ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
Final Exam
SOLUTIONS
August 8, 2008
9:00-11:00

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 6 questions.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
14	16	16	24	20	15	105

1.

a) Express $2e^{i\pi/4}$ in the standard form $a+ib$.

b) Express $\left(\frac{1-i}{\sqrt{3}+i}\right)^8$ in polar form $re^{i\theta}$.

Solution:

a)

$$2e^{i\pi/4} = 2\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 2\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = \sqrt{2} + i\sqrt{2}.$$

b)

$$\left(\frac{1-i}{\sqrt{3}+i}\right)^8 = \left(\frac{\sqrt{2}e^{-i\pi/4}}{2e^{i\pi/6}}\right)^8 = \left(\frac{1}{\sqrt{2}}\right)^8 \left(e^{-i5\pi/12}\right)^8 = \frac{1}{16}e^{-i10\pi/3} = \frac{1}{16}e^{i2\pi/3}.$$

2.
a) For what values of x, y is the function $f(x+iy) = xy + ix$ is differentiable? analytic?
b) Find a function analytic in the entire plane whose real part is $u(x, y) = x^3y - xy^3$.
Solution:
a)

$$\begin{aligned} u_x &= y, \quad u_y = x \\ v_x &= 1, \quad v_y = 0 \end{aligned}$$

Thus, by Cauchy-Riemann equations, if f is differentiable at $x + iy$, then $x = -1, y = 0$. Since all partial derivatives are continuous, f is indeed *differentiable* at $x = -1, y = 0$.
Since f is not differentiable in a neighborhood of this point, f is *nowhere* analytic.

b) Find harmonic conjugate v of u:
Since $v_y = u_x = 3x^2y - y^3$,

$$v = \int \left(3x^2y - y^3\right)\, dy = (3/2)\, x^2y^2 - y^4/4 + h(x),$$

where $h(x)$ can be determined from the equations:

$$v_x = 3xy^2 + h'(x), \quad v_x = -u_y = -x^3 + 3xy^2$$

thus, $h'(x) = -x^3$ and so $h(x) = -x^4/4 + C$, where C is a constant.
It follows that

$$v = (3/2)\, x^2y^2 - y^4/4 - x^4/4 + C,$$

is a harmonic conjugate for u and that $f(x, y) = u + iv = \left(x^3y - xy^3\right) + i\left((3/2)\, x^2y^2 - y^4/4 - x^4/4 + C\right)$ is an analytic function whose real part is $u(x, y) = x^3y - xy^3$.

3.

a) Let C be the unit circle traversed clockwise. Find the value of $\int_C z \sin z^2 \, dz$ without explicitly calculating the integral.

b) Let C be the circle of radius 1 centered at $2+i$ traversed counterclockwise. Find the value of $\int_C \frac{1}{z} \, dz$ without explicitly calculating the integral.

Solution:

a) We know that $f(z) = z \sin z^2$ is everywhere analytic so in particular, inside and on C, therefore by Cauchy-Goursat theorem, $\int_C z \sin z^2 = 0$.

b) The function $f(z) = \frac{1}{z}$ has one isolated singular point namely, $z = 0$, and it is analytic everywhere else, but $z = 0$ is outside the contour C, therefore by Cauchy-Goursat theorem, $\int_C \frac{1}{z} \, dz = 0$.

---

4. Evaluate the following integrals:

(a) $\displaystyle\int_{|z-1|=1} \frac{z}{z^2-1}\, dz$, (b) $\displaystyle\int_{|z|=2} \frac{ze^z}{(z-1)^3}\, dz$, (c) $\displaystyle\int_{|z|=1} \frac{z \sin z}{(z-2)^3}\, dz$

Solution:

a) Let $f(z) = \dfrac{z}{z+1}$. Then $f(z)$ is analytic inside and on C. Therefore, by the Cauchy Integral Formula, we have $\displaystyle\int_{|z-1|=1} \frac{z}{z^2-1}\, dz = \int_{|z-1|=1} \frac{f(z)}{z-1}\, dz = 2\pi i f(1) = 2\pi i \left[\frac{z}{z+1}\right]_{z=1} = 2\pi i \frac{1}{1+1} = \pi i.$

b) Let $g(z) = ze^z$. Then $g(z)$ is analytic inside and on C. Hence, by the Cauchy Integral Formula, we have $\displaystyle\int_{|z|=2} \frac{ze^z}{(z-1)^3}\, dz = \frac{2\pi i}{2!} g''(1) = \pi i \left[2e^z + ze^z\right]_{z=1} = \pi i \left[2e^1 + e^1\right] = 3\pi i e.$

c) $\displaystyle\int_{|z|=1} \frac{z \sin z}{(z-2)^3}\, dz = 0$, by the Cauchy-Goursat theorem since the integrand $\dfrac{z \sin z}{(z-2)^3}$ is analytic at all points in the interior and on C.

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5. Evaluate the following contour integrals

a) $\int_C (z+z^2)\ dz$ where C is the straight line segment from $z=1$ to $z=i$.

b) $\int_C \sqrt{z}\ dz$ where C is the segment of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ from $z=3$ to $z=2i$. (use the principal branch of $\sqrt{z}$).

Solution:

a)

$f(z)=z+z^2$ has antiderivative $F(z)=\frac{1}{2}z^2+\frac{1}{3}z^3$ in $\mathbb{C}$. Therefore,

$$\int_C f(z)\ dz=\left[\frac{1}{2}z^2+\frac{1}{3}z^3\right]_1^i=-\frac{1}{2}-\frac{i}{3}-\left(\frac{1}{2}+\frac{1}{3}\right)=-\frac{4}{3}-\frac{i}{3}.$$

b)

$f(z)=\sqrt{z}$ (principal branch) has antiderivative $F(z)=\frac{2}{3}z^{3/2}$ (principal branch). Therefore,

$$\begin{aligned}\int_C f(z)\ dz &= \left[\frac{2}{3}z^{3/2}\right]_3^{2i}=\frac{2}{3}\left((2i)^{3/2}-3^{3/2}\right)\\ &= \frac{2}{3}\left(2^{3/2}e^{i3\pi/4}-3^{3/2}\right)\\ &= -\frac{4}{3}-2\sqrt{3}+\frac{4}{3}i\end{aligned}$$

---

6. Find the Taylor series representation for $f(z)=\dfrac{z^2}{(2+z)^2}$, indicate its domain of convergence.

Solution:

$f(z)=\dfrac{z^2}{(2+z)^2}$;

We start with

$$\frac{1}{2+z}=\frac{1}{2}\frac{1}{1-\left(-\frac{z}{2}\right)}=\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n \text{ for } \left|\frac{z}{2}\right|<1 \text{ i.e., for } |z|<2.$$

Next we differentiate:

$$\begin{aligned}
\frac{d}{dz}\left(\frac{1}{2+z}\right) &= \frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n\right), \text{ for } |z|<2 \\
-\frac{1}{(2+z)^2} &= \sum_{n=1}^{\infty}\frac{(-1)^n}{2^{n+1}}nz^{n-1}, \text{ for } |z|<2 \\
-\frac{1}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{n+2}}(n+1)z^n, \text{ for } |z|<2 \\
\frac{1}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(n+1)z^n, \text{ for } |z|<2 \\
\frac{z^2}{(2+z)^2} &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+2}}(n+1)z^{n+2}, \text{ for } |z|<2
\end{aligned}$$

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Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
First Midterm
SOLUTIONS
July 14, 2008
9:00-10:30

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Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	20	105

1. Find all complex numbers z that are complex conjugates of their own squares i.e., $\overline{z} = z^2$.
Solution:
We need to solve the equation $\overline{z} = z^2$; that is, we need to find all pairs (x, y) of real numbers such that $x - iy = x^2 - y^2 + i \cdot 2xy$. Since 1 and i are linearly independent, this means we need to solve the two equations

$$\begin{aligned} x &= x^2 - y^2 \\ -y &= 2xy \end{aligned}$$

for x and y. For $y = 0$, we get $x = 0$ or $x = 1$. For $y \neq 0$, we get $x = -\frac{1}{2}$ and $y = \pm\frac{1}{2}$. So there are four solutions, namely, $0, 1, -\frac{1}{2} + i\frac{1}{2}, \frac{1}{2} - i\frac{1}{2}$.

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2. Find all of the roots of $(-8i)^{1/3}$ in the form $a + ib$ and point out which is the principal root.

Solution:

Since $-8i = 8\exp\left[i\left(-\frac{\pi}{2} + 2k\pi\right)\right]$ $(k = 0, 1, 2)$, the three cube roots of the number $z_0 = -8i$ are

$$(-8i)^{1/3} = 2\exp\left[i\left(-\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right] \qquad (k = 0, 1, 2)\,.$$

the principal one being

$$c_0 = 2\exp\left(i\frac{-\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} - i\frac{1}{2} = \sqrt{3} - i\right).$$

The others are

$$c_1 = 2\exp\left(i\frac{\pi}{2}\right) = 2i$$

and

$$c_2 = 2\exp\left(i\frac{7\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\left(\sqrt{3} + 1\right).$$

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3. Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \dfrac{1}{z}$;

(b) $f(z) = x^2 + iy^2$.

Solution:

(a)

$f(z) = \dfrac{1}{z} = \dfrac{1}{z} \cdot \dfrac{\overline{z}}{\overline{z}} = \dfrac{\overline{z}}{|z|^2} = \dfrac{x}{x^2+y^2} + i\dfrac{-y}{x^2+y^2}$. So

$$u = \frac{x}{x^2+y^2} \quad \text{and} \quad v = \frac{-y}{x^2+y^2}.$$

Since

$$u_x = \frac{y^2 - x^2}{(x^2+y^2)^2} = v_y \qquad \text{and} \qquad u_y = \frac{-2xy}{(x^2+y^2)^2} = -v_x \qquad x^2+y^2 \neq 0,$$

$f'(z)$ exists when $z \neq 0$. Moreover, when $z \neq 0$,

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{y^2-x^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} = -\frac{x^2 - i2xy - y^2}{(x^2+y^2)^2} \\ &= -\frac{(x-iy)^2}{(x^2+y^2)^2} = -\frac{(\overline{z})^2}{(z\overline{z})^2} = -\frac{(\overline{z})^2}{z^2(\overline{z})^2} = -\frac{1}{z^2}. \end{aligned}$$

(b)

$f(z) = x^2 + iy^2$. Hence $u = x^2$ and $v = y^2$. Now

$$u_x = v_y \Longrightarrow 2x = 2y \Longrightarrow y = x \quad \text{and} \quad u_y = -v_x \Longrightarrow 0 = 0.$$

So $f'(z)$ exists only when $y = x$, and we find that

$$f'(x+ix) = u_x(x,x) + iv_x(x,x) = 2x + i0 = 2x.$$

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4. Determine if the following functions are analytic
(a) $f(z) = 3x + y + i(3y - x)$
(b) $f(z) = 2xy + i(x^2 - y^2)$.
Solution:
(a)
$f(z) = \underbrace{3x + y} + i\underbrace{(3y - x)} = u + iv$ where $u = 3x + y$ and $v = 3y - x$ is entire since

$$u_x = 3 = v_y \quad \text{and} \quad u_y = 1 = -v_x.$$

(b) $f(z) = \underbrace{2xy} + i\underbrace{(x^2 - y^2)} = u + iv$ where $u = 2xy$ and $v = x^2 - y^2$ is nowhere analytic since

$$u_x = v_y \Longrightarrow 2y = -2y \Longrightarrow y = 0 \quad \text{and} \quad u_y = -v_x \Longrightarrow 2x = -2x \Longrightarrow x = 0,$$

which means that the Cauchy-Riemann equations hold only at the point $z = (0, 0) = 0$.

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5. Show that $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$.

Solution:

It is straightforward to show that $u_{xx} + u_{yy} = 0$. To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \Longrightarrow v_y = 2 - 3x^2 + 3y^2 \Longrightarrow v(x,y) = 2y - 3x^2y + y^3 + \phi(x).$$

Then

$$u_y = -v_x \Longrightarrow 6xy = 6xy - \phi'(x) \Longrightarrow \phi'(x) = 0 \Longrightarrow \phi(x) = c.$$

Consquently,

$$v(x,y) = 2y - 3x^2y + y^3 + c.$$

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6. Find all values of z such that $e^z = 1 + \sqrt{3}i$.

Solution:

Write $e^z = 1 + \sqrt{3}i$ as $e^x e^{iy} = 2e^{i(\pi/3)}$, from which we see that

$$e^x = 2 \quad \text{and} \quad y = \frac{\pi}{3} + 2n\pi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

That is,

$$x = \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{3}\right)\pi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

Consequently

$$z = \ln 2 + \left(2n + \frac{1}{3}\right)\pi i \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I
Second Midterm
SOLUTIONS
August 4, 2008
9:00-10:30

Surname : ______________________
Name : ______________________
ID # : ______________________
Department : ______________________
Section : ______________________
Instructor : ______________________
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Q1	Q2	Q3	Q4	Q5	Q6	TOTAL
11	18	20	20	16	20	105

1. Find all the zeros of the function $f(z) = 2 + \cos z$. (Hint: if they exist, they must be nonreal.)

Solution:

Following the hint, write $z = x + iy$ with real and imaginary parts $x, y \in \mathbb{R}$. But then

$$\cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y,$$

since $\cos iy = \cosh y$ and $\sin iy = i \sinh y$. To solve $2 + \cos z = 0$ is thus equivalent to finding $z = x + iy$ such that $\cos y \cosh y = -2$ and $\sin x \sinh y = 0$.

Now $\sin x \sinh y = 0$ if and only if either $\sinh y = 0$ or $\sin x = 0$. The first case is excluded because it requires $y = 0$, so $\cosh y = 1$, so $\cos x = -2$ which cannot happen.

The second case is equivalent to $x = k\pi$ for $k \in \mathbb{Z}$. Now $\cosh y = \frac{1}{2}\left(e^y + e^{-y}\right) \geq 1$ for all real y with equality if and only if $y = 0$; otherwise, $\cosh y = C$ has two distinct real roots for every $C > 1$. We conclude that

$$-2 = \cos x \cosh y = \cos k\pi \cosh y = (-1)^k \cosh y$$

has a solution if and only if $x = k\pi$ for some odd integer k and y is one of the two real roots of $\cosh y = 2$.

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2. Find all of values of $\tan^{-1}(1+i)$.
Solution:

$$\begin{aligned}
\tan^{-1}(1+i) &= \frac{i}{2}\log\left(\frac{i+1+i}{i-1-i}\right) \\
&= \frac{i}{2}\log(-1-2i) \\
&= \frac{i}{2}\left(\ln\sqrt{5}+i\arg(-1-2i)\right) \\
&= -\frac{1}{2}\arg(-1-2i)+i\frac{\ln\sqrt{5}}{2} \\
&= -\frac{1}{2}\arg(-1-2i)+i\ln 5
\end{aligned}$$

---

3. Evaluate the line integral $\int_C |z|^2 \; dz$ where C is the line segment from the point 0 to the point $1+i$.

Solution:

Since $f(z) := |z|^2 = x^2 + y^2$, for $z(t) = t + it, (0 \leq t \leq 1)$ is the parametrization of C then we have $z'(t) = (1+i)\; dt$, $f(z(t)) = t^2 + t^2 = 2t^2$. Therefore

$$\begin{aligned} \int_C |z|^2 \; dz &= \int_0^1 2t^2 (1+i)\; dt \\ &= 2(1+i) \int_0^1 t^2 \; dt \\ &= 2(1+i)\left[\frac{1}{3}t^3\right]_0^1 \\ &= \frac{2}{3}(1+i). \end{aligned}$$

---

4. By finding an antiderivative, evaluate the integral

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz.$$

Solution:

$$\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz = \left[2\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} = 2\sin\left(\frac{\pi+2i}{2}\right) - 2\sin\left(\frac{0}{2}\right)$$

$$= 2\frac{e^{i\left(\frac{\pi}{2}+i\right)} - e^{-i\left(\frac{\pi}{2}+i\right)}}{2i} = -i\left(e^{i\pi/2}e^{-1} - e^{-i\pi/2}e\right)$$

$$= -i\left(\frac{i}{e} + ie\right) = \frac{1}{e} + e = e + \frac{1}{e}.$$

---

5. Use Cauchy's Integral Formula to evaluate $\int_{|z-1|=1} \frac{\cos(2\pi z)}{z^2-1}\, dz$ where the integration path is oriented in the standard counterclockwise direction.

Solution:

Let $f(z) = \frac{\cos(2\pi z)}{z+1}$. Then $f(z)$ is analytic at all points both interior to and on the contour C. Therefore, by the Cauchy Integral Formula, we have

$$\begin{aligned}
\int_{|z-1|=1} \frac{\cos(2\pi z)}{z^2-1}\, dz &= 2\pi i f(1) \\
&= 2\pi i \left[\frac{\cos(2\pi z)}{z+1}\right]_{z=1} \\
&= 2\pi i \frac{\cos(2\pi(1))}{1+1} \\
&= 2\pi i \frac{1}{1+1} = \pi i.
\end{aligned}$$

6. Find the value of the integral $\int_C \frac{z-b}{z-a}\,dz$ where C is the unit circle traversed once counterclockwise. Be sure to consider the cases $|a| < 1$ and $|a| > 1$.

Solution 1:

If $|a| > 1$, then the integrand is analytic on $|z| < |a|$ and Cauchy-Goursat Theorem says that

$$\int_C \frac{z-b}{z-a}\,dz = 0.$$

If $|a| < 1$, then define $f(z) = (z-b)$ which is analytic on $\mathbb{C}$. Then Cauchy Integral Formula says that

$$\begin{aligned}\int_C \frac{z-b}{z-a}\,dz &= \int_C \frac{f(z)}{z-a}\,dz \\ &= 2\pi i f(a) \\ &= 2\pi i (a-b).\end{aligned}$$

Solution 2:

If $|a| < 1$, then we could notice that $z - b = (z-a) + (a-b)$ and therefore

$$\begin{aligned}\int_C \frac{z-b}{z-a}\,dz &= \int_C dz + (a-b)\int_C \frac{1}{z-a}\,dz \\ &= 2\pi i (a-b).\end{aligned}$$

There are other ways to do this as well, but these two methods are the simplest.

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

Practice Problems-1

First midterm

July 14, 2008

09:40

1. Complex Numbers

1.1. **Section 4.** (p. 11)

3. Verify that $\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$
Suggestion Reduce this inequality to $(|x| - |y|)^2 \geq 0$.
Solution:
Let $z = x + iy \Longrightarrow$ the inequality becomes $\sqrt{2}\sqrt{x^2+y^2} \geq |x| + |y|$

$$\begin{aligned} &\Longleftrightarrow \quad 2\left(x^2+y^2\right) \geq (|x|+|y|)^2 = x^2+y^2+2\,|x|\,|y| \\ &\Longleftrightarrow \quad x^2+y^2-2\,|x|\,|y| \geq 0 \\ &\Longleftrightarrow \quad (|x|-|y|)^2 \geq 0. \end{aligned}$$

This last form of the inequality to be verified is obviously true since the left-hand side is a perfect square.

---

4. In each case, sketch the set of points determined by the given condition:
(a) $|z-1+i| = 1$; (b) $|z+i| \leq 3$; (c) $|z-4i| \geq 4$.
Solution:
(a) $|z-1+i| = 1$; it's a circle with center $z_0 = (1, -1)$ and radius $R = 1$.
(b) $|z+i| \leq 3$; it's a disk with center $z_0 = (0, -1)$ and radius $R = 3$.
(c) $|z-4i| \geq 4$; it's the set of points outside the disk of radius $R = 4$ and center $z_0 = 4i$.

(p.13)
7. Use the established properties of moduli to show that when $|z_3| \neq |z_4|$,

$$\left|\frac{z_1+z_2}{z_3+z_4}\right| \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}.$$

Solution:
$\left|\frac{z_1+z_2}{z_3+z_4}\right| = \frac{|z_1+z_2|}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$ by triangle inequality $|z_1+z_2| \leq |z_1|+|z_2|$ and the inequality $|z_3 \pm z_4| \geq ||z_3|-|z_4||$ (p.10).

---

10. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors, show that z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4-4z^2+3}\right| \leq \frac{1}{3}$$

Solution:
Factorizing $z^4 - 4z^2 + 3 = \left(z^2 - 1\right)\left(z^2 - 3\right)$

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| = \frac{1}{|z^4 - 4z^2 + 3|} = \frac{1}{|(z^2 - 1)(z^2 - 3)|} = \frac{1}{|z^2 - 1||z^2 - 3|} \le \frac{1}{\left||z|^2 - 1\right|\left||z|^2 - 3\right|}$$

$$= \frac{1}{(4 - 1)(4 - 3)} = \frac{1}{3}.$$

(p.21)
1. Find the principal argument Arg z when
(a) $z = \dfrac{i}{-2 - 2i}$; (b) $z = \left(\sqrt{3} - i\right)^6$.
Solution:
(a)

$$z = \frac{i}{-2 - 2i} = -\frac{1}{2}\frac{i}{1 + i}\frac{1 - i}{1 - i} = -\frac{1}{2}\frac{i - i^2}{1 - i^2} = -\frac{1}{4}(1 + i) = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}e^{-3\pi i/4}$$

$\Longrightarrow \text{Arg}(z) = -\frac{3\pi}{4}$
(b)

$$z = \left(\sqrt{3} - i\right)^6$$

Observe $\xi = \sqrt{3} - i \Longrightarrow |\xi| = \sqrt{3 + 1} = 2 \Longrightarrow \xi = 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right) = 2e^{-i\pi/6}$

$z = \xi^6 = \left(2e^{-i\pi/6}\right)^6 = 2^6 e^{-i\pi} = 2^6 e^{i\pi} = -64$ (since $-\pi = \pi + 2\pi$ and $e^{2\pi i} = 1$)
$\Longrightarrow \text{Arg}(z) = \pi$.

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1. Derive the following trigonometric identities:
(a) $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, (b) $\sin 3\theta = 3\cos^2\theta - \sin^3\theta$.
Solution:
(a)

$$z = \frac{i}{-2 - 2i} = -\frac{1}{2}\frac{i}{1 + i}\frac{1 - i}{1 - i} = -\frac{1}{2}\frac{i - i^2}{1 - i^2} = -\frac{1}{4}(1 + i) = \frac{\sqrt{2}}{4}\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) = \frac{\sqrt{2}}{4}e^{-3\pi i/4}$$

$\Longrightarrow \text{Arg}(z) = -\frac{3\pi}{4}$
(b)

$$z = \left(\sqrt{3} - i\right)^6$$

Observe $\xi = \sqrt{3} - i \Longrightarrow |\xi| = \sqrt{3 + 1} = 2 \Longrightarrow \xi = 2\left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right) = 2e^{-i\pi/6}$

$z = \xi^6 = \left(2e^{-i\pi/6}\right)^6 = 2^6 e^{-i\pi} = 2^6 e^{i\pi} = -64$ (since $-\pi = \pi + 2\pi$ and $e^{2\pi i} = 1$)
$\Longrightarrow \text{Arg}(z) = \pi$.

(p.73)
1. Verify that each of these functions is entire:
(a)$f(z) = 3x + y + i(3y - x)$; (b)$f(z) = \sin x \cosh y + i \cos x \sinh y$;
(c)$f(z) = e^{-y}\sin x - ie^{-y}\cos x$; (d)$f(z) = \left(z^2 - 2\right)e^{-x}e^{-iy}$.
Solution:

(a)$f(z)=\underbrace{3x+y}+i\underbrace{(3y-x)}$ is entire since

$$u_x=3=v_y \text{ and } u_y=1=-v_x$$

(b) $f(z)=\underbrace{\sin x\cosh y}+i\underbrace{\cos x\sinh y}$ is entire since

$$u_x=\cos x\cosh y=v_y \text{ and } u_y=\sin x\sinh y=-v_x.$$

(c) $f(z)=\underbrace{e^{-y}\sin x}+i\underbrace{\left(-e^{-y}\cos x\right)}$ is entire since

$$u_x=e^{-y}\cos x=v_y \text{ and } u_y=-e^{-y}\sin x=-v_x.$$

(d) $f(z)=\left(z^2-2\right)e^{-x}e^{-iy}$ is entire since it is the product of entire functions

$$g(z)=z^2-2 \text{ and } h(z)=e^{-x}e^{-iy}=e^{-x}(\cos y-i\sin y)=\underbrace{e^{-x}\cos y}+i\underbrace{\left(-e^{-x}\sin y\right)}.$$

The function g is entire since it is a polynomial, and h is entire since

$$u_x=-e^{-x}\cos y=v_y \text{ and } u_y=-e^{-x}\sin y=-v_x.$$

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2. Show that each of these functions is nowhere analytic:
(a) $f(z)=xy+iy$
(b) $f(z)=2xy+i\left(x^2-y^2\right)$.
(c) $f(z)=e^ye^{ix}$
Solution:
(a)
$f(z)=xy+iy$ is nowhere analytic since

$$u_x=v_y \Longrightarrow y=1 \text{ and } u_y=-v_x \Longrightarrow x=0,$$

which means that the Cauchy-Riemann equations hold only at the point $z=(0,1)=i$.
(b)
$f(z)=e^ye^{ix}=e^y(\cos x+i\sin x)$ is nowhere analytic since

$$u_x=v_y \Longrightarrow -e^y\sin x=e^y\sin x \Longrightarrow 2e^y\sin x=0 \Longrightarrow \sin x=0$$

and

$$u_y=-v_x \Longrightarrow e^y\cos x=-e^y\cos x \Longrightarrow 2e^y\cos x=0 \Longrightarrow \cos x=0.$$

More precisely, the roots of the equation $\sin x=0$ are $n\pi$ $(n=0,\pm1,\pm2,\cdots)$, and $\cos(n\pi)=(-1)^n\neq 0$. Consequently, the Cauchy-Riemann equations are not satisfied anywhere.

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4. In each case, determine the singular points of the function and state why the function is analytic everywhere except at those points:
(a) $f(z)=\dfrac{2z+1}{z\left(z^2+1\right)}$; (b) $f(z)=\dfrac{z^3+i}{z^2-3z+2}$; (c) $f(z)=\dfrac{z^2+1}{(z+2)\left(z^2+2z+2\right)}$.
Solution: (a)
$f(z)=\dfrac{2z+1}{z\left(z^2+1\right)}$; this function is the quotient of two polynomials (a) $f(z)=\dfrac{P(z)}{Q(z)}$, hence it's analytic in any domain throughout which (a) $Q(z)\neq 0$.
$\longrightarrow$ $z\left(z^2+1\right)=0$ iff (a) $z=0$ or (a) $z=\pm i$ (ant the numerator does not vanish at these points)
$\Longrightarrow$ singular points: $z=0,\pm i$ (They are poles, i.e., $\lim_{z\to 0,\pm i}|f(z)|=+\infty$)
(b)
$f(z)=\dfrac{z^3+i}{z^2-3z+2}$ similarly as above, check check where the denominator vanishes:

$z^2 - 3z + 2 = 0 \iff z = \dfrac{3 \pm \sqrt{9-8}}{2} = \dfrac{3 \pm 1}{2} \Longrightarrow z_1 = 2, z_2 = 1$
and the numerator does not vanish at these points.
$\Longrightarrow$ singular points $z = 1, 2$ (poles)

(c) $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$
$(z+2)(z^2+2z+2) = 0$ iff $z = -2$ or $z^2+2z+2 = 0 \iff z = -1 \pm \sqrt{1-2} = -1 \pm i$
$\Longrightarrow$ singular points (poles) $z = -2, -1 \pm i$

(p.78)
7. Let a function $f(z)$ be analytic in a domain D. Prove that $f(z)$ must be constant throughout D if
(a) $f(z)$ is real-valued for all $z \in D$;
(b) $|f(z)|$ is constant throughout D.
Solution:
(a)
Suppose $f(z) \in \mathbb{R}$ for all $z \in D \Longrightarrow v(x,y) = 0$ on D
$\Longrightarrow u_x(x,y) = v_y(x,y) = 0$
and
$u_y(x,y) = -v_x(x,y) = 0$ on D.
$\Longrightarrow \nabla u(x,y) = 0$ on D.
$u(x,y) =$ constant on $D \Longrightarrow f(z) =$ constant on D.
(b)
Suppose
$|f(z)| = c$ for all $z \in D$
If $c = 0 \Longrightarrow f(z) = 0$ on D, hence it's constant.
If $c \neq 0 \Longrightarrow |f(z)|^2 = c^2 \iff f(z)\overline{f(z)} = c^2 \iff \overline{f(z)} = \dfrac{c^2}{f(z)}$
$\Longrightarrow$ both f and $\overline{f}$ are analytic in D (since $\overline{f} = \dfrac{c^2}{f}$ and $f \neq 0$)
$\Longrightarrow f(z) =$ constant on D.
(otherway to solve it)
$f(z) = c$ and suppose $c \neq 0 \Longrightarrow u^2 + v^2 = c^2$
$\Longrightarrow 2uu_x + 2vv_x = 0$
$2uu_y + 2vv_y = 0$
$\longrightarrow 0 = (uu_x + vv_x)^2 + (uu_y + vv_y)^2 = u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y$
$\longrightarrow (u^2+v^2)(u_x^2+u_y^2) = 0 \Longrightarrow u_x^2 + u_y^2 = 0$
$\Longrightarrow u_x = u_y = 0$ & (by C-R equations) $\Longrightarrow v_x = v_y = 0$
$\Longrightarrow u =$ constant $v =$ constant $\Longrightarrow f(z) =$ constant

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