Math 351
Old Exam Questions

2005-2006 Fall, Midterm 1

1. Let $z_1 = 4 - 3i$, $z_2 = 1 + i$ and $z_3 = -1 + 2i$.
   (a) Find $|z_1|$.
   (b) Find $\mathrm{Im}(z_1 z_2)$.
   (c) Write $\dfrac{z_1}{z_2 \bar{z}_3}$ in the standard form.
2. Write $\dfrac{-2i}{1+i}$ in the exponential form. What is the principal argument?
3. Find all roots of the equation $z^3 + 1 = 0$, write them in the standard form and locate them in the complex plane.
4. Find the regions in the complex plane in which the following functions are analytic and determine their derivative in those regions.
   (a) $f(z) = y^2 + ix^2$.
   (b) $f(z) = \dfrac{1}{z} + 1$.
5. Show that the function $u = x + e^y \cos x$ is harmonic and find its harmonic conjugate.

2005-2006 Fall, Midterm 2

1. A polynomial $p(z)$ of degree four has a zero with multiplicity 2 at the point $z = -1$ and two zeros, each with multiplicity 1, at points $z = i$ and $z = -i$. Find $p(z)$ if $p(1) = 16$.
2. Write the polynomial $p(z) = z^4 - z^2 + z + 1$ in the Taylor form centered at $z = -1$.
3. Find all solutions of the equation $\cos z = \sin z$.
4. Solve the equation $\mathrm{Log}(z^2 - 1) = \dfrac{i\pi}{2}$ for z.
5. Find the principal value of $4^{\frac{1}{2}}$.
6. Find the derivative of the principal branch of z^{i+1} at $z = i$.
7. Find all roots of the equation $\cosh z = -2$.
8. Indicate a region in which the function $f(z) = \dfrac{z^{\frac{1}{2}}}{z^2 + 4}$ is analytic.

2005-2006 Fall, Final

1. Let C denote the positively oriented circle $|z+\pi i| = 2$. Use the Cauchy integral formula to evaluate the integral $\displaystyle\int_C \frac{e^z\,dz}{(z^2+\pi^2)^2}$.

2. Evaluate the contour integral $\displaystyle\int_C 2\bar{z}\,dz$ where $C = C_1 + L$ is the contour consisting of the parabolic arc $y = x^2$ from the origin $z = 0$ to the point $z = 1+i$, joined by the straight line from the point $z = 1+i$ to the point $z = -1$, as shown in the figure below.

3. Let C denote the circle $|z - z_0| = R$, taken counterclockwise. Use the principal branch of the integrand to evaluate $\displaystyle\int_C (z-z_0)^{a-1}\,dz$ where a is any real number other than zero. What is the value of the integral when $a = 1, 2, 3, \cdots$?

4. Let C denote the circle $|z| = \rho$, $(0 < \rho < 1)$, oriented counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-\frac{1}{2}}$ represents any particular branch of that power of z,then there is a nonnegative constant M, independent of ρ, such that $\left|\displaystyle\int_C z^{-\frac{1}{2}} f(z)\,dz\right| \leq 2\pi M\sqrt{\rho}$. Thus, show that the value of the integral approaches zero as ρ tends to zero.

5. Find all roots of the equation $\sinh z = i$.

6. Show that $u(x,y) = \sin x \sinh y$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$.

7. Show that the function $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$, $(r > 0,\ 0 < \theta < 2\pi)$ is analytic in the indicated domain and satisfies $zf'(z) = if(z)$.

2004-2005 Fall, Midterm 1

1. (a) Prove that $\mathrm{Re}(iz) = -\mathrm{Im}(z)$ and $\mathrm{Im}(iz) = \mathrm{Re}(z)$ for any complex number z.
   (b) If a is a real and z is a complex number, prove $\mathrm{Re}(az) = a\,\mathrm{Re}(z)$ and $\mathrm{Im}(az) = a\,\mathrm{Im}(z)$.
   (c) Prove that $\mathrm{Re} : \mathbb{C} \to \mathbb{R}$ is a linear map.

2. (a) Using polar coordinates, prove that $z \to z + \dfrac{1}{z}$ maps the unit circle $|z| = 1$ to the interval $[-2, 2]$ on the x-axis.
   (b) Calculate the n^{th} root of the unity. Explain clearly.

3. (a) Are the real and imaginary parts of $f(z) = z^4$ harmonic or not? Explain clearly.
   (b) If f is analytic on $A = \{z \,|\, \mathrm{Re}\, z > 1\}$ and $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ on A, then prove f' is constant on A.

4. (a) Write the following functions in the form $w = u(x,y) + iv(x,y)$.
      i. $f(z) = \dfrac{1}{z}$.
      ii. $g(z) = \dfrac{z+i}{z^2+1}$.
   (b) If $J(z) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$, then calculate $J\left(\dfrac{1}{z}\right)$. Explain clearly.

2004-2005 Fall, Midterm 2

1. Prove that there does not exist an analytic function defined on $\mathbb{C}\backslash\{0\}$ such that $f'(z) = \frac{1}{z}$. Explain clearly.

2. (a) Calculate $\int_C z^3\, dz$, where C is the portion of the ellipse $x^2 + 4y^2 = 1$, that joins $z = 1$ to $z = \frac{i}{2}$.

   (b) Find an upper bound for $\left|\int_C \frac{e^z}{z^2+1}\, dz\right|$, where C is the circle $|z| = 2$ traversed once in the counterclockwise direction. Explain clearly.

3. (a) Prove that $\cosh^2 z - \sinh^2 z = 1$.

   (b) Calculate $\sinh(z_1 + z_2)$.

   (c) Calculate $\sin(2i)$.

   Explain clearly.

4. (a) Differentiate the following functions, giving the appropriate region on which the functions are analytic.

      i. e^{e^z}.

      ii. $\sin(e^z)$

   (b) Calculate.

      i. $\log i$.

      ii. $\log(1 - i)$.

2004-2005 Fall, Final

1. (a) Prove that $|z_1 z_2 z_3| = |z_1||z_2||z_3|$.

   (b) Calculate $e^{z+\pi i}$.

   (c) Solve the following equation: $\sin z = 0$.

   Explain clearly.

2. Evaluate $\int_C \bar{z}^2\, dz$ along the two paths joining $(0, 0)$ to $(1, 1)$ as follows.

   (a) C is the straight line from $(0, 0)$ to $(1, 1)$.

   (b) C is the broken line joining $(0, 0)$ to $(1, 0)$, then $(1, 0)$ to $(1, 1)$.

   Explain clearly.

3. (a) Prove that $\sin^{-1} z = -i \log\left[iz + (1 - z^2)^{\frac{1}{2}}\right]$.

   (b) Prove that $\cos^{-1} z = -i \log\left[z + (z^2 - 1)^{\frac{1}{2}}\right]$.

   Explain clearly.

4. (a) Is $f(z) = |z|$ analytic or not? Explain clearly.

   (b) Let f be an analytic function on an open connected set A and suppose $f^{(n+1)}(z)$ exists and is 0 on A. Then show that f is a polynomial of degree $\leq n$. Explain clearly.

5. Prove that: A map $f : A \to \mathbb{C}$, when A is open, is continuous if and only if for every open set $U \subset \mathbb{C}$, $f^{-1}(U)$ is open. Use this to prove that if $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ are continuous and $f(A) \subset B$, then $g \circ f$ is continuous. Explain clearly.

2003-2004 Fall, Midterm 1

1. (a) Express $2i^{27} - \dfrac{5}{i^{127}} + 3i^{-5}$ in the form $a + bi$.

   (b) Express $(1-i)^3$ in the form $re^{i\theta}$.

2. Find all the values of $\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)^{\frac{1}{4}}$.

3. Sketch the following regions:

   (a) $|z - 2 - i| < 5$

   (b) $-2 \leq \operatorname{Im} z \leq 2$

4. Find the region where $f(z) = \dfrac{1}{z+1}$ is analytic by using Cauchy-Riemann equations. Is this function entire?

5. (a) Show that the function $u = x^3 - 3xy^2 - 2x^2 + 2y^2 + 5x$ is harmonic.

   (b) Find a harmonic conjugate to u.

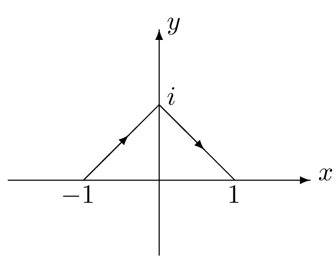
2003-2004 Fall, Midterm 2

1. Find all values of z which satisfies the equation $\sin z = i$.

2. Find the principal value of z^w where $z = 16i$ and $w = \dfrac{1}{4i}$.

3. Let $f(z) = 3\tanh(2z)$. Find the explicit form of $f^{-1}(z)$.

4. Find two different parametrizations for the contour seen in the figure.



5. Evaluate the integral $\displaystyle\int_\Gamma (3z + 5\bar{z})\, dz$ where Γ is the line segment from origin to $2 + i$.

2003-2004 Fall, Final

1. Are the following functions entire? Explain.

   (a) $f(z) = z^2 + \bar{z}$.

   (b) $g(z) = \dfrac{1}{1+z^2}$.

   (c) $h(z) = x^3 - 3xy^2 + 3x^2 - 3y^2 + i(3x^2y - y^3 + 6xy)$.

2. Find all values of $(-1)^{\frac{1}{6}}$.

3. Find all the solutions of the equations
   (a) $\sinh z = 0$
   (b) $\cosh z = 0$

4. Evaluate the integral $\int_C z^2\,dz$ where C is the line segment from origin to $1+3i$.

5. Evaluate the integral $\int_C \frac{1-z^2}{z(1+z^2)}\,dz$ where C is
   (a) The circle $|z-5-i| = 1$.
   (b) The circle $|z| = \frac{1}{2}$.

6. Evaluate the integral $\int_C \frac{dz}{(z^2+9)^2}$ where C is the rectangle with vertices $z_1 = 6,\ z_2 = -6,\ z_3 = -6-6i,\ z_4 = 6-6i$.

   Hint: If f is analytic inside and on the simple closed contour Γ and if z is any point inside Γ, then $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(s)}{(s-z)^{n+1}}\,ds$.

2002-2003 Fall, Midterm 1

1. Either prove the following claim, or give a counter example: An open set contains no accumulation points.

2. Sketch the region $|z-i| + |z+i| < 5,\ \operatorname{Im} z > -1$.

3. Evaluate $(-24-24\sqrt{3}i)^{\frac{1}{4}}$.

4. Find the image of $|z-3| = 1$ under the mapping $w = 4iz + 2i$.

5. Let $f(z) = \frac{x^3+x^2+xy^2-y^2+i(x^2y+2xy+y^3)}{x^2+y^2+2x+1}$. Is f differentiable?

6. Determine where the derivative of $f(z) = x^2 + iy^2$ exists and evaluate the derivative there.

2002-2003 Fall, Midterm 2

1. Show that $u(x,y) = 4x^3y - 4xy^3$ is harmonic on any domain and find a harmonic conjugate.

2. Find all roots of the equation $e^z = -1-\sqrt{3}i$.

3. Find all roots of the equation $\cos z = 2$.

4. Find the principal value of $(\sqrt{2}+i\sqrt{2})^{\frac{2}{3}}$.

5. Let $z = f(w) = \sinh(2w)$. Find the inverse function $w = f^{-1}(z)$ as an explicit function of z.

6. Evaluate $\int_C \frac{dz}{\sqrt{z-i}}$ where C is the circle $|z-i| = 4$, counterclockwise.

   Hint: Use the principal branch $-\pi < \theta \leq \pi$.

2002-2003 Fall, Final

1. Find all the roots of the equation $z^5 + 1 = 0$.
2. Is the function $f(z) = \dfrac{1}{\cosh z}$ entire? Why, why not?
3. Find all roots of the equation $\tanh z = \dfrac{1}{2}$.
4. Find the principal value of $(-2\sqrt{3} + 2i)^{2-i}$.
5. Evaluate the integral $\int_C \bar{z}\, dz$ where C is the part of the curve $y = x^3$ from $z_1 = -2-8i$ to $z_2 = 1+i$.
6. Evaluate $\int_C \dfrac{z^4 + 64}{z^4 - 64}\, dz$ where C is the circle given by
   (a) $C: |z + 5i| = 3$.
   (b) $C: |z + 5i| = 1$.
7. Let $P(z) = z + z^2 + z^3$. Show that, when $|z| > 2$, $|P(z)| < 2|z|^3$.

2001-2002 Summer, Midterm 1

1. (a) Evaluate $\sqrt[3]{-27}$.
   (b) Simplify the expression $\dfrac{(3-i)(1+i)}{4+2i}$.
2. Express $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$.
3. Sketch the region $|z - 2| < 2$, $\operatorname{Im} z < 0$.
4. Find the image of the line segment $x = 4$, $-1 < y < 1$ under the transformation $w = z^2$.
5. Find the points where the function $f(z) = |z|^2 + \bar{z}^2$ is differentiable.
6. Is the function $f(z) = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy)$ entire? (analytic everywhere)

2001-2002 Summer, Midterm 2

1. Let $u(x, y) = \dfrac{1}{\sqrt{x^2 + y^2}}$. Is u harmonic?
2. Let $\tanh z = \dfrac{1}{3}$. Find z.
3. Evaluate $\left(\dfrac{1-i}{1+i}\right)^{2i}$.
4. Find real and imaginary parts of $\sin(3 + 5i)$.
5. Let $e^{x+iy} = \operatorname{Log}(\sqrt{2} + i\sqrt{2})$. Find x and y.

2001-2002 Summer, Final

1. Evaluate $(-8-8\sqrt{3}i)^{\frac{1}{4}}$.
2. Let $f(z) = 1 + x^2 - y^2 + 3xy^2 - x^3 + i(y^3 - 3yx^2 + 2xy)$. Is f entire?
3. Find the real and imaginary parts of $\mathrm{Log}(\mathrm{Log}(1+i))$.
4. Evaluate $(i-1)^{i-1}$.
5. Evaluate $\displaystyle\int_C \frac{1}{\sqrt{z-1}}\,dz$ where $C:\ |z-1| = 1$.
6. Evaluate $\displaystyle\int_C \frac{z^2+1}{(z^2+9)(z^2+25)}\,dz$ where $C:\ |z-5i| = 1$.

2001-2002 Fall, Midterm 1

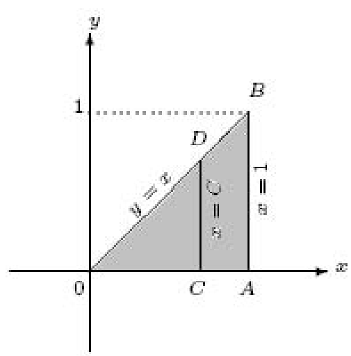
1. Find all roots of the equation $z^4 + 1 = 0$ in rectangular coordinates, exhibit them geometrically, and point out which is the principal root.
2. Show that when $w = f(z) = z^2 = (x^2 - y^2) + i(2xy) = u(x,y) + iv(x,y)$, the image of the closed triangular region

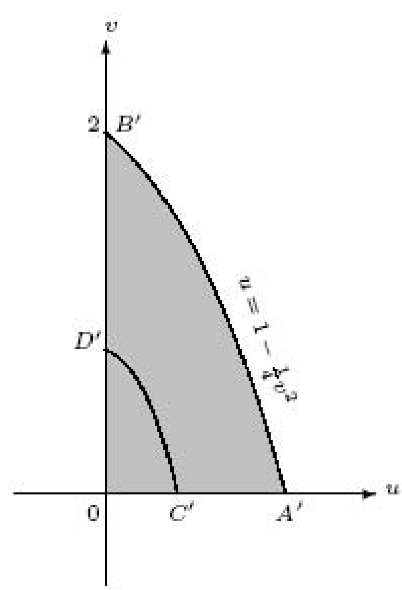
$$S = \{(x,y)\big|\, 0 \le x \le 1, 0 \le y \le x\}$$

is the closed semiparabolic region

$$S' = \{(u,v)\big|\, 0 \le v \le 2, 0 \le u \le 1 - \frac{1}{4}v^2\}$$

Verify the corresponding points on the two boundaries shown in the figure.





3. (a) Give the definition of $\lim_{z\to\infty} f(z) = w_0$, and prove that

$$\lim_{z\to\infty} f(z) = w_0 \quad \text{if and only if} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0.$$

(b) Use part (a) to show that when

$$f(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0), \quad \lim_{z\to\infty} f(z) = \frac{a}{c} \text{ if } c \neq 0.$$

4. Use the Theorem about the existence of $f'(z)$ to show that the function

$$f(z) = \sin x \cosh y + i \cos x \sinh y$$

has a derivative at every point $z = (x, y)$ and find $f'(z)$.

5. Suppose the component functions u and v of

$$w = f(z) = u(x, y) + iv(x, y)$$

have continuous first-order partial derivatives in a neighborhood $|z - z_0| < \delta$ of a nonzero point $z_0 = r_0 e^{i\theta_0}$. The real and imaginary parts of $w = u + iv$ are expressed in terms of r and θ by means of the coordinate transformation

$$x = r\cos\theta, \; y = r\sin\theta.$$

(a) Use the chain rule to show that

$$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \; u_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$\left(\text{Similarly, then } v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r}, \; v_y = v_r \sin\theta + v_\theta \frac{\cos\theta}{r}\right)$$

(b) If the partial derivatives with respect to x and y satisfy the Cauchy-Riemann equations at $z_0 = x_0 + iy_0$

$$u_x = v_, \; u_y = -v_x$$

show that the partial derivatives of u and v with respect to r and θ satisfy the polar form of the Cauchy-Riemann equations at $z_0 = r_0 e^{i\theta_0}$

$$u_r = \frac{1}{r} v_\theta, \; \frac{1}{r} u_\theta = -v_r.$$

Hint: Use part (a).

2001-2002 Fall, Midterm 2

1. Let $f(z) = e^{\frac{1}{z}}$.

   (a) State why f is analytic in any domain D that does not contain the origin.

   (b) Write $f(z)$ in the form $f(z) = u(x, y) + iv(x, y)$. Why is $\text{Re}(e^{\frac{1}{z}})$ harmonic in any domain D that does not contain the origin? Is $\text{Im}(e^{\frac{1}{z}})$ also harmonic in such a domain?

2. (a) Show that $\sin z = -i \sinh(iz)$, $\cos z = \cosh(iz)$.

   (b) Derive the identity: $|\cosh z|^2 = \sinh^2 x + \cos^2 y$, for all $z = x + iy$, and use it to find all zeros of $\cosh z$.

3. Find all values of $(1 - i)^{4i}$ and indicate which one is the principal value.

4. (a) Derive the formula $\cos^{-1} z = -i \log[z + i(1 - z^2)^{\frac{1}{2}}]$.

   (b) Use the formula in part (a) to find all values of $\cos^{-1} \sqrt{2}$.

5. Let $f(z)$ be the principal branch

$$z^{-1+i} = e^{(-1+i)\,\mathrm{Log}\,z} \quad (|z| > 0,\ -\pi < \mathrm{Arg}\,z < \pi)$$

of the indicated power function.

(a) Write $f(z)$ in the form $f(z) = u(r,\theta) + iv(r,\theta)$ for $z = re^{i\theta}$ in the domain $D : r > 0$, $-\pi < \theta < \pi$ and show that $|f(z)| = \dfrac{1}{r}e^{-\theta}$.

(b) Evaluate $f'(z)$.

2001-2002 Fall, Final

1. Let $f(z)$ be the branch

$$z^{-1+i} = e^{(-1+i)\,\mathrm{Log}\,z} \quad (|z| > 0,\ -\pi < \mathrm{Arg}\,z < \pi)$$

of the indicated power function.

(a) Use the parametric representation $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$ for the unit circle $C : |z| = 1$ to evaluate the integral $I = \displaystyle\int_C f(z)\,dz$.

(b) Show that $f(z) = z^{-1+i} = e^{(-1+i)\,\mathrm{Log}\,z}$ has an antiderivative $F(z) = -iz^i$ in the domain $D :\ |z| > 0,\ -\pi < \mathrm{Arg}\,z < \pi$, and evaluate $\displaystyle\int_C f(z)\,dz = \int_{-i}^{i} f(z)\,dz$ where C is any contour from $-i$ to i that except for its end points, lies in the right half plane $x > 0$.

2. (a) Evaluate $\displaystyle\int_C \frac{1}{z}\,dz$ on the circle $C : |z| = R$, taken in the positive sense.

(b) Show that the integral of part (a) is 0 for every simple closed contour C not enclosing the origin and not through the origin.

(c) Show that $\displaystyle\int_C \frac{1}{z^2}\,dz = 0$ for every simple closed contour C not through the origin.

Hint: $f(z) = \dfrac{1}{z^2}$ has an antiderivative $F(z) = -\dfrac{1}{z}$ in the domain D which consists of all complex numbers $z \neq 0$.

3. Let $C : |z| = 2$, $C_1 : |z-1| = \dfrac{1}{2}$, and $C_2 : |z+1| = \dfrac{1}{2}$, all described in the counterclockwise direction.

(a) Apply the extension of the Cauchy-Goursat Theorem to integrals along the boundary of a multiply connected domain to show that

$$\int_C \frac{1}{z^2-1}\,dz = \int_{C_1} \frac{1}{z^2-1}\,dz + \int_{C_2} \frac{1}{z^2-1}\,dz.$$

(b) Evaluate $\displaystyle\int_{C_1} \frac{1}{z^2-1}\,dz = \int_{C_1} \frac{\frac{1}{z+1}}{z-1}\,dz$.

(c) Evaluate $\displaystyle\int_{C_2} \frac{1}{z^2-1}\,dz = \int_{C_2} \frac{\frac{1}{z-1}}{z-(-1)}\,dz$.

(d) Evaluate $\int_C \frac{1}{z^2-1}\,dz$ by using the parametric representation $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$ for C, and check your answer by the result in part (a).

4. Suppose $f(z)$ is entire and there is a nonnegative constant M such that $|f(z)| \le M$ for all z. Let C be a circle $|z - z_0| = R$, taken in the positive sense, where z_0 is any fixed complex number.

   (a) Show that $|f'(z_0)| = \left| \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^2}\,dz \right| \le \frac{M}{R},\ R > 0.$

   (b) Use part (a) to show that $f'(z_0) = 0$ for every complex number z_0.

   (c) Give an example of an entire and bounded function in the complex plane.

5. Let f be the function $f(z) = e^z$ and R the rectangular region $0 \le x \le 1,\ 0 \le y \le \pi$.

   (a) Show that $|f(z)|$ is continuous on R and it takes its maximum value at some point z_0 in R.

   (b) Illustrate the use of the Maximum Modulus Principle by finding points in R where $|f(z)|$ reaches its maximum value.

2000-2001 Fall, Midterm 1

1. By writing $1+\sqrt{3}i$ in the exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

$$(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i).$$

2. Find all roots $(-8-8\sqrt{3}i)^{\frac{1}{4}}$ in rectangular coordinates, exhibit them geometrically, and point out which is the principal root.

3. Show, indicating corresponding orientations, that the mapping $w = f(z) = z^2$ transforms lines $y = c$ $(c > 0)$ into parabolas $v^2 = 4c^2(u + c^2)$, all with foci at $w = 0$.

4. Show that the function $f(z) = e^y(\cos x - i \sin x)$ is differentiable at every point $z = x + iy = (x, y)$, and evaluate $f'(z)$.

5. Suppose that the function $f(z) = u(r,\theta) + iv(r,\theta)$ has the derivative $f'(z)$ at (r,θ) so that the polar form of the Cauchy-Riemann equations

$$u_r = \frac{1}{r}v_\theta,\ \frac{1}{r}u_\theta = -v_r$$

are satisfied, by the partial derivatives of u and v, at (r,θ). Show that

$$u_x = u_r\cos\theta - u_\theta\frac{\sin\theta}{r}, \qquad u_y = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$$
$$v_x = v_r\cos\theta - v_\theta\frac{\sin\theta}{r}, \qquad v_y = v_r\sin\theta + v_\theta\frac{\cos\theta}{r}$$

Then use this equations to show that the partial derivatives satisfy the Cauchy-Riemann equations in Cartesian coordinates $u_x = v_y,\ u_y = -v_x$ at $z = (x, y)$.

2000-2001 Fall, Midterm 2

1. (a) Show in two ways that $u(x,y) = \dfrac{y}{x^2+y^2}$ is harmonic in any domain D which doesn't contain $0=(0,0)$. (Hint: Show that $u(x,y) = \text{Re}[f(z)]$, where $f(z) = \dfrac{i}{z}$.)

   (b) Find a harmonic function conjugate $v(x,y)$ of $u(x,y) = \dfrac{y}{x^2+y^2}$.

2. (a) Use the reflection principle to show that

$$\overline{\sinh z} = \sinh \bar{z} \qquad \text{for all } z.$$

   (b) Show that $\sinh z = \sinh x \cos y + i \cosh x \sin y$, where $z = x+iy$. Find all roots of the equation $\sinh z = i$ by equating real parts and imaginary parts in that equation.

3. (a) Show that $\cos z = \cos x \cosh y - i \sin x \sinh y$, where $z = x+iy$. With the aid of the identity above show that $\sin x \sinh y$ is everywhere harmonic.

   (b) Solve the equation $\cos z = \sqrt{2}$ for z, by using the identity obtained in part (a).

4. Find all values of the following powers:

   (a) $\left(\dfrac{1-i}{\sqrt{2}}\right)^{1+i}$,

   (b) i^i.

5. Let $f(z)$ be the principal branch

$$z^{-1+i} = e^{(-1+i)\,\text{Log}\, z} \qquad (|z|>0,\ -\pi < \text{Arg}\, z < \pi)$$

of the indicated power function, and let C be the positively oriented unit circle $|z|=1$. Use a parametric representation for C to evaluate the integral $I = \displaystyle\int_C f(z)\,dz$.

2000-2001 Fall, Final

1. Apply the Cauchy-Goursat Theorem to show that $\displaystyle\int_C f(z)\,dz = 0$ when the contour C is the circle $|z|=1$, in either direction, and when

   (a) $f(z) = \text{Log}(z+2)$.
   (Hint: Show that $f(z)$ is analytic everywhere except on the half line $x \le -2,\ y=0$.)

   (b) $f(z) = \tan z$.
   (Hint: Show that none of the singularities of $f(z)$ lies within and on C.)

2. (a) Let C denote the positively oriented boundary of the square region R bounded by the lines $x=\pm 2$, and $y = \pm 2$. Evaluate the integral:

$$\int_C \frac{\cos z}{z(z^2+8)}\,dz.$$

   (Hint: Show that $f(z) = \dfrac{\cos z}{z^2+8}$ is analytic inside and on C.)

(b) Find the value of the integral $\int_C \frac{1}{(z^2+4)^2}\,dz$, where C is the circle $|z-i|=2$ in the positive sense.

(Hint: Show that $f(z) = \frac{1}{(z+2i)^2}$ is analytic within and on C, and then evaluate $\int_C \frac{f(z)}{(z-2i)^2}\,dz$.)

3. (a) Suppose that $z = z(t)$, $a \le t \le b$, represents contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. Let the function $f(z)$ be piecewise continuous on C. Let L be the length of C, and M be the maximum of $|f(z)|$ on C. Show that

$$\left|\int_C f(z)\,dz\right| \le ML.$$

(b) Let z_0 be a fixed point in the plane. If $f(z)$ is analytic within and on a circle $C: |z-z_0| = R$, taken in the positive sense we know that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz,\ n = 0, 1, 2, \cdots.$$

Let M_R be the maximum of $|f(z)|$ on C. Apply the inequality in part (a) to obtain the Cauchy's inequality:

$$|f^{(n)}(z_0)| \le \frac{n! M_R}{R^n} \quad (n = 0, 1, 2, \cdots.)$$

4. (a) State the Liouville's Theorem.

(b) Suppose that $f(z) = u(x,y) + iv(x,y)$ is entire and that $u(x,y)$ has an upper bound; that is, $u(x,y) \le u_0$ for all points (x,y) in the xy-plane. Show that $u(x,y)$ must be constant throughout the plane.

(Hint: Apply Liouville's Theorem to the function $g(z) = e^{f(z)}$.)

5. (a) State the Theorem which is known as the Maximum Modulus Principle.

(b) Let a function f be continuous in a closed and bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a minimum value in R which occurs on the boundary of R and never in the interior.

(Hint: Apply the Corollary of the Maximum Modulus Principle to the function $g(z) = \frac{1}{f(z)}$.)

1999-2000 Fall, Midterm 1

1. (a) Simplify $\frac{2}{(1-i)(3-i)(i+2)}$.

(b) Show that $|z_1 z_2| = |z_1||z_2|$ where z_1 and z_2 are any two complex numbers.

2. (a) Find all the roots of $z^4 = -2(1+\sqrt{3}i)$.

(b) Write the de Moivre's formula, then use it to prove any trigonometric formula you like.

3. (a) Write the Cauchy-Riemann equations in Cartesian coordinates for the function $f(z)$.

(b) Derive the Cauchy-Riemann equations in polar coordinates for the function $f(z)$ assuming that $f'(z)$ exists at any point z_0.

4. (a) Give an example of a function of two variables $u(x,y)$ that is polynomial in x and y of degree at least three and is harmonic.

(b) Find a harmonic conjugate of $u(x,y)$ obtained in part (a).

1999-2000 Fall, Midterm 2

1. Evaluate $\int_C \pi e^{\pi \bar{z}}\, dz$, where C is the boundary of the square with vertices at the points 0, 1, $1+i$ and i, the orientation of C being in the counterclockwise direction.

2. (a) Find all the values of $\sinh^{-1}(i)$.
   (b) Show that $\sin^{-1}(-i) = n\pi + i(-1)^{n+1}\ln(1+\sqrt{2}),\ n = 0, \pm 1, \pm 2, \cdots$.

3. (a) Show that

$$\cos z = \cos x \cosh y - i \sin x \sinh y \quad \text{and} \quad \sin z = \sin x \cosh y - i \cos x \sinh y.$$

   (b) Show that

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \quad \text{and} \quad |\sin z|^2 = \sin^2 x + \sinh^2 y.$$

4. (a) Show that $(1+i)^{2-i} = 2e^{\frac{\pi}{4} \pm 2n\pi}\left[\sin\left(\frac{1}{2}\ln 2\right) + i\cos\left(\frac{1}{2}\ln 2\right)\right]$.
   (b) (Cauchy's integral formula) Let $f(z)$ be analytic in a simply connected domain D. Then for any point z_0 in D and any simple closed path C in D that encloses z_0

$$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0),$$

   where the integration being taken counterclockwise.
   Use Cauchy formula to evaluate $\int_C \frac{z^2+1}{z^2-1}\, dz$, where C is the circle $|z-1| = 1$ in the counterclockwise direction.

1999-2000 Fall, Final

1. Show that when $|z_3| \neq |z_4|$, $\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$.

2. Find the four roots of the equation $z^4 + 4 = 0$ and use them to factor $z^4 + 4$ into quadratic factors with real coefficients.

3. Solve the equation $\cos z = \sqrt{2}$ for z.

4. Show that $\mathrm{Log}(1+i)^2 = 2\,\mathrm{Log}(1+i)$ but that $\mathrm{Log}(-1+i)^2 \neq 2\,\mathrm{Log}(-1+i)$.

5. Show that $|p_n(x)| \leq 1$ for all $x \in [-1, 1]$, where

$$p_n(x) = \frac{1}{\pi}\int_0^\pi [x + i\sqrt{1-x^2}\cos\theta]^n\, d\theta,\ n = 0, 1, 2, \cdots.$$

6. Use the Cauchy-Goursat theorem to show that $\int_C \frac{dz}{z^2 + 2z + 2} = 0$, where the contour C is the circle $|z| = 1$.

7. Let C denote the boundary of the square whose sides along the lines $x = \pm 2$ and $y = \pm 2$ in the positive sense. Evaluate.

   (a) $\int_C \frac{\cos z}{z(z^2+8)}\, dz$.

(b) $\displaystyle\int_C \frac{\sin z}{(z-3)(z+i)^2}\,dz.$

8. Show that $e^{nz} = (\cosh z + \sinh z)^n = \cosh nz + \sinh nz$. Then use this to find formulas for $\cosh 2z$ and $\sinh 2z$ in terms of $\sinh z$ and $\cosh z$.

9. Prove that $\cos\theta + \cos 3\theta + \cos 5\theta + \cdots + \cos(2n-1)\theta = \dfrac{\sin 2n\theta}{2\sin\theta}$.

10. (a) State Morera's theorem.

    (b) Evaluate $\left|\dfrac{\sqrt{5}+3i}{1-i}\right|$.