ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

1st Midterm

November 6, 2006

08:40-10:30

Surname : ____________________
Name : ____________________
ID # : ____________________
Department : ____________________
Section : ____________________
Instructor : ____________________
Signature : ____________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1.

(a) Find all solutions of the equation $z^4 + 16i = 0$ in polar coordinates and mark them on the complex plane. (10 points)

(b) Let $f(z)$ be the principle square root function. Compute $f(1+i)$. (10 points)

Answer 1.

(a) Note that $z^4 + 16i = 0$ if and only if $z^4 = -16i = 16e^{i(-\frac{\pi}{2})}$. We have shown in class that $z^4 = 16e^{i(-\frac{\pi}{2})}$ has four distinct solutions, namely

$$z_k = 16^{\frac{1}{4}} e^{i((-\frac{\pi}{2}+2k\pi)/4)} = 2e^{i\frac{(4k-1)\pi}{8}}, \quad k = 0, 1, 2, 3.$$

If $k = 0$,

$$z_0 = 2e^{i(-\frac{\pi}{8})} = \boxed{2\cos\left(-\tfrac{\pi}{8}\right) + 2i\sin\left(-\tfrac{\pi}{8}\right)} = 2\cos\frac{\pi}{8} - 2i\sin\frac{\pi}{8}.$$

If $k = 1$,

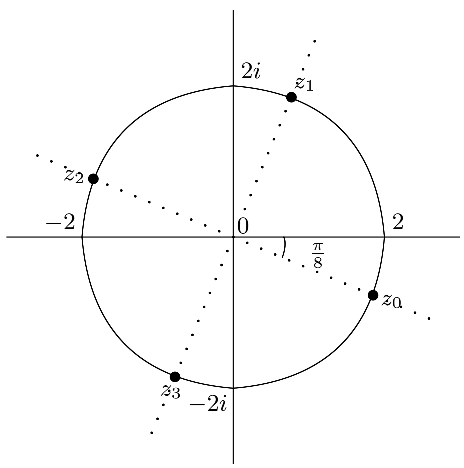
$$z_1 = 2e^{i\frac{3\pi}{8}} = \boxed{2\cos\tfrac{3\pi}{8} + 2i\sin\tfrac{3\pi}{8}} = 2\sin\frac{\pi}{8} + 2i\cos\frac{\pi}{8} = iz_0.$$

If $k = 2$,

$$z_2 = 2e^{i\frac{7\pi}{8}} = \boxed{2\cos\tfrac{7\pi}{8} + 2i\sin\tfrac{7\pi}{8}} = -2\cos\frac{\pi}{8} + 2i\sin\frac{\pi}{8} = -z_0.$$

If $k = 3$,

$$z_3 = 2e^{i\frac{11\pi}{8}} = \boxed{2\cos\tfrac{11\pi}{8} + 2i\sin\tfrac{11\pi}{8}} = -2\sin\frac{\pi}{8} - 2i\cos\frac{\pi}{8} = -iz_0.$$



(b) The principal square root function is $f(re^{i\theta}) = \sqrt{r}e^{i\frac{\theta}{2}}$, $r > 0, -\pi < \theta < \pi$. Clearly, $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$. Therefore

$$f(1+i) = \sqrt{\sqrt{2}\,e^{i\frac{\pi}{8}}} = \boxed{\sqrt[4]{2}e^{i\frac{\pi}{8}}} = \sqrt[4]{2}\cos\frac{\pi}{8} + i\sqrt[4]{2}\sin\frac{\pi}{8} \approx 1.10 + 0.46\,i.$$

Question 2. Compute

(a) $\left(\frac{2+i}{3-2i}\right)^2$. (5 points)

(b) $\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right|$. (5 points)

(c) $\lim_{z\to i} \frac{iz^3-1}{z+i}$. (5 points)

(d) $\lim_{z\to 1+i\sqrt{3}} \frac{z^6-64}{z^3+8}$. (5 points)

Answer 2.

(a)

$$\left(\frac{2+i}{3-2i}\right)^2 = \frac{(2+i)^2}{(3-2i)^2} = \frac{(2^2-1^2)+(2\cdot 2\cdot 1)i}{(3^2-2^2)-(2\cdot 3\cdot 2)i} = \frac{3+4i}{5-12i} = \frac{(3+4i)(5+12i)}{(5-12i)(5+12i)}$$
$$= \frac{15+36i+20i-48}{25+144} = \frac{-33+56i}{169} = -\frac{33}{169}+\frac{56}{169}i.$$

(b)

$$\left|\frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}\right| = \frac{|3+4i|\,|-1+2i|}{|-1-i|\,|3-i|} = \frac{\sqrt{3^2+4^2}\sqrt{1^2+2^2}}{\sqrt{1^2+1^2}\sqrt{3^2+1^2}} = \frac{5\sqrt{5}}{\sqrt{2}\sqrt{10}} = \frac{5}{2}.$$

(c)

$$\lim_{z\to i} \frac{iz^3-1}{z+i} = \frac{\lim_{z\to i} iz^3-1}{\lim_{z\to i} z+i} = \frac{i(i)^3-1}{i+i} = \frac{i^4-1}{2i} = \frac{1-1}{2i} = 0.$$

(d) Let $f(z) = z^6-64$ and $g(z) = z^3+8$. Then, we have $f'(z) = 6z^5$ and $g'(z) = 3z^2$. Since $1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}$, we have

$$f(1+i\sqrt{3}) = (1+i\sqrt{3})^6 - 64 = \left(2e^{i\frac{\pi}{3}}\right)^6 - 64 = 64e^{i2\pi} - 64 = 64-64 = 0,$$
$$g(1+i\sqrt{3}) = (1+i\sqrt{3})^3 + 8 = \left(2e^{i\frac{\pi}{3}}\right)^3 + 8 = 8e^{i\pi}+8 = -8+8 = 0,$$
$$g'(1+i\sqrt{3}) = 3(1+i\sqrt{3})^2 = 3\left(2e^{i\frac{\pi}{3}}\right)^2 = 12e^{i\frac{2\pi}{3}} \neq 0.$$

Therefore L'Hospital's rule is applicable, and

$$\lim_{z\to 1+i\sqrt{3}} \frac{z^6-64}{z^3+8} = \lim_{z\to 1+i\sqrt{3}} \frac{6z^5}{3z^2} = \lim_{z\to 1+i\sqrt{3}} 2z^3 = 2(1+i\sqrt{3})^3 = 2\left(2e^{i\frac{\pi}{3}}\right)^3 = 16e^{i\pi} = -16.$$

Question 3. Find the image of the set $\mathscr{D} = \{z \in \mathbb{C} \ : \ |z-1| < 1\}$ under

(a) the mapping $f(z) = (3+4i)z - 2 + i$. (10 points)

(b) the mapping $f(z) = \dfrac{1}{z}$. (10 points)

Answer 3.

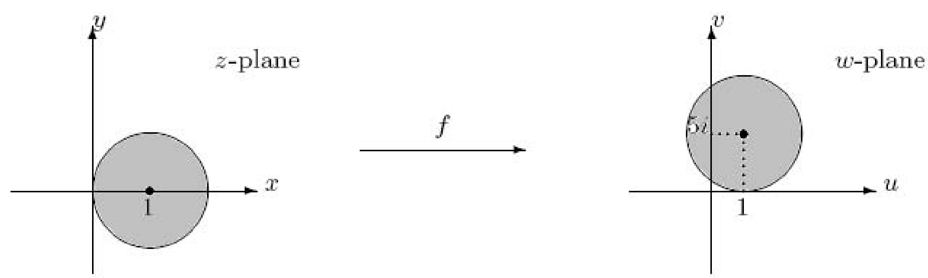
(a) Clearly, $w = u + iv \in f(\mathscr{D})$ if and only if $f^{-1}(w) \in \mathscr{D}$. One can easily compute that

$$f^{-1}(w) = \frac{w + 2 - i}{3 + 4i}.$$

Therefore

$$\begin{aligned} f^{-1}(w) \in \mathscr{D} &\Leftrightarrow \left| \frac{w+2-i}{3+4i} - 1 \right| < 1 \\ &\Leftrightarrow |w + 2 - i - 3 - 4i| < |3 + 4i| = \sqrt{3^2 + 4^2} = 5 \\ &\Leftrightarrow |w - (1 + 5i)| < 5. \end{aligned}$$

Hence, the image of $\mathscr{D}$ under $f(z) = (3+4i)z - 2 + i$ is $\{w \in \mathbb{C} \ : \ |w - (1+5i)| < 5\}$:



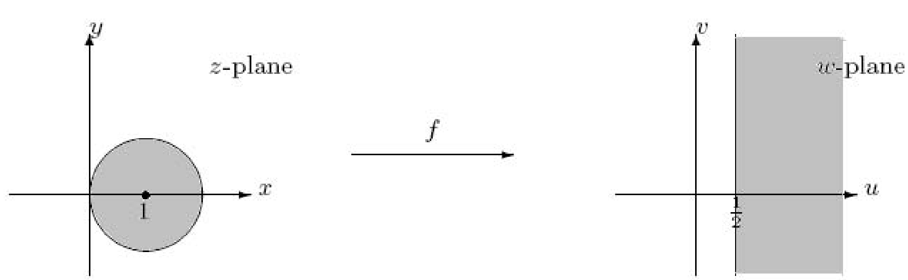
(b) Similarly, $w = u + iv \in f(\mathscr{D})$ if and only if $f^{-1}(w) \in \mathscr{D}$. Clearly

$$f^{-1}(w) = \frac{1}{w}.$$

Therefore

$$\begin{aligned} f^{-1}(w) \in \mathscr{D} &\Leftrightarrow \left| \frac{1}{w} - 1 \right| < 1 \Leftrightarrow |1 - w| < |w| \\ &\Leftrightarrow |(u-1) + iv| < |u + iv| \Leftrightarrow \sqrt{(u-1)^2 + v^2} < \sqrt{u^2 + v^2} \\ &\Leftrightarrow (u-1)^2 + v^2 < u^2 + v^2 \Leftrightarrow u^2 - 2u + 1 < u^2 \\ &\Leftrightarrow 1 < 2u \Leftrightarrow \frac{1}{2} < u. \end{aligned}$$

Hence, the image of $\mathscr{D}$ under $f(z) = \dfrac{1}{z}$ is $\{w \in \mathbb{C} \ : \ \mathrm{Re}(w) > \frac{1}{2}\}$:



Question 4.

(a) Let $f(z) = u + iv$ be analytic on a domain $\mathscr{D}$. If $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ holds on $\mathscr{D}$, then show that f' is constant there. (10 points)

(b) Write the function $f(z) = \dfrac{z+i}{z^2+1}$ in the form $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, where u and v are real-valued functions of the real variables x and y. (10 points)

Answer 4.

(a) Since f is analytic, u and v satisfy Cauchy-Riemann equations, namely

$$u_x = v_y \tag{1}$$

$$v_x = -u_y \tag{2}$$

If $u_x + v_y = 0$, then by (1), we get $u_x = v_y = 0$. Therefore u depends only on y and v depends only on x. That is $u(x,y) = \phi(y)$ and $v(x,y) = \psi(x)$. By (2)

$$\psi'(x) = -\phi'(y). \tag{3}$$

Since the left hand side of (3) does not depend on y, and the right hand side of (3) does not depend on x, (3) holds if and only if

$$\psi'(x) = -\phi'(y) = c,$$

for some real constant c. Hence

$$f'(z) = u_x(x,y) + iv_x(x,y) = 0 + ic = ic,$$

a complex constant.

(b)

$$\begin{aligned}
f(z) &= f(x+iy) \\
&= \frac{(x+iy)+i}{(x+iy)^2+1} = \frac{x+i(y+1)}{(x^2-y^2)+2ixy+1} = \frac{x+i(y+1)}{(x^2-y^2+1)+2ixy} \\
&= \frac{(x+i(y+1))((x^2-y^2+1)-2ixy)}{((x^2-y^2+1)+2ixy)((x^2-y^2+1)-2ixy)} \\
&= \frac{x(x^2-y^2+1)-2ix^2y+i(y+1)(x^2-y^2+1)+2xy(y+1)}{(x^2-y^2+1)^2+4x^2y^2} \\
&= \frac{x(x^2-y^2+1)+2xy(y+1)}{x^4+y^4+1+2x^2-2x^2y^2-2y^2+4x^2y^2} + i\frac{(y+1)(x^2-y^2+1)-2x^2y}{x^4+y^4+1+2x^2-2x^2y^2-2y^2+4x^2y^2} \\
&= \frac{x^3+xy^2+x+2xy}{x^4+y^4+1+2x^2+2x^2y^2-2y^2} + i\frac{-x^2y-y^3+y+x^2-y^2+1}{x^4+y^4+1+2x^2+2x^2y^2-2y^2}.
\end{aligned}$$

Question 5.

(a) Determine the region of the complex plane in which the function

$$f(z) = x^2 - x + y + i(y^2 - 5y - x), \quad z = x + iy$$

(i) is differentiable. (7 points)

(ii) is analytic. (3 points)

(b) Let $g(z) = e^{-x}e^{-iy}$, $z = x + iy$. Show that $g'(z)$ and its derivative $g''(z)$ exist everywhere, and find $g''(z)$. (10 points)

Answer 5.

(a) (i) $f(z) = u(x,y) + iv(x,y)$, where $u(x,y) = x^2 - x + y$ and $v(x,y) = y^2 - 5y - x$. Note that $u_x(x,y) = 2x - 1$, $u_y(x,y) = 1$, $v_x(x,y) = -1$ and $u_y(x,y) = 2y - 5$.
Since u, v, u_x, u_y, v_x, v_y are all continuous, f is differentiable at (x,y) if and only if the Cauchy-Riemann equations

$$u_x = v_y \tag{4}$$

$$v_x = -u_y \tag{5}$$

hold at (x,y).
Clearly (5) holds for all (x,y), but (4) holds only if $2x - 1 = 2y - 5$, or equivalently $y = x + 2$.
Therefore f is only differentiable on the line $\mathscr{D} = \{z = x + iy \in \mathbb{C} \; : \; y = x + 2\}$.

(ii) Since $\mathscr{D}$ has no interior points, f is nowhere analytic.

(b) Note that $g(z) = e^{-x}e^{-iy} = e^{-x}(\cos(-y) + i\sin(-y)) = e^{-x}\cos y - ie^{-x}\sin y$. Therefore $g(z) = u(x,y) + iv(x,y)$ where $u(x,y) = e^{-x}\cos y$ and $v(x,y) = -e^{-x}\sin y$. Clearly,

$$u_x(x,y) = -e^{-x}\cos y = v_y(x,y)$$
$$v_x(x,y) = e^{-x}\sin y = -u_y(x,y).$$

Since u, v, u_x, u_y, v_x, v_y are continuous everywhere and Cauchy-Riemann equations are satisfied for all (x,y), g is everywhere differentiable. Moreover,

$$g'(z) = u_x(x,y) + iv_x(x,y) = -e^{-x}\cos y + ie^{-x}\sin y.$$

Thus $g'(z) = U_x(x,y) + iV_x(x,y)$ where $U(x,y) = -e^{-x}\cos y$ and $V(x,y) = e^{-x}\sin y$. Note that,

$$U_x(x,y) = e^{-x}\cos y = V_y(x,y)$$
$$V_x(x,y) = -e^{-x}\sin y = -U_y(x,y).$$

Since U, V, U_x, U_y, V_x, V_y are continuous everywhere and Cauchy-Riemann equations are satisfied for all (x,y), g' is everywhere differentiable, and,

$$g''(z) = U_x(x,y) + iV_x(x,y) = e^{-x}\cos y - ie^{-x}\sin y = g(z).$$

ÇANKAYA UNIVERSITY
Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

2nd Midterm

December 21, 2006

12:40-14:30

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

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GOOD LUCK!

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Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1. Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u + iv$ as an analytic function of z.

(a) $u(x, y) = 2xy + 3xy^2 - 2y^3$. (10 points)

(b) $u(x, y) = xe^x \cos y - ye^x \sin y$. (10 points)

Answer 1.

(a) Clearly,

$$\begin{aligned} u_x(x, y) &= 2y + 3y^2 \\ u_{xx}(x, y) &= 0 \end{aligned}$$

and,

$$\begin{aligned} u_y(x, y) &= 2x + 6xy - 6y^2 \\ u_{yy}(x, y) &= 6x - 12y. \end{aligned}$$

Thus,

$$u_{xx}(x, y) + u_{yy}(x, y) = 6x - 12y \neq 0$$

in any domain, and hence u is not harmonic.

(b) Clearly,

$$\begin{aligned} u_x(x, y) &= e^x \cos y + xe^x \cos y - ye^x \sin y, \qquad (1) \\ u_{xx}(x, y) &= 2e^x \cos y + xe^x \cos y - ye^x \sin y, \end{aligned}$$

and,

$$\begin{aligned} u_y(x, y) &= -xe^x \sin y - e^x \sin y - ye^x \cos y, \qquad (2) \\ u_{yy}(x, y) &= -2e^x \cos y - xe^x \cos y + ye^x \sin y. \end{aligned}$$

Thus, $u_{xx}(x, y) + u_{yy}(x, y) = 0$ for all $x + iy \in \mathbb{C}$, and hence u is harmonic in $\mathbb{C}$.

Let v be a harmonic conjugate of u. Then u and v should satisfy

$$\begin{aligned} u_x(x, y) &= v_y(x, y), \qquad (3) \\ u_y(x, y) &= -v_x(x, y). \qquad (4) \end{aligned}$$

It follows from (1) and (3) that $v_y(x, y) = e^x \cos y + xe^x \cos y - ye^x \sin y$ for all (x, y), and so

$$\begin{aligned} v(x, y) &= \int (e^x \cos y + xe^x \cos y - ye^x \sin y)\, dy \\ &= e^x \int \cos y\, dy + xe^x \int \cos y\, dy - e^x \int y \sin y\, dy \\ &= e^x \sin y + xe^x \sin y - e^x \left(-y \cos y + \int \cos y\, dy \right) \\ &= e^x \sin y + xe^x \sin y - e^x \left(-y \cos y + \sin y\, dy \right) + \phi(x) \\ &= xe^x \sin y + e^x y \cos y + \phi(x). \qquad (5) \end{aligned}$$

Using (2) and (5), we obtain

$$e^x \sin y + xe^x \sin y + e^x y \cos y + \phi'(x) = xe^x \sin y + e^x \sin y + ye^x \cos y.$$

Thus, $\phi'(x) = 0$ for all (x, y), and so $\phi(x) = c$ for some $c \in \mathbb{R}$, and

$$v(x, y) = xe^x \sin y + e^x y \cos y + c, \ c \in \mathbb{R}.$$

Hence

$$\begin{aligned} u(x, y) + iv(x, y) &= xe^x \cos y - ye^x \sin y + ixe^x \sin y + ie^x y \cos y + ic \\ &= e^x \cos y(x + iy) + e^x \sin y(-y + ix) + ic \\ &= e^x \cos y(x + iy) + ie^x \sin y(x + iy) + ic \\ &= (x + iy)e^x(\cos y + i \sin y) + ic \\ &= (x + iy)e^x e^{iy} = (x + iy)e^{x+iy} = ze^z + C, \ z = x + iy, \ C \in \mathbb{C}. \end{aligned}$$

Question 2.

(a) Show that for $|z-i|<\sqrt{2}$, $\dfrac{1}{1-z}=\displaystyle\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}$. (10 points)

(b) Find all values of z^c and show which one is the principal value, where z and c are given as $z=-1-\sqrt{3}i$, $c=\sqrt{3}i$. (10 points)

Answer 2.

(a) We have proved in class that

$$\frac{1}{1-w}=\sum_{n=0}^{\infty}w^n, \quad \text{whenever} \quad |w|<1. \tag{6}$$

Clearly,

$$\frac{1}{1-z}=\frac{1}{1-i+i-z}=\frac{1}{(1-i)-(z-i)}=\frac{1}{1-i}\left(\frac{1}{1-\frac{z-i}{1-i}}\right). \tag{7}$$

Let $w=\dfrac{z-i}{1-i}$. Clearly,

$$|w|=\left|\frac{z-i}{1-i}\right|=\frac{|z-i|}{\sqrt{1^2+(-1)^2}}=\frac{|z-i|}{\sqrt{2}}<1 \text{ if and only if } |z-i|<\sqrt{2}.$$

Then, it follows from (6) that,

$$\frac{1}{1-\frac{z-i}{1-i}}=\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^n}, \quad \text{for} \quad |z-i|<\sqrt{2}. \tag{8}$$

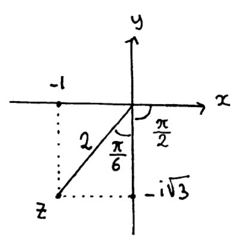
Combining (7) and (8) together, we obtain

$$\frac{1}{1-z}=\frac{1}{1-i}\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^n}=\sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}, \quad \text{for} \quad |z-i|<\sqrt{2}.$$

(b) The power function z^c is defined by

$$z^c=e^{c\log z}=e^{c(\ln|z|+i\arg(z))}, \ z\neq 0. \tag{9}$$

If $z=-1-\sqrt{3}i$, clearly, $|z|=2$, $\operatorname{Arg}(z)=-\dfrac{2\pi}{3}$, and so, $\arg(z)=-\dfrac{2\pi}{3}+2n\pi,\ n\in\mathbb{Z}$. (Remind that for any $z\in\mathbb{C}\backslash\{0\}$, $-\pi<\operatorname{Arg}(z)\leq\pi$.)



Therefore, putting $c=\sqrt{3}i$, $|z|=2$, and $\arg(z)=-\dfrac{2\pi}{3}+2n\pi,\ n\in\mathbb{Z}$ in (9), we obtain

$$\begin{aligned}(-1-\sqrt{3}i)^{\sqrt{3}i}&=e^{\sqrt{3}i(\ln 2+i(-\frac{2\pi}{3}+2n\pi))}=e^{i\sqrt{3}\ln 2+\frac{2\sqrt{3}\pi}{3}-2n\sqrt{3}\pi}\\&=e^{\frac{2\pi}{\sqrt{3}}-2n\sqrt{3}\pi}e^{i\sqrt{3}\ln 2}\\&=e^{\frac{2\pi}{\sqrt{3}}-2n\sqrt{3}\pi}(\cos(\sqrt{3}\ln 2)+i\sin(\sqrt{3}\ln 2)) \quad n\in\mathbb{Z}.\end{aligned} \tag{10}$$

The principal value occurs if we use $\operatorname{Arg} z$ instead of $\arg z$ in (9), so it corresponds to $n=0$ in (10), namely, the principal value of $(-1-\sqrt{3}i)^{\sqrt{3}i}$ is

$$e^{\frac{2\pi}{\sqrt{3}}}(\cos(\sqrt{3}\ln 2)+i\sin(\sqrt{3}\ln 2)).$$

Question 3. Find all solutions of the equation $\sin z = i$ by using

(a) the expression $\sin z = \sin x \cosh y + i \cos x \sinh y$. (10 points)

(b) the inverse function $\arcsin z = -i \log\left(iz + (1-z^2)^{\frac{1}{2}}\right)$. (10 points)

Answer 3.

(a) Clearly,

$$\sin z = \sin x \cosh y + i \cos x \sinh y = i$$

if and only if

$$\sin x \cosh y = 0, \tag{11}$$

and

$$\cos x \sinh y = 1. \tag{12}$$

Since $\cosh y \geq 1$ for all $y \in \mathbb{R}$, it follows from (11) that $\sin x = 0$. Thus $x = n\pi,\ n \in \mathbb{Z}$. Putting $x = n\pi$ in (12), we obtain

$$\cos(n\pi) \sinh y = (-1)^n \sinh y = 1. \tag{13}$$

And, hence

$$\sinh y = (-1)^n.$$

Since $\sinh y = \dfrac{e^y - e^{-y}}{2}$,

$$\begin{aligned} \sinh y = (-1)^n &\Leftrightarrow e^y - e^{-y} = 2(-1)^n \\ &\Leftrightarrow e^{2y} - 2(-1)^n e^y - 1 \\ &\Leftrightarrow e^y = \frac{2(-1)^n \pm \sqrt{8}}{2} = (-1)^n \pm \sqrt{2}. \end{aligned}$$

Since $(-1)^n - \sqrt{2} < 0$ for all $y \in \mathbb{R}$, $e^y = (-1)^n - \sqrt{2}$ never holds. Therefore

$$\sinh y = (-1)^n \Leftrightarrow e^y = (-1)^n + \sqrt{2} \Leftrightarrow y = \ln(\sqrt{2} + (-1)^n).$$

Therefore,

$$\sin z = i \Leftrightarrow z = n\pi + i \ln(\sqrt{2} + (-1)^n),\ n \in \mathbb{Z}. \tag{14}$$

Note that $\sqrt{2} - 1 = \dfrac{1}{\sqrt{2}+1}$ implies $\ln(\sqrt{2}-1) = -\ln(\sqrt{2}+1)$. And so,

$$\ln(\sqrt{2} + (-1)^n) = (-1)^n \ln(\sqrt{2}+1).$$

Thus, we may write (14) in the form:

$$\sin z = i \Leftrightarrow z = n\pi + (-1)^n \ln(\sqrt{2}+1)i = n\pi + i^{2n+1} \ln(\sqrt{2}+1),\ n \in \mathbb{Z}. \tag{15}$$

(b) Clearly,

$$\begin{aligned} \sin z = i \Leftrightarrow z &= \arcsin i \\ &= -i \log(i^2 + (1-i^2)^{\frac{1}{2}}) \\ &= -i \log(-1 + 2^{\frac{1}{2}}) \\ &= -i \log(-1 \pm \sqrt{2}) \\ &= -i(\ln|-1 \pm \sqrt{2}| + i \arg(-1 \pm \sqrt{2})) \\ &= \arg(-1 \pm \sqrt{2}) - i \ln|-1 \pm \sqrt{2}|. \end{aligned}$$

Since $|-1 \pm \sqrt{2}| = \sqrt{2} \mp 1$, $\arg(-1+\sqrt{2}) = 2n\pi,\ n \in \mathbb{Z}$, and $\arg(-1-\sqrt{2}) = (2n+1)\pi,\ n \in \mathbb{Z}$, we have $\sin z = i$ if and only if

$$z = 2n\pi - i\ln(\sqrt{2}-1) = 2n\pi + i\ln(\sqrt{2}+1),\ n \in \mathbb{Z}$$

or

$$z = (2n+1)\pi - i\ln(\sqrt{2}+1),\ n \in \mathbb{Z}.$$

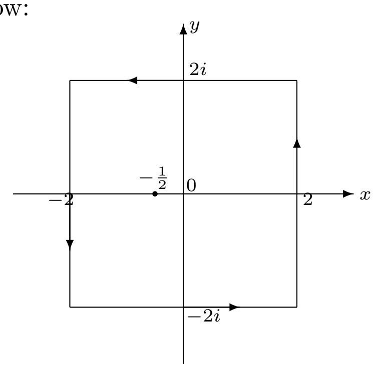
(Compare this result with (15).)

Question 4. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate the following integrals.

(a) $\displaystyle\oint_C \frac{z}{2z+1}\,dz.$ (10 points)

(b) $\displaystyle\oint_C \frac{e^{-3z}\sin(z^2)}{z+5}\,dz.$ (10 points)

Answer 4. The contour C is shown below:



(a) Using the method of partial fractions, we get

$$\frac{z}{2z+1} = \frac{z+\frac{1}{2}-\frac{1}{2}}{2z+1} = \frac{1}{2} - \frac{1}{2(2z+1)} = \frac{1}{2} - \frac{1}{4\left(z-\left(-\frac{1}{2}\right)\right)}.$$

Therefore

$$\oint_C \frac{z}{2z+1}\,dz = \oint_C \frac{1}{2}\,dz - \frac{1}{4}\oint_C \frac{1}{z-\left(-\frac{1}{2}\right)}\,dz. \tag{16}$$

Since $f(z) = \dfrac{1}{2}$ is analytic in a simply-connected region containing C, the first integral on the right hand side of (16) is 0 by the Cauchy-Goursat theorem.

On the other hand, we have proved in class that: *If C is a simple closed contour and z_0 is a fixed complex number such that z_0 lies interior to C, then*

$$\oint_C \frac{dz}{z-z_0} = 2\pi i.$$

Therefore, the second integral on the right hand side of (16) is $2\pi i$, and so

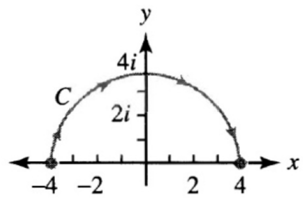
$$\oint_C \frac{z}{2z+1}\,dz = -\frac{1}{4}\cdot 2\pi i = -\frac{\pi i}{2}.$$

(b) Clearly, e^{-3z}, $\sin(z^2)$, and $z+5$ are entire functions, and so $f(z) = \dfrac{e^{-3z}\sin(z^2)}{z+5}$ is analytic everywhere except at $z=-5$. Since -5 does not lie on C or in the region interior to C, $\displaystyle\oint_C \frac{e^{-3z}\sin(z^2)}{z+5}\,dz$ is 0 by the Cauchy-Goursat theorem.

Question 5.

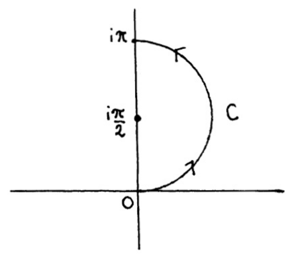
(a) Evaluate $\int_C \sinh 5z\,dz$, where the path C is parametrized as $z(t) = \frac{\pi}{2}\cos t + i\frac{\pi}{2}(\sin t + 1)$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. (10 points)

(b) Evaluate $\int_C x\,dz$ where C is the upper half of the circle $|z| = 4$, oriented clockwise as shown in the figure below. (10 points)



Answer 5.

(a) Note that $z(t) = \frac{\pi}{2}\cos t + i\frac{\pi}{2}(\sin t + 1) = \frac{\pi}{2}(\cos t + i\sin t) + i\frac{\pi}{2} = \frac{\pi}{2}e^{it} + i\frac{\pi}{2}$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Thus, C is the simple contour shown below:



Since C is a simple closed contour and $\sinh z$ is analytic in a simply connected region containing C, we can use the theorem proved in class on definite integrals involving antiderivatives to evaluate $\int_C \sinh 5z\,dz$. Clearly, $\frac{d}{dz}\left(\frac{\cosh 5z}{5}\right) = \sinh 5z$. Thus, $\frac{\cosh 5z}{5}$ is an antiderivative of $\sinh 5z$ and so we have

$$\int_C \sinh 5z\,dz = \int_0^{i\pi} \sinh z\,dz = \frac{\cosh 5z}{5}\Big|_0^{i\pi} = \frac{\cosh 5\pi i}{5} - \frac{\cosh 0}{5}.$$

Using the formula $\cosh z = \frac{e^z + e^{-z}}{2}$, one can easily evaluate that $\cosh 5\pi i = -1$ and $\cosh 0 = 1$, and so,

$$\int_C \sinh 5z\,dz = \int_0^{i\pi} \sinh z\,dz = -\frac{1}{5} - \frac{1}{5} = -\frac{2}{5}.$$

(b) We can use the parametrization $z(t) = 4e^{-it} = 4\cos t - 4i\sin t$, $\pi \le t \le 2\pi$ for C. Hence,

$$\begin{aligned}
\int_C x\,dz &= \int_\pi^{2\pi} 4\cos t(-4\sin t - 4i\cos t)\,dt \\
&= -16\int_\pi^{2\pi} \sin t\cos t\,dt - 16i\int_\pi^{2\pi} \cos^2 t\,dt \\
&= -8\int_\pi^{2\pi} \sin 2t\,dt - 8i\int_\pi^{2\pi} (\cos 2t + 1)\,dt \\
&= 4\cos 2t\Big|_\pi^{2\pi} - 4i\sin 2t\Big|_\pi^{2\pi} - 8it\Big|_\pi^{2\pi} \\
&= -8\pi i.
\end{aligned}$$

ÇANKAYA UNIVERSITY

Department of Mathematics and Computer Science

MATH 351 Complex Analysis I

Final

January 17, 2007

14:00-16:00

Surname : ______________________

Name : ______________________

ID # : ______________________

Department : ______________________

Section : ______________________

Instructor : ______________________

Signature : ______________________

- The exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding questions. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full credit.
- Calculators are not allowed.

GOOD LUCK!

Please do not write below this line.

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Question 1.

(a) Evaluate $\oint_C \frac{e^z \cos z}{z^4}\, dz$, where $C = \{z : |z| = 1\}$. (10 points)

(b) Evaluate $\frac{1}{2\pi}\int_0^{2\pi} \sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right)\, d\theta$. *Hint*: Use Gauss' mean value theorem. (10 points)

Answer 1.

(a) Let $f(z) = e^z \cos z$. Since f is entire, by the Cauchy integral formula for derivatives

$$f'''(0) = \frac{3!}{2\pi i}\oint_C \frac{e^z \cos z}{z^4}\, dz.$$

Clearly,

$$\begin{aligned} f'(z) &= e^z \cos z - e^z \sin z \\ f''(z) &= e^z \cos z - e^z \sin z - e^z \sin z - e^z \cos z = -2e^z \sin z \\ f'''(z) &= -2e^z \sin z - 2e^z \cos z. \end{aligned}$$

Therefore, $f'''(0) = -2$, and so

$$\oint_C \frac{e^z \cos z}{z^4}\, dz = -2\frac{2\pi i}{3!} = -\frac{2\pi i}{3}.$$

(b) Gauss' mean value theorem asserts that if f is a function analytic in a domain containing the disk $\{z : |z - z_0| \leq R\}$, then

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + Re^{i\theta})\, d\theta.$$

Since $\sin^2 z$ is entire we can apply the above formula for any $z_0 \in \mathbb{C}$ and any $R > 0$. Putting $f(z) = \sin^2 z$, $z_0 = \frac{\pi}{6}$ and $R = 2$, we get

$$\sin^2\left(\frac{\pi}{6}\right) = \frac{1}{2\pi}\int_0^{\pi} \sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right)\, d\theta.$$

Since $\sin^2\left(\frac{\pi}{6}\right) = \frac{1}{4}$, we have

$$\frac{1}{2\pi}\int_0^{\pi} \sin^2\left(\frac{\pi}{6} + 2e^{i\theta}\right)\, d\theta = \frac{1}{4}.$$

Question 2.

(a) Find all functions $f(z)$ which are entire and satisfy the conditions:

(i) $f(2-i) = 4i$.

and

(ii) $|f(z)| < e^2$ for all z.

(10 points)

(b) Find all functions $f(z)$ which are analytic in $R = \{z : |z| \leq 1\}$ and satisfy the conditions:

(i) $f(0) = 3$.

and

(ii) $|f(z)| \leq 3$ for all $z \in R$.

(10 points)

Answer 2.

(a) By (ii), f is bounded. Since f is entire, it is constant by the Liouville theorem. Since $f(2-i) = 4i$, this constant value should be $4i$. Therefore the only function satisfying (i) and (ii) is the constant function $f(z) = 4i$.

(b) Maximum modulus theorem asserts that a nonconstant analytic function on a bounded region cannot take its maximum modulus value at an interior point. By (i) and (ii), the maximum modulus value of f is 3 and taken at the origin, which is an an interior point. Therefore f should be constant. Since $f(0) = 3$, this constant value should be 3. Therefore the only function satisfying (i) and (ii) is the constant function $f(z) = 3$.

Question 3. Express the following quantities in $u + iv$ form

(a) $\tan\left(\frac{\pi + i}{2}\right)$. (10 points)

(b) $\text{Log}(i\sqrt{2} - \sqrt{2})$. (10 points)

Answer 3.

(a) Recall that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y.$$

Therefore,

$$\tan\left(\frac{\pi + i}{2}\right) = \frac{\sin\left(\frac{\pi+i}{2}\right)}{\cos\left(\frac{\pi+i}{2}\right)} = \frac{\sin\left(\frac{\pi}{2}\right)\cosh\left(\frac{1}{2}\right) + i\cos\left(\frac{\pi}{2}\right)\sinh\left(\frac{1}{2}\right)}{\cos\left(\frac{\pi}{2}\right)\cosh\left(\frac{1}{2}\right) - i\sin\left(\frac{\pi}{2}\right)\sinh\left(\frac{1}{2}\right)}.$$

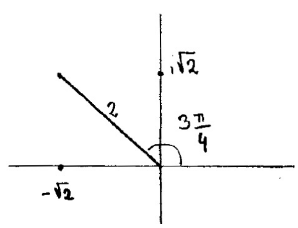
Noting that $\sin\left(\frac{\pi}{2}\right) = 1$, $\cos\left(\frac{\pi}{2}\right) = 0$, $\sinh\left(\frac{1}{2}\right) = \frac{e^{\frac{1}{2}} - e^{-\frac{1}{2}}}{2}$, and $\cosh\left(\frac{1}{2}\right) = \frac{e^{\frac{1}{2}} + e^{-\frac{1}{2}}}{2}$, we get

$$\tan\left(\frac{\pi + i}{2}\right) = \frac{\cosh\left(\frac{1}{2}\right)}{-i\sinh\left(\frac{1}{2}\right)} = i\frac{\cosh\left(\frac{1}{2}\right)}{\sinh\left(\frac{1}{2}\right)} = i\frac{\frac{e^{\frac{1}{2}}+e^{-\frac{1}{2}}}{2}}{\frac{e^{\frac{1}{2}}-e^{-\frac{1}{2}}}{2}} = i\frac{\sqrt{e} + \frac{1}{\sqrt{e}}}{\sqrt{e} - \frac{1}{\sqrt{e}}} = i\frac{e+1}{e-1}.$$

(b) By the definition,

$$\text{Log}(i\sqrt{2} - \sqrt{2}) = \ln|i\sqrt{2} - \sqrt{2}| + i\,\text{Arg}(i\sqrt{2} - \sqrt{2}),$$

where $-\pi < \text{Arg}(i\sqrt{2} - \sqrt{2}) \leq \pi$. Clearly, $|i\sqrt{2} - \sqrt{2}| = 2$ and $\text{Arg}(i\sqrt{2} - \sqrt{2}) = \frac{3\pi}{4}$ (See the figure below.).



Hence

$$\text{Log}(i\sqrt{2} - \sqrt{2}) = \ln 2 + i\frac{3\pi}{4}.$$

Question 4.

(a) Without evaluating the integral, show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}$$

where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. (10 points)

(b) Evaluate $\oint_C \frac{dz}{z^2 - 1}$ where $C = \{z : |z| = r\}$ with $r < 1$ or $r > 1$. (10 points)

Answer 4.

(a) On C, $|z| = 2$, and so $|z^2 - 1| \geq |z|^2 - 1 = 3$ and $\left| \frac{1}{z^2 - 1} \right| \leq \frac{1}{3}$. Since the length of C is π, using the ML-inequality, we obtain

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{\pi}{3}.$$

(b) Using the simple fractions method, we obtain

$$\frac{1}{z^2 - 1} = \frac{1}{2} \frac{1}{z - 1} - \frac{1}{2} \frac{1}{z + 1}.$$

Thus,

$$\oint_C \frac{dz}{z^2 - 1} = \frac{1}{2} \oint_C \frac{dz}{z - 1} - \frac{1}{2} \oint_C \frac{dz}{z + 1}.$$

If $r > 1$,

$$\oint_C \frac{dz}{z - 1} = \oint_C \frac{dz}{z + 1} = 2\pi i,$$

and if $r < 1$,

$$\oint_C \frac{dz}{z - 1} = \oint_C \frac{dz}{z + 1} = 0.$$

Hence in any case

$$\oint_C \frac{dz}{z^2 - 1} = 0.$$

Question 5.

(a) Determine where $f(z) = -2(xy + x) + i(x^2 - 2y - y^2)$ is differentiable, and evaluate the derivative at those points where it exists. (10 points)

(b) Determine whether $u(x, y) = 2x(1 - y)$ is harmonic in $\mathbb{C}$, and find its harmonic conjugate if it is harmonic. (10 points)

Answer 5.

(a) $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = -2(xy + x)$ and $v(x, y) = x^2 - 2y - y^2$. Since u and v are polynomials, they are continuously differentiable everywhere. Moreover

$$u_x(x, y) = -2y - 2,\ u_y(x, y) = -2x,\ v_x(x, y) = 2x,\ v_y(x, y) = -2 - 2y.$$

Since $u_x(x, y) = v_y(x, y)$ and $v_x(x, y) = -u_y(x, y)$ everywhere, Cauchy-Riemann equations are satisfied in $\mathbb{C}$ and hence f is differentiable everywhere. The derivative of f is

$$f'(z) = u_x(x, y) + iv_x(x, y) = -2y - 2 + i2x = 2i(x + iy) - 2 = 2iz - 2.$$

(b) Clearly, $u_x(x, y) = 2(1-y)$, $u_{xx}(x, y) = 0$, $u_y(x, y) = -2x$, and $u_{yy}(x, y) = 0$. Since u is two times continuously differentiable and satisfies the Laplace equation $u_{xx}(x, y)+v_{yy}(x, y) = 0$, it is harmonic in the whole complex plane. Let v be a harmonic conjugate of u. By the Cauchy-Riemann equations,

$$v_y(x, y) = u_x(x, y) = 2(1 - y),$$

and so

$$v(x, y) = \int 2(1 - y)\, dy = 2y - y^2 + \phi(x).$$

Since $v_x(x, y) = -u_y(x, y)$, we get

$$\phi'(x) = 2x,$$

and hence

$$\phi(x) = x^2 + C$$

for some $C \in \mathbb{R}$. Therefore

$$v(x, y) = 2y - y^2 + x^2 + C,$$

for some $C \in \mathbb{R}$.

MATH 351 Complex Analysis I

Make-up

January 23, 2007

15:00-17:00

Questions

(1) (a) Find all solutions of the equation $(-8-8\sqrt{3}i)^{\frac{1}{4}}$ in polar coordinates and mark them on the complex plane. (10 points)

(b) Find the square roots of $2i$. Indicate which one is the principal. (10 points)

(2) (a) Perform the required calculations and express your answers in the form $a+bi$.

(i) i^{275}. (5 points)

(ii) $\overline{(1+i\sqrt{3})(i+\sqrt{3})}$. (5 points)

(b) Find the following limits.

(i) $\lim_{z\to 1+i} \dfrac{z^2+z-2+i}{z^2-2z+1}$. (5 points)

(ii) $\lim_{z\to 1+i} \dfrac{z^2+z-1-3i}{z^2-2z+2}$. (5 points)

(3) Find the image of the right half-plane $\text{Re}(z) > 1$ under

(a) the mapping $f(z) = z^2+2z+1$. *Hint*: $z^2+2z+1 = (z+1)^2$. (10 points)

(b) the mapping $f(z) = \dfrac{1}{z}$. (10 points)

(4) (a) Assume that f is analytic in a region and that at every point, either $f=0$ or $f'=0$. Show that f is constant. *Hint*: Consider the function $(f(z))^2$. (10 points)

(b) Express the function $f(z) = z^5 + \bar{z}^3$ in the polar coordinate form $u(r,\theta)+iv(r,\theta)$. For what values of z is this expression valid? Why? (10 points)

(5) (a) Determine the region of the complex plane in which the function

$$f(z) = x^2 + iy^2, \quad z = x+iy$$

(i) is differentiable. (7 points)

(ii) is analytic. (3 points)

(b) Let $g(z) = \cos x \cosh y - i \sin x \sinh y,\ z = x+iy$. Show that $g'(z)$ and its derivative $g''(z)$ exist everywhere, and find $g''(z)$. (10 points)

MATH 351 Complex Analysis I

Make-up
January 23, 2007
15:00-17:00

QUESTIONS

(1) (a) Does an analytic function $f(z) = u(x, y) + iv(x, y)$ exist for which $v(x, y) = x^3 + y^3$? Why or why not? (10 points)

(b) Show that $u(x, y) = 2x - x^3 + 3xy^2$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$. (10 points)

(2) (a) Use the ratio test to find a disk in which the following series converges and find its sum in that disk.

$$\sum_{n=0}^{\infty} \frac{(z-i)^n}{(3+4i)^n}.$$

(10 points)

(b) Find the principal value of $(1+i)^{\pi i}$. (10 points)

(3) Find all solutions of the equation $\sin z = i$ by using

(a) the expression $\sin z = \sin x \cosh y + i \cos x \sinh y$. (10 points)

(b) the inverse function $\arcsin z = -i \log\left(iz + (1-z^2)^{\frac{1}{2}}\right)$. (10 points)

(4) Let $C = \{z : |z| = 1\}$. Evaluate the following integrals.

(a) $\displaystyle\oint_C \frac{z}{2z+1}\, dz.$ (10 points)

(b) $\displaystyle\oint_C \frac{1}{4z^2 - 4z + 5}\, dz.$ (10 points)

(5) (a) Evaluate $\displaystyle\int_C e^z\, dz$, where C is the line segment from 2 to $i\frac{\pi}{2}$. (10 points)

(b) Evaluate the integral $\displaystyle\int_C \overline{z}\, dz$ where C is the part of the circle $|z| = 2$ in the right half-plane from $z = -2i$ to $z = 2i$. (10 points)

MATH 351 Complex Analysis I

Make-up
January 23, 2007
15:00-17:00

Questions

(1) (a) Evaluate $\oint_C \frac{e^z}{z^2(z^2-16)}\,dz$, where $C = \{z : |z| = 1\}$. (10 points)

(b) Evaluate $\oint_C \frac{\sin z}{z^2+1}\,dz$, where $C = \{z : |z-i| = 1\}$. (10 points)

(2) (a) Let f be an entire function with the property that $|f(z)| \geq 1$ for all z. Show that f is constant. (10 points)

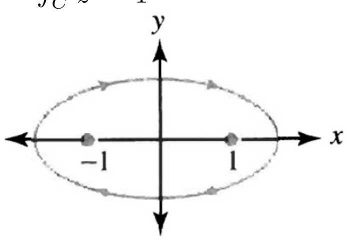
(b) Let f be a nonconstant analytic function in the closed disk $\{z : |z| \leq 1\}$. Suppose that $|f(z)| = 3$ for $z \in \{z : |z| = 1\}$. Show that f has a zero in D. (10 points)

(3) Express the following quantities in $u + iv$ form

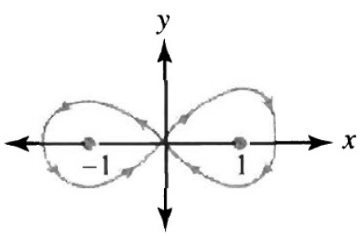
(a) $\sinh(1 + i\pi)$. (10 points)

(b) $(1+i)^{\pi i}$. (10 points)

(4) Evaluate $\int_C \frac{1}{z^2-1}\,dz$ for the contours shown below



(a) (10 points)



(b) (10 points)

(5) (a) Determine where $f(z) = x^3 - 3x^2 - 3xy^2 + 3y^2 + i(3x^2y - 6xy - y^3)$ is differentiable, and evaluate the derivative at those points where it exists. (10 points)

(b) Determine whether $u(x,y) = 2x - x^3 + 3xy^2$ is harmonic in $\mathbb{C}$, and find its harmonic conjugate if it is harmonic. (10 points)