{\pi/2} \sin^4\theta \, d\theta$$

We note that by symmetry

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} \sin^4\theta \, d\theta$$

Now we can express the integral on the right hand side using the formula

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz} \quad C:$$



where $C \equiv \{z = re^{i\theta} : r=1,\ \theta = 0 \text{ to } \theta = 2\pi\}$. Thus,

$$\frac{1}{4}\int_0^{2\pi} \sin^4\theta \, d\theta = \frac{-i}{4}\int_C \frac{(z-z^{-1})^4}{16z}\, dz$$

$$= \frac{-i}{4} 2\pi i \operatorname*{Res}_{z=0} \frac{(z-z^{-1})^4}{16z}$$

$$= \frac{\pi}{2} \operatorname{Res} \frac{(z-z^{-1})^{-1}}{16z}$$

since $z=0$ is a pole which is enclosed positively by C.

Since

$$\frac{\left(z-\frac{1}{z}\right)^4}{16z} = \frac{\frac{1}{z^4} - \frac{4}{z^2} + 6 - 4z^2 + z^4}{16z}$$

the residue is $\frac{6}{16} = \frac{3}{8}$. Therefore

$$\int_0^{\pi/2} \sin^4\theta \, d\theta = \frac{\pi}{2}\cdot\frac{3}{8} = \boxed{\frac{3\pi}{16}}$$

3. (20 pts.) Given the transformation

$$f(z) = z^2 + 2z$$

(a) Find the points at which $f(z)$ is conformal

(b) Find the scale factor of $f(z)$ when $z = i$.

(c) Explain neatly why $f(z)$ is **not** conformal at $z = -1$.

(d) What angle of rotation is produced by $f(z)$ when $z = i$?

(a) $f'(z) = 2z+2$, so f is conformal for every $z \neq -1$.

(b) When $z = i$, $f'(i) = 2i+2 = \sqrt{8}e^{i\pi/4}$. Thus the scale factor is $|f'(i)| = |\sqrt{8}e^{i\pi/4}| = \sqrt{8}$.

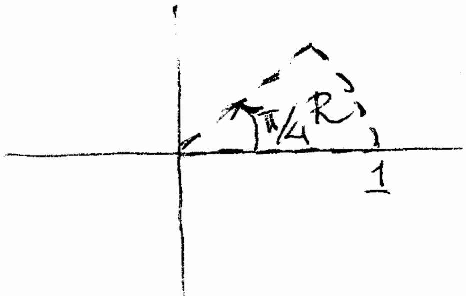
(c) We have $f(z-1) = z^2-1$, so that $f(-1+ae^{it}) = a^2e^{2it}-1$. This shows explicitly how the map is not conformal: angles are doubled

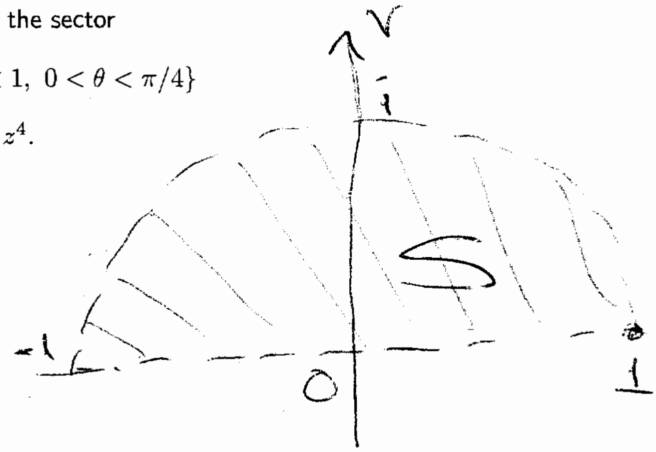
(d) The angle of rotation is $\operatorname{Arg}(\sqrt{8}e^{i\pi/4}) = \frac{\pi}{4}$.

4. (20 pts) Describe the region on which the sector

$$R = \{(r, \theta) : \ r < 1, \ 0 < \theta < \pi/4\}$$

is mapped by the transformation $w = z^4$.





$$R = \{ r e^{i\theta} : r < 1, \ 0 < \theta < \pi/4 \}$$

$$w = z^4 = (r e^{i\theta})^4 = r^4 e^{i4\theta}$$

$$0 < r < 1 \Rightarrow 0 < r^4 < 1$$

$$0 < \theta < \frac{\pi}{4} \Rightarrow 0 < 4\theta < \pi .$$

$$\Rightarrow S = \{ (r, \theta) : 0 < r < 1, \ 0 < \theta < \pi \}.$$

5. (20 pts.) Consider $f(z) = z^{1/3}$ and the inverse $g(w) = w^3$.

(a) Describe a region of the w-plane that $g(w) = w^3$ maps one-to-one onto the whole z-plane.

(b) Identify the branch points of $f(z)$ and introduce a branch cut to make $f(z)$ single-valued.

$$z^{1/3} = |z|^{1/3} e^{i\left(\frac{\operatorname{Arg} z}{3} + \frac{2\pi k}{3}\right)}, \quad k = 0, 1, 2$$

So, $f(z) = z^{1/3}$ has <u>three</u> values for $z \neq 0$.

(a) Let $w = re^{i\theta}$. Then,

$$z = g(w) = w^3 = r^3 e^{i3\theta}.$$

To cover the whole z-plane, we must have

$0 \leq r^3 < \infty$

$\rightarrow 0 \leq r < \infty$

and

$\alpha \leq 3\theta < \alpha + 2\pi$

$\rightarrow \frac{\alpha}{3} \leq \theta < \frac{\alpha}{3} + \frac{2\pi}{3}$

Therefore, any wedge in the w-plane of the form

$$\left\{ w = re^{i\theta} : 0 \leq r < \infty, \ \theta_0 \leq \theta < \theta_0 + \frac{2\pi}{3} \right\}$$

maps one-to-one to the whole z-plane.

(b) We write $f(z)$ in polar form:

$$f(z) = z^{1/3} = |z|^{1/3} e^{i\frac{\arg z}{3}}$$

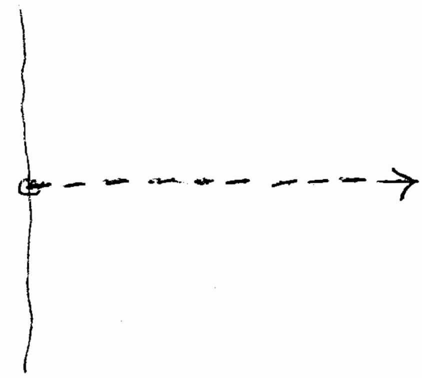
If we start at the point $z = z_0 = r_0 e^{i\theta_0}$ and travel counterclockwise in a circuit which encloses $z = 0$, we end up at

$$z = z_0 e^{i(\theta_0 + 2\pi)} = z_0$$

But the image of this circuit under $w=f(z)$ goes from $w=w_0=r_0^{1/3}e^{i\theta_0/3}$ to $w=r_0^{1/3}e^{i\frac{\theta_0+2\pi}{3}} \neq w_0$,

Thus $z=0$ is a branch point.

If we travel in a circuit which does not enclose $z=0$, $\arg(z)$ is the same at the end as it is at the beginning. Thus the value of w remains unchanged, implying that there are no other finite branch points.



The cut chosen to make $f(z)=z^{1/3}$ single-valued.

In summary,

$z=0$, z_∞ are the branch points of $f(z)=z^{1/3}$

Now, if we set $\theta_0 < \arg z < \theta_0+2\pi$ for some particula θ_0, then we cannot make a circuit about either branch point, and the problem of function discontinuit is resolved. For example, choosing $\theta_0=0$ (or 2π, 4π, etc.) corresponds to a branch cut along the positive real axis, as shown in figure.

6. (20 pts.) Consider the multiple-valued function

$$f(z) = \left(\frac{z+1}{z-6}\right)^{2/3}$$

(a) Find all the branch points in the extended complex plane.

(b) Define a branch that is continuous at $z = \pm 7$ with $f(7) = 4$ and find $f(-7)$.

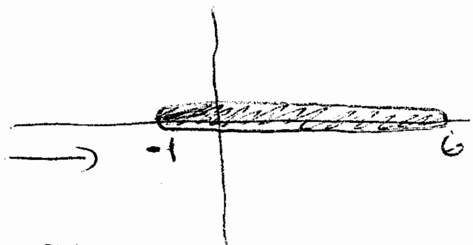
(a) Let $\underset{z_1}{\underset{\parallel}{z+1}} = r_1 e^{i\theta_1}$ and $z_2 = z-6 = r_2 e^{i\theta}$. Then

$$f(z) = \left(\frac{r_1}{r_2}\right)^{3/2} e^{i\frac{2}{3}(\theta_1-\theta_2)}$$

Adding 2π to either θ_1 or θ_2 alone changes the value of $f(z)$ by the factor $e^{\pm i\frac{4\pi}{3}} \neq 1$. Thus $z=-1, 6$ are the finite branch points. Adding 2π to both θ_1 and θ_2 (travelling about both finite branch points, which is equivalent to travelling about z_∞) leaves $f(z)$ unchanged. So z_∞ is not a branch point. Therefore, $\boxed{z=-1, 6}$ are the branch points.

(b) Let

$\boxed{0<\theta_i<2\pi, \quad i=1,2}$ $\xrightarrow{\text{its}}$ branch cut is $\longrightarrow$



This branch is continuous at $z=\pm 7$.

At $z=7$, we have $r_1 = 8$, $r_2 = 1$, $\theta_1 = \theta_2 = 0$ (or 2π)

$\longrightarrow f(7) = 8^{2/3} e^{i0} = 4$ as desired.

At $z=-7$, $r_1 = 6$, $r_2 = 13$, $\theta_1 = \theta_2 = \pi$.

Thus,

$$f(-7) = \left(\frac{6}{13}\right)^{\frac{2}{3}} e^{i0} = \boxed{\left(\frac{6}{13}\right)^{\frac{2}{3}}}$$

7. (20 pts.) Find a LFT that maps the circle $|z| < 1$ onto the circle $|w| < 1$ so that

(a) $w\left(\frac{1}{2}\right) = 0$ and $\arg w'\left(\frac{1}{2}\right) = 0$

(b) $w\left(\frac{i}{2}\right) = 0$ and $\arg w'\left(\frac{i}{2}\right) = \frac{\pi}{2}$.

10 (a) To map $|z|<1$ onto $|w|<1$, $w(1/2)=0$, $\arg w'(\frac{1}{2}) = 0$

Here $\left(\alpha = 0,\ a = \frac{1}{2}\right)$

$$w = e^{i0}\frac{z - 1/2}{1 - \frac{1}{2}z} = \frac{2z-1}{2-z}$$

10 (b) To map $|z|<1$ onto $|w|<1$, $w(\frac{i}{2}) = 0$, $\arg w'(\frac{i}{2}) = \frac{\pi}{2}$

$$w = e^{i\pi/2}\frac{z - i/2}{1 + \frac{i}{2}z} = i\,\frac{2z - i}{2 + iz} = \boxed{\frac{2iz+1}{iz+2}}$$

(Here $\left(\alpha = \frac{\pi}{2},\ a = \frac{i}{2}\right)$)

We have used $w = e^{i\alpha}\dfrac{z-a}{1-\bar{a}z}$