ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

1$^{\text{st}}$ Midterm
November 8, 2007
12:40-14:30

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1.

(a) Find all the values of $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{\frac{1}{4}}$. (10 points)

(b) Sketch the set of points satisfying

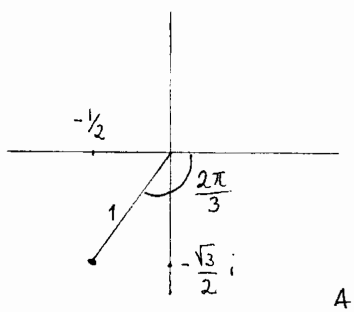
(i) $1 < |2z-6| < 2$, (5 points)

(ii) $\mathrm{Re}(iz+2) > 0$. (5 points)

Answer 1.

(a) we need to solve $z^4 = \left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)$ or equivalently,

$z^4 = (re^{i\theta})^4 = r^4 e^{4i\theta} = e^{-\frac{2\pi i}{3}}$.



Now, $r^4 = 1$ and $4\theta = -\frac{2\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$

$\Rightarrow$ $r = 1$ and $\theta = -\frac{2\pi}{12} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$

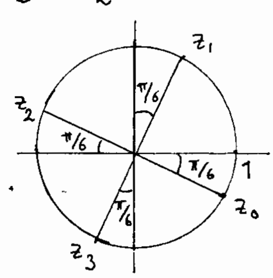
$\Rightarrow$ $r = 1$ and $\theta = -\frac{\pi}{6} + \frac{k\pi}{2}$, $k \in \mathbb{Z}$

Actually, there are only four different values of $z = re^{i\theta}$ which correspond to $k = 0, 1, 2, 3$. Other values of k repeats previous solutions.

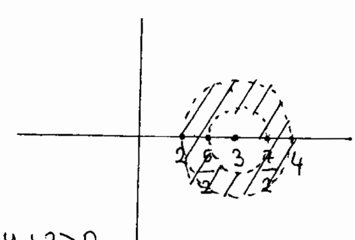
So, $\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)^{\frac{1}{4}}$ has four different values:

$z_0 = e^{-\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} - \frac{i}{2}$, $z_1 = e^{i\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, $z_2 = e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{i}{2}$,

$z_3 = e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

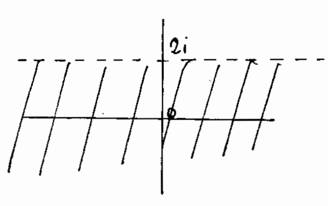


(b) (i) $1 < |2z-6| < 2 \Leftrightarrow \frac{1}{2} < |z-3| < 1$



(ii) $\mathrm{Re}(iz+2) = \mathrm{Re}(i(x+iy)+2) = \mathrm{Re}(ix-y+2) = -y+2>0$

$\Rightarrow 2 > y$



Question 2. Simplify

(a) $\left(\dfrac{1+i}{1-i}\right)^{16}+\left(\dfrac{1-i}{1+i}\right)^{8}$, (5 points)

(b) $\dfrac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}$, (5 points)

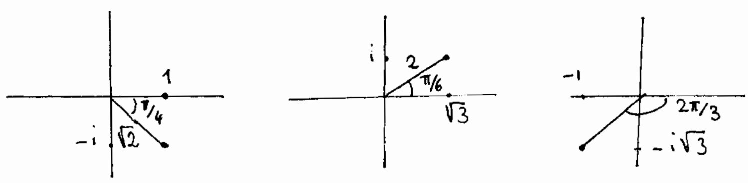
(c) $\dfrac{2}{(1-i)(3-i)(i+2)}$, (5 points)

(d) $\left(\dfrac{2+i}{3-2i}\right)^{2}$. (5 points)

Answer 2. (a) $\dfrac{1+i}{1-i}\,\underset{(1+i)}{} = \dfrac{1+i+i-1}{1+i-i+1} = \dfrac{2i}{2} = i$ and $\dfrac{1-i}{1+i} = \dfrac{1}{\frac{1+i}{1-i}} = \dfrac{1}{i}\,\underset{(-i)}{} = -i$

So, $\left(\dfrac{1+i}{1-i}\right)^{16}+\left(\dfrac{1-i}{1+i}\right)^{8} = (i)^{16} + (-i)^{8} = (-1)^{8} + (-1)^{4} = 1+1 = 2.$

(b) Note that $1-i = \sqrt{2}\,e^{-i\frac{\pi}{4}}$, $\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$ and $-1-i\sqrt{3} = 2e^{-i\frac{2\pi}{3}}$



So, $\dfrac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}} = \dfrac{\left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{10}\left(2e^{i\frac{\pi}{6}}\right)^5}{\left(2e^{-\frac{2\pi}{3}i}\right)^{10}} = \dfrac{2^{10}\,e^{-\frac{5\pi}{2}i}\,e^{\frac{5\pi}{6}i}}{e^{-\frac{20\pi}{3}i}}$

$= e^{\left(-\frac{5\pi}{2}+\frac{5\pi}{6}+\frac{20\pi}{3}\right)i} = e^{5\pi i} = -1$

(c) $\dfrac{2}{(1-i)(3-i)(i+2)} = \dfrac{2}{(3-i-3i-1)(i+2)} = \dfrac{2}{(2-4i)(i+2)} = \dfrac{2}{2i+4+4-8i}$

$= \dfrac{2}{8-6i} = \dfrac{1}{4-3i} = \dfrac{4+3i}{25}$

(d) $\left(\dfrac{2+i}{3-2i}\right)^2 = \left(\dfrac{(2+i)(3+2i)}{13}\right)^2 = \left(\dfrac{6+4i+3i-2}{13}\right)^2$

$= \left(\dfrac{4+7i}{13}\right)^2 = \dfrac{16}{169} - \dfrac{49}{169} + \dfrac{56i}{169} = -\dfrac{33}{169} + \dfrac{56i}{169}.$

Question 3. Find the image of the set $\mathscr{D} = \{z \in \mathbb{C} \ : \ |z-3| = 1\}$ under the mapping

(a) $f(z) = 4iz + 2i$, (10 points)

(b) $f(z) = \dfrac{1}{z}$. (10 points)

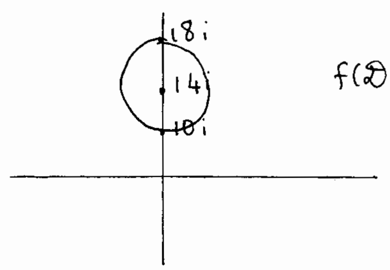
Answer 3.

(a) $w \in f(\mathscr{D}) \Leftrightarrow f^{-1}(w) \in \mathscr{D}$, and $w = 4iz + 2i \Rightarrow$

$\dfrac{w-2i}{4i} = z \Rightarrow f^{-1}(w) = \dfrac{w-2i}{4i}$ So, $w \in f(\mathscr{D}) \Rightarrow$

$$|f^{-1}(w) - 3| = \left| \frac{w-2i}{4i} - 3 \right| = 1 \Rightarrow |w - 2i - 12i| = |4i| = 4 \Rightarrow$$

$|w - 14i| = 4.$



$f(\mathscr{D}) = \{ w \in \mathbb{C} : |w - 14i| = 4 \}$

(b) $w = \dfrac{1}{z} \Rightarrow z = \dfrac{1}{w} \Rightarrow f^{-1}(w) = \dfrac{1}{w}$ and $w \in f(\mathscr{D}) \Rightarrow$

$f^{-1}(w) \in \mathscr{D} \Rightarrow |f^{-1}(w) - 3| = 1 \Rightarrow \left| \dfrac{1}{w} - 3 \right| = 1.$

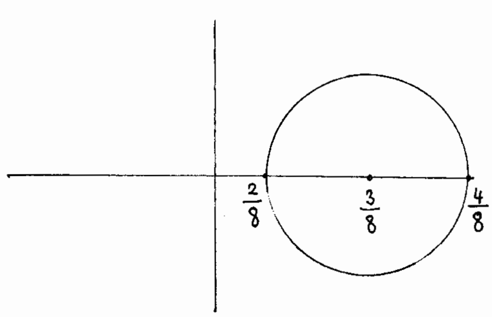
Say $w = u + iv$. So $w \in f(\mathscr{D}) \Rightarrow \left| \dfrac{1}{u+iv} - 3 \right| = 1 \Rightarrow$

$$|1 - 3u - 3iv| = |u + iv| \Rightarrow (1-3u)^2 + (3v^2) = u^2 + v^2 \Rightarrow$$

$$1 - 6u + 9u^2 + 9v^2 = u^2 + v^2 \Rightarrow 8u^2 + 8v^2 - 6u + 1 = 0 \Rightarrow$$

$$u^2 + v^2 - \frac{6}{8}u + \frac{1}{8} = 0 \Rightarrow \left(u - \frac{3}{8}\right)^2 + v^2 = \frac{9}{64} - \frac{1}{8} = \frac{1}{64}$$

$$\Rightarrow f(\mathscr{D}) = \left\{ w \in \mathbb{C} : \left(u - \frac{3}{8}\right)^2 + v^2 = \left(\frac{1}{8}\right)^2 \right\}$$



Question 4.

(a) Let

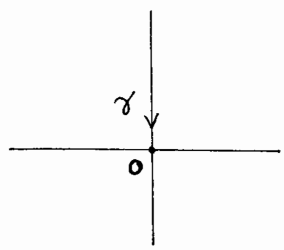
$$f(z) = \begin{cases} \dfrac{z^2}{|z|^2} & \text{if } \; z \neq 0 \\ 1 & \text{if } \; z = 0 \end{cases}$$

Decide whether f is continuous at $z = 0$. (10 points)

(b) Evaluate $\lim\limits_{z \to i} \dfrac{z^2 + iz + 2}{z - i}$ if it exits. (10 points)

Answer 4.

(a) Let $\gamma = \{ z = iy \; : \; 0 < y < r \;\; \text{for some } r > 0 \}$.



Since

$$\lim_{\substack{z \to 0 \\ z \in \gamma}} f(z) = \lim_{y \to 0^+} \frac{(iy)^2}{y^2} = \lim_{y \to 0^+} -\frac{y^2}{y^2} = -1 \neq 1 = f(0),$$

f is not continuous at $z = 0$.

(b) $$\lim_{z \to i} \frac{z^2 + iz + 2}{z - i} \qquad \left(\frac{i^2 - 1 + 2}{i - i} = \frac{0}{0} \right) \text{ indeterminate form.}$$

$$\begin{array}{r|l} z^2 + iz + 2 & z - i \\ \cline{2-2} - \; z^2 - iz & z + 2i \\ \hline 2iz + 2 & \\ - \; 2iz + 2 & \\ \hline 0 & \end{array} \qquad \Rightarrow \qquad z^2 + iz + 2 = (z - i)(z + 2i).$$

And so,

$$\lim_{z \to i} \frac{z^2 + iz + 2}{z - i} = \lim_{z \to i} \frac{(z - i)(z + 2i)}{z - i}$$

$$= \lim_{z \to i} z + 2i = 3i.$$

Question 5. Let $f(z) = z^4$.

(a) Show that f is differentiable everywhere and find $f'(z)$. (5 points)

(b) Write $f(z)$ in the form $u(x,y) + iv(x,y)$. (5 points)

(c) Show that $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$. (5 points)

(d) Show that $f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$. (5 points)

Answer 5.

(a) $$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{(z+h)^4 - z^4}{h} = \lim_{h\to 0} \frac{\cancel{z^4} + 4z^3h + 6z^2h^2 + 4zh^3 + h^4 - \cancel{z^4}}{h}$$

$$= \lim_{h\to 0} (4z^3 + 6z^2h + 4zh^2 + h^3) = 4z^3 \quad \text{for all } z \in \mathbb{C}.$$

(b) $$f(z) = z^4 = (x+iy)^4 = x^4 + 4x^3iy + 6x^2(iy)^2 + 4x(iy)^3 + (iy)^4$$

$$= x^4 + i\,4x^3y - 6x^2y^2 - i\,4xy^3 + y^4$$

$$= x^4 - 6x^2y^2 + y^4 + i(4x^3y - 4xy^3) \Rightarrow$$

$u(x,y) = x^4 - 6x^2y^2 + y^4$ and $v(x,y) = 4x^3y - 4xy^3$

(c) $u_x(x,y) = 4x^3 - 12xy^2$ $\quad$ $v_y(x,y) = 4x^3 - 12xy^2 \Rightarrow u_x(x,y) = v_y(x,y)$

and $u_y(x,y) = -12x^2y + 4y^3$ $\quad$ $v_x(x,y) = 12x^2y - 4y^3 \Rightarrow u_y(x,y) = -v_x(x,y)$

(d) $$u_x(x,y) + iv_x(x,y) = 4x^3 - 12xy^2 + i(12x^2y - 4y^3)$$

$$= 4(x^3 + 3x^2iy - 3xy^2 - iy^3)$$

$$= 4(x+iy)^3$$

$$= 4z^3 = f'(z).$$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

2$^{\text{nd}}$ Midterm

December 17, 2007

08:40-10:30

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

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GOOD LUCK!

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Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1.

(a) Find the most general harmonic function u of the form

$$u(x, y) = ax^3 + bx^2y + cxy^2 + dy^3,$$

where a, b, c, d are real coefficients. (9 points)

(b) What is the harmonic conjugate v of u? (9 points)

(c) Express $f = u + iv$ as a function of z. (2 points)

Answer 1.

(a) $u_x = 3ax^2 + 2bxy + cy^2$ $\qquad$ $u_y = bx^2 + 2cxy + 3dy^2$

$u_{xx} = 6ax + 2by$ $\qquad$ $u_{yy} = 2cx + 6dy$

$u_{xx} + u_{yy} = 0 \Rightarrow \quad x(6a + 2c) + y(2b + 6d) = 0 \Rightarrow c = -3a, \quad b = -3d$

$\Rightarrow$ u has the form

$$u(x,y) = ax^3 - 3dx^2y - 3axy^2 + dy^3, \qquad a, d \in \mathbb{R}.$$

(b) $u_x = v_y \quad \Rightarrow \quad v = \int (3ax^2 - 6dxy - 3ay^2)\, dy$

$= 3ax^2y - 3dxy^2 - ay^3 + \phi(x)$

where ϕ is a function depending only on x.

$v_x = -u_y \quad \Rightarrow \quad 6axy - 3dy^2 + \phi'(x) = 3dx^2 + 6axy - 3dy^2$

and so, $\quad \phi'(x) = 3dx^2$ and hence $\phi(x) = dx^3 + \alpha$, $\alpha \in \mathbb{R}$.

Putting all things together

$$v(x,y) = 3ax^2y - 3dxy^2 - ay^3 + dx^3 + \alpha, \qquad \alpha \in \mathbb{R}.$$

(c) $f(z) = u(x,y) + iv(x,y)$

$= ax^3 - 3dx^2y - 3axy^2 + dy^3 + i\,3ax^2y - 3idxy^2 - iay^3 + idx^3 + i\alpha$

$= a(x^3 - 3xy^2 + 3ix^2y - iy^3) + id(x^3 + 3ix^2y - 3xy^2 - iy^3) + i\alpha$

$= a(x+iy)^3 + id(x+iy)^3 + i\alpha = (a+id)z^3 + i\alpha, \quad \alpha \in \mathbb{R}$

If we denote $a+id$ by A and $i\alpha$ by B, we get

$$f(z) = Az^3 + B, \qquad A \in \mathbb{C},\ B \in i\mathbb{R}.$$

Question 2. In each part, determine all possible values and give the principle value of the following numbers (put in the form $x + iy$).

(a) $i^{\frac{1}{2}}$ (5 points)

(b) $\dfrac{1}{(1+i)^{\frac{1}{2}}}$ (5 points)

(c) $\log i^3$ (5 points)

(d) $i^{\sqrt{3}}$ (5 points)

Answer 2.

(a) $i^{\frac{1}{2}} = e^{\frac{1}{2}\log i} = e^{\frac{1}{2}(\ln|i| + i\arg(i))} = e^{\frac{1}{2}\left(i\left(\frac{\pi}{2}+2n\pi\right)\right)} = e^{i\frac{\pi}{4}} e^{in\pi}, \quad n \in \mathbb{Z}$

$e^{in\pi} = \begin{cases} 1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$. So $i^{\frac{1}{2}} = \{ e^{i\frac{\pi}{4}}, -e^{i\frac{\pi}{4}} \}$

$= \{ \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} \}$

$= \{ \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \}$

The principal one is: $\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$.

(b) $\dfrac{1}{(1+i)^{\frac{1}{2}}} = (1+i)^{-\frac{1}{2}} = e^{-\frac{1}{2}\log(1+i)} = e^{-\frac{1}{2}(\ln|1+i| + i\arg(1+i))}$

$= e^{-\frac{1}{2}\left(\ln\sqrt{2} + i\left(\frac{\pi}{4}+2n\pi\right)\right)}$

$= 2^{-\frac{1}{4}} e^{-i\frac{\pi}{8}} e^{-n\pi i} = (-1)^n 2^{-\frac{1}{4}} e^{-i\frac{\pi}{8}}, \quad n \in \mathbb{Z},$

and $\dfrac{1}{(1+i)^{\frac{1}{2}}} = \left\{ 2^{-\frac{1}{4}}\cos\frac{\pi}{8} - i2^{-\frac{1}{4}}\sin\frac{\pi}{8}, -2^{-\frac{1}{4}}\cos\frac{\pi}{8} + i2^{-\frac{1}{4}}\sin\frac{\pi}{8} \right\}$

The principal one is: $2^{-\frac{1}{4}}\cos\frac{\pi}{8} - i2^{-\frac{1}{4}}\sin\frac{\pi}{8}$.

(c) $\log i^3 = \log(-i) = \ln|-i| + i\arg(-i) = i\left(-\frac{\pi}{2} + 2n\pi\right) = -\frac{i\pi}{2} + 2n\pi i, \quad n \in \mathbb{Z}$

The principal one is: $-\frac{i\pi}{2}$.

(d) $i^{\sqrt{3}} = e^{\sqrt{3}\log i} = e^{\sqrt{3}(\ln|i| + i\arg(i))} = e^{\sqrt{3}\left(i\left(\frac{\pi}{2}+2n\pi\right)\right)}$

$= e^{i\left(\frac{\sqrt{3}\pi}{2} + 2n\sqrt{3}\pi\right)}$

$= \cos\left(\frac{\sqrt{3}\pi}{2} + 2n\sqrt{3}\pi\right) + i\sin\left(\frac{\sqrt{3}\pi}{2} + 2n\sqrt{3}\pi\right), \quad n \in \mathbb{Z}$

The principal one is: $\cos\frac{\sqrt{3}\pi}{2} + i\sin\frac{\sqrt{3}\pi}{2}$.

Question 3.

(a) Find all roots of the equation $\log z = i\frac{\pi}{2}$. (10 points)

(b) Show that $\cos z = 0$ if and only if $z = \left(n + \frac{1}{2}\right)\pi$ for some $n \in \mathbb{Z}$. (10 points)

Answer 3.

Actually, part (a) of this question should be corrected in a way:

(a') Find all roots of the equation $\operatorname{Log} z = i\frac{\pi}{2}$

or

(a'') Find all z with $\log z \ni i\frac{\pi}{2}$.

Now, solutions are:

(a') $\operatorname{Log} z = i\frac{\pi}{2} \Rightarrow \ln|z| + i \operatorname{Arg} z = i\frac{\pi}{2} \Rightarrow \ln|z| = 0$ and $\operatorname{Arg} z = i\frac{\pi}{2}$

Thus $|z|=1$ and $\operatorname{Arg} z = i\frac{\pi}{2}$ and so $z = |z| e^{i \operatorname{Arg} z} = e^{i\frac{\pi}{2}} = i$.

That is, there is only one solution, which is $z = i$.

(a'') $\ln|z| + i \arg z \ni i\frac{\pi}{2} \Rightarrow$

$\ln|z| + i \operatorname{Arg} z + 2n\pi i = i\frac{\pi}{2}$ for some $n \in \mathbb{Z}$

So $|z|=1$ and $\operatorname{Arg} z = i\left(\frac{\pi}{2} - 2n\pi\right)$ for some $n \in \mathbb{Z}$.

Since $-\pi < \operatorname{Arg} z \leq \pi$, n must be 0 and so

$z = e^{i\frac{\pi}{2}} = i$.

(b) $\cos z = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y = 0 \Rightarrow$

i) $\cos x \cosh y = 0$ and ii) $\sin x \sinh y = 0$

Since $\cosh y$ is never zero for any real y, $\cos x$ should be zero and we know that $x = \left(n + \frac{1}{2}\right)\pi$ for some $n \in \mathbb{Z}$.

Then, by (ii) $\sin x \sinh y = (-1)^n \sinh y = 0$ and this happens only when $y=0$, and hence $z = \left(n+\frac{1}{2}\right)\pi$.

Reverse implication is trivial.

Question 4.

(a) Find $\lim_{n\to\infty} \frac{3+ni}{n+2ni}$, if it exits. (10 points)

(b) Determine whether the series $\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}}$ is convergent or divergent. If convergent, find its sum. (10 points)

Answer 4.

(a) $$\lim_{n\to\infty} \frac{3+ni}{n+2ni} = \lim_{n\to\infty} \frac{3n-6ni+n^2i+2n^2}{n^2+4n^2}$$

$$= \lim_{n\to\infty} \frac{3n+2n^2}{5n^2} + i \lim_{n\to\infty} \frac{n^2-6n}{5n^2} = \frac{2}{5} + \frac{i}{5}.$$

(b) $$\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}} = \sum_{n=0}^{\infty} \frac{i^{n+2}}{(1+i)^{n+1}} = \frac{i^2}{(1+i)} \sum_{n=0}^{\infty} \left(\frac{i}{1+i}\right)^n = \frac{-1}{1+i} \sum_{n=0}^{\infty} \left(\frac{i}{1+i}\right)^n$$

We know that the geometric series

$$\sum_{n=0}^{\infty} z^n = \begin{cases} \frac{1}{1-z} & \text{if } |z|<1 \\ \text{diverges} & \text{if } |z| \geq 1. \end{cases}$$

Put $z = \frac{i}{1+i}$. Since $|z| = \frac{1}{\sqrt{2}} < 1$,

$$\sum_{k=2}^{\infty} \frac{i^k}{(1+i)^{k-1}} = -\frac{1}{1+i} \sum_{n=0}^{\infty} z^n = -\frac{1}{1+i} \cdot \frac{1}{1-z}$$

$$= -\frac{1}{1+i} \cdot \frac{1}{1-\frac{i}{1+i}} = -\frac{1}{1+i} \cdot \frac{1+i}{1} = -1.$$

Question 5.

(a) Determine where f is

(i) differentiable (7 points)

(ii) analytic (3 points)

when $f(z) = (x+\alpha y)^2 + 2i(x - \alpha y)$ for α real and constant.

(b) Let $f(x,y) = y^3 - 3x^2y + i(x^3 - 3xy^2 + 2)$.

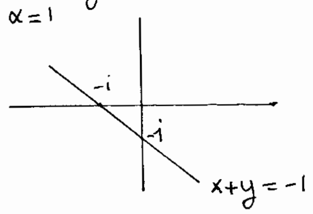
(i) Find the region D where f is analytic. (5 points)

(ii) Find the derivative of f and express it as a function of z in D. (5 points)

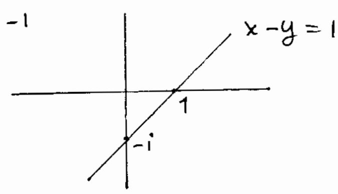
Answer 5.

(a) (i) Let $u(x,y) = (x+\alpha y)^2$ and $v(x,y) = 2(x-\alpha y)$. Since u and v are polynomials they belong to C^1 and so f is differentiable if and only if u and v satisfy Cauchy-Riemann conditions. Since $u_x = 2(x+\alpha y)$, $v_y = -2\alpha$, $u_y = 2(x+\alpha y)\alpha$, $v_x = 2$
x, y, α must satisfy (i) $x + \alpha y = -\alpha$ and (ii) $(x+\alpha y)\alpha = -1$
multiplying (i) by $-\alpha$ and adding to (ii), we obtain $\alpha^2 - 1 = 0$.
Thus, $\alpha = 1$ or $\alpha = -1$. If $\alpha = 1$, then C-R equations hold only on the line $x + y = -1$ and if $\alpha = -1$, C-R equations hold only on the line $x - y = 1$.

$\alpha = 1$



$\alpha = -1$



(ii) In any case ($\alpha = 1$ or $\alpha = -1$), f is nowhere analytic because the lines $x+y=-1$ and $x-y=1$ has no interior points in $\mathbb{R}^2$.

(b) (i) Let $u(x,y) = y^3 - 3x^2y$, $v(x,y) = x^3 - 3xy^2 + 2$
Since $u_x = -6xy = v_y$ and $u_y = 3y^2 - 3x^2 = -v_x$ for all $(x,y) \in \mathbb{R}^2$ and $u, v \in C^1(\mathbb{R}^2)$, f is everywhere differentiable and so it is an entire function.

(ii)
$$
\begin{aligned}
f'(z) &= u_x + i v_x = -6xy + i(3x^2 - 3y^2) \\
&= 3i(x^2 - y^2 + 2ixy) \\
&= 3i(x+iy)^2 = 3iz^2.
\end{aligned}
$$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

Final

January 14, 2008

11:00-13:00

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might <u>not</u> get full credit.
- Calculators are <u>not</u> allowed.

GOOD LUCK!

Please do <u>not</u> write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Evaluate

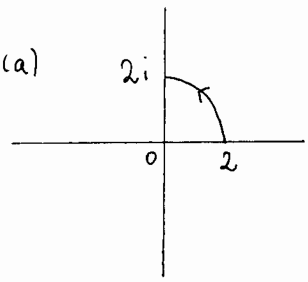
$$\int_C (z^2 + 3z)\, dz$$

along

(a) the circle $|z| = 2$ from 2 to $2i$ in a counterclockwise direction, (6 points)
(b) the straight line from 2 to $2i$, (7 points)
(c) the straight lines from 2 to $2 + 2i$ and then from $2 + 2i$ to $2i$. (7 points)

Answer 1.

(a)



$$z(t) = 2e^{it}, \quad 0 \leq t \leq \frac{\pi}{2}.$$

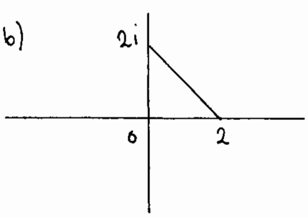
$$\int_C (z^2 + 3z)dz = \int_0^{\pi/2} (4e^{2it} + 6e^{it})2ie^{it}\, dt$$

$$= 8i \int_0^{\frac{\pi}{2}} e^{3it}\, dt + 12i \int_0^{\frac{\pi}{2}} e^{2it}\, dt$$

$$= 8i \left. \frac{e^{3it}}{3i} \right|_0^{\frac{\pi}{2}} + 12i \left. \frac{e^{2it}}{2i} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{8}{3}(-i-1) + 6(-1-1) = -\frac{44}{3} - \frac{8i}{3}.$$

(b)



$$z(t) = 2(1-t) + 2it, \quad 0 \leq t \leq 1$$

$$\int_C (z^2 + 3z)dz = \int_0^1 \Big((2(1-t) + 2it)^2 + 3(2(1-t) + 2it)\Big)(-2+2i)\,dt$$

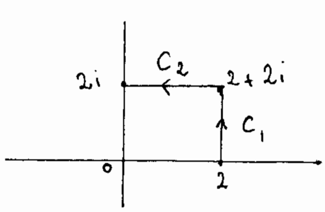
$$= \int_0^1 (-2+2i)(-8it^2 - 14t + 14it + 10)\,dt = (-2+2i)\left(-8i \frac{t^3}{3}\Big|_0^1 - 14\frac{t^2}{2}\Big|_0^1 + 14i\frac{t^2}{2}\Big|_0^1 + 10t\Big|_0^1\right)$$

$$= (-2+2i)\left(-\frac{8i}{3} - 7 + 7i + 10\right) = -\frac{44}{3} - \frac{8i}{3}.$$

(c) $C = C_1 + C_2$ where C_1: $z(t) = 2(1-t) + (2+2i)t, \quad 0 \leq t \leq 1$

$$= 2 + 2it, \quad 0 \leq t \leq 1$$

and C_2: $z(t) = (2+2i)(1-t) + 2it = 2 - 2t + 2i, \quad 0 \leq t \leq 1$



so, $$\int_C (z^2+3z)dz = \int_{C_1} (z^2+3z)dz + \int_{C_2} (z^2+3z)dz.$$

$$\int_{C_1} (z^2+3z)\,dz = \int_0^1 \left((2+2it)^2 + 3(2+2it) \right)(2i)\,dt$$

$$= 2i \int_0^1 (-4t^2 + 14it + 10)\,dt = 2i \left(-4\frac{t^3}{3}\Big|_0^1 + 14i\frac{t^2}{2}\Big|_0^1 + 10t\Big|_0^1 \right)$$

$$= 2i\left(-\frac{4}{3} + 7i + 10\right) = \frac{52}{3}i - 14.$$

$$\int_{C_2} (z^2+3z)\,dz = \int_0^1 \left((2-2t+2i)^2 + 3(2-2t+2i) \right)(-2)\,dt$$

$$= -2 \int_0^1 (4t^2 + 14i - 14t - 8it + 6)\,dt$$

$$= -2\left(4\frac{t^3}{3}\Big|_0^1 + 14it\Big|_0^1 - 14\frac{t^2}{2}\Big|_0^1 - 8i\frac{t^2}{2}\Big|_0^1 + 6t\Big|_0^1 \right)$$

$$= -2\left(\frac{4}{3} + 14i - 7 - 4i + 6\right) = -\frac{2}{3} - 20i$$

$$\Rightarrow \qquad \int_C (z^2+3z)\,dz = \frac{52}{3}i - 14 - \frac{2}{3} - 20i = -\frac{8i}{3} - \frac{44}{3}.$$

Question 2. Let C_R denote the upper half of the circle $|z| = R$, $R > 2$, taken in the counter-clockwise direction. Show that

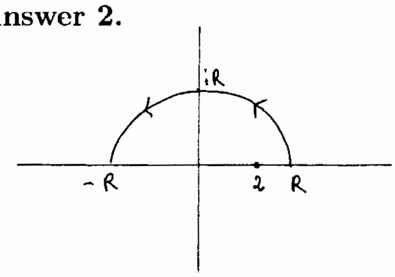
$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, show that the value of the integral

$$\int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz$$

tends to zero as R tends to infinity. (20 points)

Answer 2.



$$\left| \int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \right| \leqslant ML$$

where M is a number satisfying

$$\left| \frac{2z^2-1}{z^4+5z^2+4} \right| \leqslant M \quad \text{for all} \quad z \in C_R,$$

and L is the length of C_R.

Clearly, $L = \pi R$. If $|z| = R$, $|2z^2-1| \leqslant 2|z|^2+1 = 2R^2+1$,

$z^4+5z^2+4 = (z^2+1)(z^2+4)$, $|z^2+1| \geqslant |z|^2-1 = R^2-1$,

$|z^2+4| \geqslant |z|^2-4 = R^2-4$ and hence

$$|z^4+5z^2+4| = |z^2+1||z^2+4| \geqslant (R^2-1)(R^2-4).$$

So

$$\frac{1}{|z^4+5z^2+4|} \leqslant \frac{1}{(R^2-1)(R^2-4)} \quad \text{and}$$

$$\left| \frac{2z^2-1}{z^4+5z^2+4} \right| \leqslant \frac{2R^2+1}{(R^2-1)(R^2-4)}.$$ That is, we can choose

$M = \dfrac{2R^2+1}{(R^2-1)(R^2-4)}$. Therefore, $\left| \displaystyle\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \right| \leqslant \dfrac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}$

Since, $0 \leqslant \left| \displaystyle\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \right| \leqslant \dfrac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)} \to 0$ as $R \to \infty$,

$\displaystyle\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\, dz \to 0$ as $R \to \infty$ by the Sandwich theorem.

Question 3. Find all solutions of the equation $\sin z = 2$ by using

(a) the expression $\sin z = \sin x \cosh y + i \cos x \sinh y$. (10 points)

(b) the inverse function $\arcsin z = -i \log\left(iz + (1-z^2)^{\frac{1}{2}}\right)$. (10 points)

Answer 3.

(a) $\sin z = \sin x \cosh y + i \cos x \sinh y = 2 \Rightarrow$

$\sin x \cosh y = 2$ and $\cos x \sinh y = 0$. Since $\cos x \sinh y = 0$, we have $x = (n+\frac{1}{2})\pi$ or $y = 0$ for $n \in \mathbb{Z}$.

When $y=0$, it follows from $\sin x \cosh y = 2$ that $\sin x = 2$, which is impossible because $-1 \le \sin x \le 1$ for all $x \in \mathbb{R}$.

So $x = (n+\frac{1}{2})\pi$. Then

$\sin x \cosh y = (-1)^n \cosh y = 2$. Since $\cosh y \geq 0$ for all $y \in \mathbb{R}$, n should be an even integer, and $\cosh y = 2$.

Since $\cosh y = \frac{e^y + e^{-y}}{2} = 2$, we have $e^y + e^{-y} = 4$ and $e^{2y} + 1 = 4e^y$ or equivalently $(e^y)^2 - 4e^y + 1 = 0$. Solving this quadratic equation for e^y, we obtain $e^y = 2 \pm \sqrt{3}$ and hence $y = \ln(2\pm\sqrt{3})$. So $z = (2n+\frac{1}{2})\pi + i\ln(2\pm\sqrt{3})$, $n \in \mathbb{Z}$.

(b) $\sin z = 2 \Rightarrow z = \arcsin 2 \Rightarrow$

$$
\begin{aligned}
z &= -i \log\left(2i + (1-4)^{\frac{1}{2}}\right) \\
&= -i \log(2i \pm \sqrt{3}\, i) \\
&= -i \left(\ln|2i \pm \sqrt{3}i| + i \arg(2i \pm \sqrt{3} i)\right) \\
&= -i \left(\ln(2\pm\sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right)\right), \quad n \in \mathbb{Z} \\
&= \left(2n+\frac{1}{2}\right)\pi - i\ln(2\pm\sqrt{3}), \quad n \in \mathbb{Z} \\
&= \left(2n+\frac{1}{2}\right)\pi + i\ln(2\pm\sqrt{3}), \quad n \in \mathbb{Z}
\end{aligned}
$$

Question 4. Find the domain of convergence of

(a) $\displaystyle\sum_{n=0}^{\infty} n^2(2z-1)^n$. (10 points)

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z+1)^n$. (10 points)

Answer 4.

(a) $$\sum_{n=0}^{\infty} n^2(2z-1)^n = \sum_{n=0}^{\infty} n^2\, 2^n \left(z-\frac{1}{2}\right)^n$$

$\hookrightarrow$ power series

Let $a_n = n^2 2^n$, then $$\rho = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{(n+1)^2 2^{n+1}}{n^2 2^n}} = \frac{1}{2}.$$

If $|z-\frac{1}{2}| < \frac{1}{2}$ series converges absolutely.

If $|z-\frac{1}{2}| = \frac{1}{2}$, $\left|n^2 2^n (z-\frac{1}{2})^n\right| = n^2 \not\to 0$ as $n\to\infty$ $\Rightarrow$

series diverges by n^{th} term test. So, the domain of convergence is $\{z\in\mathbb{C} \mid |z-\frac{1}{2}| < \frac{1}{2}\}$

(b) Let $a_n = \frac{(-1)^n}{n!}$. Then,

$$\rho = \frac{1}{\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}} = \frac{1}{\lim\limits_{n\to\infty} \frac{n!}{(n+1)!}} = \infty.$$

Thus, the series converges everywhere. That is, the domain of convergence is $\mathbb{C}$.

Question 5.

(a) Use any method to show that the function $f(z) = z + \bar{z}$ is nowhere differentiable. (10 points)

(b) Determine whether $u(x, y) = e^{-x}(x \sin y - y \cos y)$ is harmonic in $\mathbb{C}$, and find its harmonic conjugate if it is harmonic. (10 points)

Answer 5.

(a) $f(z) = z + \bar{z} = x + iy + x - iy = 2x$. So $u(x,y) = 2x$ and $v(x,y) = 0$.

Then, $u_x = 2$, $u_y = 0$, $v_x = 0$, $v_y = 0$. Since $u_x = v_y$ is never satisfied, f is nowhere differentiable.

(b) $u_x = -e^{-x} x \sin y + e^{-x} \sin y + e^{-x} y \cos y$

$u_{xx} = e^{-x} x \sin y - 2e^{-x} \sin y - e^{-x} y \cos y$

$u_y = e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y$

$u_{yy} = -e^{-x} x \sin y + 2e^{-x} \sin y + e^{-x} y \cos y$.

Since, $u_{xx} + u_{yy} = 0$ for all (x,y), u is harmonic in $\mathbb{C}$.

Let v be a harmonic conjugate of u, so $u_x = v_y$ implies

$$v = -e^{-x} x \int \sin y \, dy + e^{-x} \int \sin y \, dy + e^{-x} \int y \cos y \, dy$$
$$= e^{-x} x \cos y - e^{-x} \cos y + e^{-x} y \sin y + e^{-x} \cos y + \phi(x)$$
$$= e^{-x} x \cos y + e^{-x} y \sin y + \phi(x).$$

$u_y = -v_x \Rightarrow$

$$v_x = -e^{-x} x \cos y + e^{-x} \cos y - e^{-x} y \sin y + \phi'(x)$$
$$= -e^{-x} x \cos y + e^{-x} \cos y - e^{-x} y \sin y \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = C,$$

for some $C \in \mathbb{R}$.

Therefore, $v = e^{-x} x \cos y + e^{-x} y \sin y + C$.

MATH 351 Complex Analysis I

Make-up for the first midterm
January 24, 2008
13:00-15:00

Questions

(1) (a) Find all the values of $(-16)^{\frac{1}{4}}$ in rectangular coordinates and locate them in the complex plane. (10 points)

(b) Sketch the set of points satisfying

(i) $|z-2| > |z-3|$, (5 points)

(ii) $\mathrm{Re}(\overline{z} - i) = 2$. (5 points)

(2) (a) Compute and express in Cartesian form

(i) $\dfrac{1}{i^{2015}}$, (5 points)

(ii) $(\sqrt{3}+i)^6$, (5 points)

(b) Represent the following complex numbers in polar form.

(i) $\dfrac{6}{i+\sqrt{3}}$, (5 points)

(ii) $(5+5i)^3$. (5 points)

(3) (a) Find the image of the set $\mathscr{D} = \left\{re^{i\theta} \;\middle|\; r > 3,\ \frac{2\pi}{3} < \theta < \frac{3\pi}{4}\right\}$ the mapping $w = z^3$. (10 points)

(b) Find the image of the set $\mathscr{D} = \left\{z = x+iy \;\middle|\; x > 1,\ y > 1\right\}$ under the mapping $\dfrac{1}{z}$. (10 points)

(4) (a) Let

$$f(z) = \begin{cases} \dfrac{(\mathrm{Re}(z))^2}{|z|} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$$

Decide whether f is continuous at $z = 0$. (10 points)

(b) Evaluate $\displaystyle\lim_{z \to 1+i} \frac{z^2+z-1-3i}{z^2-2z+2}$ if it exits. Do not use L'Hospital's rule. (10 points)

(5) Let $f(z) = z^3 + 1$.

(a) By using the definition of derivative show that f is differentiable everywhere and find $f'(z)$. (5 points)

(b) Write $f(z)$ in the form $u(x,y) + iv(x,y)$. (5 points)

(c) Show that $u_x(x,y) = v_y(x,y)$ and $u_y(x,y) = -v_x(x,y)$. (5 points)

(d) Show that $f'(z) = u_x(x,y) + iv_x(x,y) = v_y(x,y) - iu_y(x,y)$. (5 points)

MATH 351 Complex Analysis I

Make-up for the second midterm
January 24, 2008
13:00-15:00

Questions

(1) (a) Determine whether $u(x, y) = y^3 - 3x^2y$ is harmonic. If it is harmonic find its harmonic conjugate. (10 points)

(b) Does an analytic function $f(z) = u(x, y) + iv(x, y)$ exist for which $v(x, y) = x^3 + y^3$? Why or why not? (10 points)

(2) (a) Find all values of z for which the following equation $e^z = 1 + i\sqrt{3}$ hold. (10 points)

(b) Find all values of $(1 + i)^{2-i}$. Indicate which one is the principle. (10 points)

(3) (a) Express $\cos(1 + i)$ in Cartesian form. (10 points)

(b) Show that $\sinh z = 0$ if and only if $z = n\pi i,\ n \in \mathbb{Z}$. (10 points)

(4) (a) Evaluate $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2+i}\right)^n$. (10 points)

(b) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty}\frac{n!z^n}{n^n}$. (10 points)

(5) (a) Determine where $f(z) = 8x - x^3 - xy^2 + i(x^2y + y^3 - 8y)$ is

(i) differentiable (7 points)

(ii) analytic (3 points)

(b) Let $f(z) = 3x + y + i(3y - x),\ z = x + iy$.

(i) Find the region D where f is analytic. (5 points)

(ii) Find the derivative of f and express it as a function of z in D. (5 points)