Complex Numbers

The Algebra of Complex Numbers

The complex field $\mathbb{C}$ is the set of ordered pairs of real numbers (a,b) with addition and multiplication defined by

$$(a,b) + (c,d) = (a+c,\ b+d)$$
$$(a,b)\cdot(c,d) = (ac-bd,\ ad+bc).$$

Example If $z_1 = (3,7)$ and $z_2 = (5,-6)$, then find z_1+z_2 and $z_1 z_2$.

Solution $z_1 + z_2 = (3+5,\ 7+(-6)) = (8,1)$

and $z_1 z_2 = (3.5 - 7(-6),\ 3(-6)+7.5) = (57, 17)$.

Properties For any $z_1, z_2, z_3 \in \mathbb{C}$

(1) $z_1 + z_2 = z_2 + z_1$ (commutativity of addition)

$z_1 z_2 = z_2 z_1$ (" " multiplication)

(2) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associativity of addition)

$z_1(z_2 z_3) = (z_1 z_2) z_3$ (" " multiplication)

(3) $z_1(z_2+z_3) = z_1 z_2 + z_1 z_3$ (distributivity of multiplication through addition)

(4) (Existence of additive and multiplicative identities).

There is a complex number w such that $z + w = z$ for all $z \in \mathbb{C}$.

(The number w is obviously the ordered pair $(0,0)$.)

There is a complex number ζ such that

$$z\zeta = z \quad \text{for each } z \in \mathbb{C}.$$

(Clearly, $(1,0)$ is such a ζ. Indeed, $(a,b)(1,0) = (a.1 - b.0,\ a.0 + b.1) = (a,b)$)

(5) (Existence of additive and multiplicative inverses)

For any $z \in \mathbb{C}$, there is a unique complex number η such that $z + \eta = (0,0)$.

(If $z = (x,y)$, obviously $\eta = (-x,-y)$.)

For any complex number $z \neq (0,0)$, there is a unique complex number, which we denote by z^{-1} such that $zz^{-1} = (1,0)$.

Q: How to find such a z^{-1} if $z = (x,y)$?

A: we need to find $z^{-1} = (a,b)$ such that

$$(x,y)(a,b) = (1,0).$$

Clearly, $(x,y)(a,b) = (xa - yb,\ xb + ya) = (1,0)$

that is $xa - yb = 1$ (*) and $xb + ya = 0$ (**)

Solving (**) for b and inserting it in (*), we get

$$xa - y \cdot \frac{-ya}{x} = 1 \iff xa + \frac{y^2 a}{x} = 1 \Rightarrow a = \frac{x}{x^2+y^2}$$

and $b = -\frac{ay}{x} = \frac{-y}{x^2+y^2}$. Thus

$$z^{-1} = (a,b) = \left(\frac{x}{x^2+y^2},\ \frac{-y}{x^2+y^2}\right).$$

Remark Additive, multiplicative identities and inverses (for certain numbers) are unique!

For example, suppose there are two multiplicative inverses η and ζ for the same complex number z. Then $z\eta = \eta z = (1,0)$ and $z\zeta = \zeta z = (1,0)$, and so

$$\eta = \eta(1,0) = \eta(z\zeta) = (\eta z)\zeta = (1,0)\zeta = \zeta.$$

And suppose that there is a complex number ξ such that $\xi z = z\xi = z$ for all $z \in \mathbb{C}$. Then

$\xi = \xi(1,0) = (1,0)$ and this proves the uniqueness of multiplicative identity.

uniqueness of additive inverse and additive identity can be proven similarly.

We denote the additive inverse of z by $-z$, and the multiplicative inverse of $z \neq (0,0)$ by z^{-1}.

We define the subtraction and division in $\mathbb{C}$ as follows:

$$z_1 - z_2 = z_1 + (-z_2) \quad \text{and} \quad \frac{z_1}{z_2} = z_1 z_2^{-1}.$$

Thus, if $z_1 = (a,b)$ and $z_2 = (c,d)$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a,b) + (-c,-d) = (a+(-c), b+(-d))$$
$$= (a-c, b-d) \quad \text{and}$$

$$\frac{z_1}{z_2} = (a,b) \cdot \left(\frac{c}{c^2+d^2}, \frac{-d}{c^2+d^2} \right)$$

$$= \left(\frac{ac}{c^2+d^2} - \frac{b(-d)}{c^2+d^2}, \frac{a(-d)}{c^2+d^2} + \frac{bc}{c^2+d^2} \right)$$

$$= \left(\frac{ac+bd}{c^2+d^2}, \frac{-ad+bc}{c^2+d^2} \right).$$

<u>Example.</u> If $z_1 = (3,7)$ and $z_2 = (5,-6)$, find $z_1 - z_2$ and $\frac{z_1}{z_2}$.

<u>Solution</u> $z_1 - z_2 = (3-5, 7-(-6)) = (-2, 13)$, and

$$\frac{z_1}{z_2} = \left(\frac{3.5 + 7.(-6)}{25+36}, \frac{-3.(-6) + 7.5}{25+36} \right) = \left(\frac{-27}{61}, \frac{53}{61} \right).$$

<u>Remark</u> We can think of the field of real numbers as a subset of the field of complex numbers.

indeed, any complex number of the form $(a,0)$ can be identified with the real number a. But we must check that our definitions for addition, subtraction, multiplication and division of real numbers are consistent with the corresponding operations on real numbers. For example, if x_1 and x_2 are real numbers

$$x_1+x_2 = (x_1,0)+(x_2,0) = (x_1+x_2,0) = x_1+x_2$$

confirms the consistence, and

$$x_1 x_2 = (x_1,0)\cdot(x_2,0) = (x_1x_2-0.0,\ x_1.0+0.x_2) = (x_1x_2,0) = x_1x_2.$$

Similarly, one can check the consistence of subtraction and division.

<u>Remark</u> From now on, we'll use the symbol i for the point $(0,1)$.

Note that $i^2 = (0,1)(0,1) = (0.0-1.1,\ 0.1+1.0) = (-1,0) = -1$.

And each number $(a,b)\in\mathbb{C}$ can be written in the form

$$\begin{aligned}(a,b) &= (a,0)+(0,b)\\ &= (a,0)+(b,0)(0,1)\\ &= a+b\cdot i = a+bi\end{aligned}$$

Thus, we may write

$$\mathbb{C} = \{a+bi \mid a,b\in\mathbb{R} \text{ and } i^2=-1\}.$$

Writing (a,b) in the form $a+bi$ provides us convenience in our computations.

<u>Example</u> Find $\dfrac{(4,-1)(1,-3)}{(-1,2)}$ (or, find $\dfrac{(4-i)(1-3i)}{-1+2i}$.

solution. method (i) (Using the original definition)

$$\frac{(4,-1)(1,-3)}{(-1,2)} = \frac{(4\cdot 1-(-1)(-3),\ 4(-3)+(-1)\cdot 1)}{(-1,2)}$$

$$= \frac{(1,-13)}{(-1,2)} = \left(\frac{1(-1)+(-13)\cdot 2}{1+4},\ \frac{-1\cdot 2+(-13)(-1)}{1+4}\right)$$

$$= \left(-\frac{27}{5},\ \frac{11}{5}\right)$$

method (ii) (Using $a+bi$ form and using $i^2=-1$)

$$\frac{(4-i)(1-3i)}{-1+2i} = \frac{4-13i+3i^2}{-1+2i} = \frac{4-13i-3}{-1+2i} = \frac{1-13i}{-1+2i}\,(-1-2i)$$

$$= \frac{-1+11i+26i^2}{(-1)^2-(2i)^2} = \frac{-1+11i-26}{1-4i^2} = \frac{-27+11i}{1+4}$$

$$= -\frac{27}{5}+\frac{11}{5}i .$$

The real part of $z=(x,y)=x+iy$, denoted by $\operatorname{Re} z$ is the real number x.

The imaginary part of $z=(x,y)=x+iy$, denoted by $\operatorname{Im} z$ is the real number y.

The conjugate of $z=(x,y)=x+iy$, denoted by $\bar{z}$ is the complex number $(x,-y)=x-iy$.

Example. Find $\overline{(1+i)(2+i)}\,(3+i)$.

solution. $\overline{(1+i)(2+i)} = \overline{2+3i-1} = \overline{1+3i} = 1-3i$

$(1-3i)(3+i) = 3-8i+3 = 6-8i$.

Example. Find $\operatorname{Im}\left(\dfrac{1+2i}{3-4i}\right)$.

solution. $\dfrac{1+2i}{3-4i}\,(3+4i) = \dfrac{3+10i-8}{9+16} = \dfrac{-5+10i}{25} \Rightarrow \operatorname{Im}\left(\dfrac{1+2i}{3-4i}\right) = \dfrac{10}{25} = \dfrac{2}{5}$.

Properties. For any complex numbers z, z_1, z_2, we have

(1) $\bar{\bar{z}} = z$

(2) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(3) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(4) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$ if $z_2 \neq 0$

(5) $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$

(6) $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$

(7) $\operatorname{Re}(iz) = -\operatorname{Im} z$

(8). $\operatorname{Im}(iz) = \operatorname{Re} z$

Proof of (3). Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

and $\overline{z_1 z_2} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1)$.

On the other hand,

$$\begin{aligned} \bar{z}_1 \bar{z}_2 &= (x_1 - iy_1)(x_2 - iy_2) \\ &= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1). \end{aligned}$$

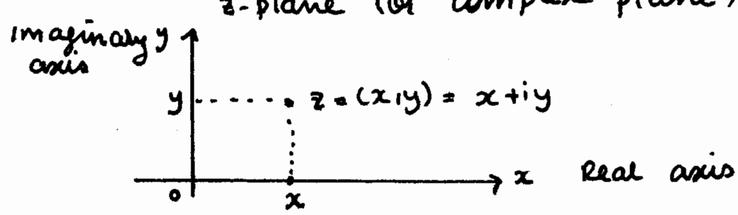
Clearly, $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$.

Proofs of the remaining items are left as exercises.

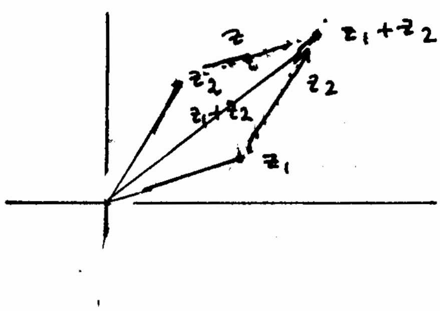
The geometry of complex numbers

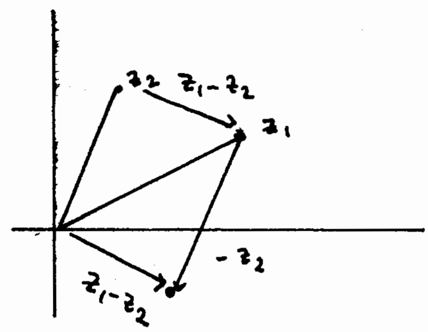
Complex numbers are ordered pairs of real numbers so they can be represented by points in the plane.

z-plane (or complex plane)



Addition and subtraction of complex numbers are analogous to addition and subtraction of vectors in the plane.





The <u>modulus</u> (norm, absolute value) of the complex number $z=x+iy$ is a nonnegative real number denoted by $|z|$ and defined by the relation

$$|z|=\sqrt{x^2+y^2}$$

that is, $|z|$ is the distance between the origin and the point z.

<u>Example</u>. Find $|(1+i)(2+i)|$.

<u>Solution</u>.

$$|(1+i)(2+i)| = |2+3i-1| = |1+3i| = \sqrt{1+9} = \sqrt{10} .$$

<u>Properties</u> For any $z, z_1, z_2 \in \mathbb{C}$

(1) $|\operatorname{Re} z| \leq |z|$

(2) $|\operatorname{Im} z| \leq |z|$

(3) $|z_1-z_2| = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ for any $z_1=x_1+iy_1$, $z_2=x_2+iy_2$.

(that is, $|z_1-z_2|$ is the distance between the points z_1, z_2 in the plane)

(4) $|z|^2 = z\bar{z}$

(5) $\left.\begin{array}{l} |z_1+z_2| \leq |z_1|+|z_2| \\ |z_1+z_2| \geq |z_1|-|z_2| \end{array}\right\}$ Triangle inequalities.

(6) $\quad |z_1 z_2| = |z_1|\,|z_2|$

(7) $\quad \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ provided that $z_2 \neq 0$.

Proof of (4) $\quad |z|^2 = x^2+y^2$ for $z = x+iy$.

$$z\bar{z} = (x+iy)(x-iy) = x^2 + \cancel{iyx} - \cancel{xiy} - i^2y^2$$
$$= x^2+y^2$$
$$\Rightarrow \quad |z|^2 = x^2+y^2 = z\bar{z}$$

Proof of (5)

$$|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2}) \quad \text{by (4)}$$
$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$
$$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$$
$$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$$
$$\le |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \quad \text{by (1)}$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \quad \text{by } (|z| = |\bar{z}|) \text{ \& by (6)}$$

prove hint: use (4)

$$= (|z_1|+|z_2|)^2 \Rightarrow |z_1+z_2| \le |z_1|+|z_2|$$

For the second inequality,

$$|z_1| = |z_1+z_2-z_2| = |z_1+z_2+(-z_2)| \le |z_1+z_2| + |-z_2|$$
$$= |z_1+z_2| + |z_2| \qquad (\text{why } |-z| = |z|?)$$
$$\Rightarrow \quad |z_1| \le |z_1+z_2| + |z_2| \Rightarrow |z_1| - |z_2| \le |z_1+z_2|$$

Proof of (6)

$$|z_1 z_2|^2 = z_1 z_2\,\overline{z_1 z_2} \quad \text{by (4)}$$
$$= z_1 z_2 \bar{z}_1 \bar{z}_2$$
$$= z_1 \bar{z}_1 z_2 \bar{z}_2$$
$$= |z_1|^2 |z_2|^2 \Rightarrow |z_1 z_2| = |z_1||z_2|$$

Remaining proofs are left as exercises!

<u>Example</u>. Let $z_1 = 1+2i$, $z_2 = 3+2i$. Verify (7) by using these two numbers.

<u>Solution</u>. $\quad |z_1| = \sqrt{1+4} = \sqrt{5} \qquad |z_2| = \sqrt{9+4} = \sqrt{13}$

So, $\dfrac{|z_1|}{|z_2|} = \sqrt{\dfrac{5}{13}}$. On the other hand,

$$\frac{z_1}{z_2} = \frac{1+2i}{3+2i} \underset{3-2i}{} = \frac{3+4i+4}{9+4} = \frac{7}{13} + i\frac{4}{13} \Rightarrow \left|\frac{z_1}{z_2}\right| = \sqrt{\frac{49}{169} + \frac{16}{169}}$$

$$= \sqrt{\frac{65}{169}} = \sqrt{\frac{5}{13}}.$$

This is consistent with (7).

<u>Example</u>. Find $|(1+i)^{50}|$.

<u>Solution</u> Applying (4) for $z_1 = z_2 = z$ for several times, one can easily see that $|z^n| = |z|^n$.

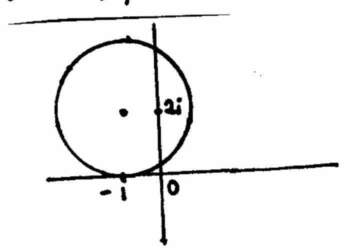
So, $|(1+i)^{50}| = |1+i|^{50} = (\sqrt{1+1})^{50} = \sqrt{2}^{50} = 2^{25}$.

<u>Example</u>. Sketch the sets of points determined by

a) $|z+1-2i| = 2$ b) $\mathrm{Re}(z+1) = 0$ c) $|z+2i| \leq 1$

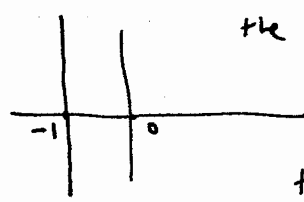
d) $\mathrm{Im}(z-2i) > 6$

<u>Solution</u> a)



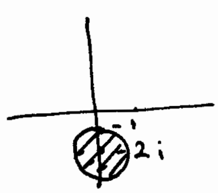
the disk of radius 2 centered at $(-1,2)$ or $-1+2i$.

b) $\mathrm{Re}(z+1) = \mathrm{Re}\,z + 1 = 0 \Rightarrow \mathrm{Re}\,z = -1$



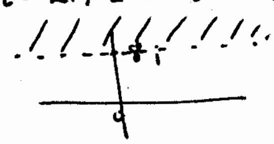
the straight line parallel to the imaginary axis and passing through the point -1.

c)



the closed disk of radius 1 centered at $z = -2i$

d) $\mathrm{Im}(z-2i) = \mathrm{Im}\,z - 2 > 6 \Rightarrow \mathrm{Im}\,z > 8$



the open region above the line $y = 8$.

<u>Example</u> Show that z_1 and z_2 are parallel if and only if $\mathrm{Im}(z_1 \bar{z}_2) = 0$

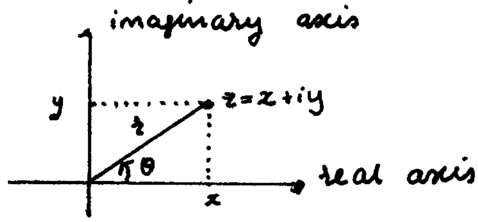
<u>Proof</u> Exercise! Hint: proof of (5).

<u>The Geometry of Complex Numbers, Continued</u>

Let $z = x + iy$ be a given complex number and r be its modulus, and θ be the angle between the line from the origin to the point z and the positive real axis. Then clearly,

$x = r\cos\theta$

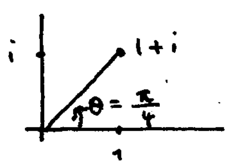
$y = r\sin\theta$



and so z has the form $z = r\cos\theta + ir\cos\theta = r(\cos\theta + i\sin\theta)$, which is known as a <u>polar representation</u> of z, and the values r, θ are called <u>polar coordinates</u> of z.

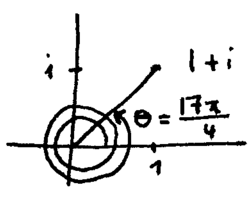
<u>Example</u>. Let $z = 1+i$, write z in its polar coordinates.

<u>Solution</u> $r = |z| = \sqrt{1+1} = \sqrt{2}$



$\theta = \frac{\pi}{4}$ $\qquad z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

<u>Remark</u>. A complex number may have several polar representations, for example, $1+i = \sqrt{2}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$ is also true.



For $z \neq 0$, the collection of all values of θ for which $z = r(\cos\theta + i\sin\theta)$ is denoted by $\arg z$, formally

$$\arg z = \{\theta : z = r(\cos\theta + i\sin\theta)\}$$

if $\theta \in \arg z$, we say that θ is an <u>argument</u> of z.

Remark. If $\theta_1, \theta_2 \in \arg z$, then there exists $n \in \mathbb{Z}$ such that $\theta_1 = \theta_2 + 2n\pi$.

Example. Find $\arg(1+i)$.

Solution. $\arg(1+i) = \{ \frac{\pi}{4} + 2n\pi, \ n \in \mathbb{Z} \}$.

Let $z \neq 0$ and $\theta \in \arg z$; if $-\pi < \theta \leq \pi$, then θ is denoted by $\operatorname{Arg} z$ and it is called the argument of z (or, the principal argument of z)

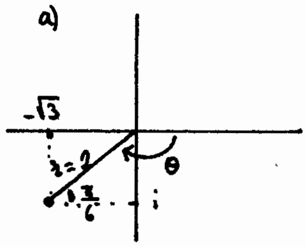
Example. Find $\operatorname{Arg}(1+i)$.

Solution. $\operatorname{Arg}(1+i) = \frac{\pi}{4}$.

Example. Find $\arg z$ and $\operatorname{Arg} z$ if

a) $z = -\sqrt{3} - i$ b) $z = i$ c) $z = -i$

Solution. a)

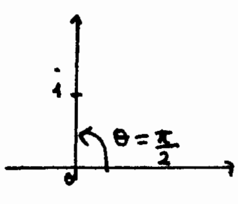


$$\theta = -\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = -\frac{5\pi}{6}$$

$$\operatorname{Arg} z = -\frac{5\pi}{6}$$

$$\arg z = \{ -\frac{5\pi}{6} + 2\pi n, \quad n \in \mathbb{Z} \}$$

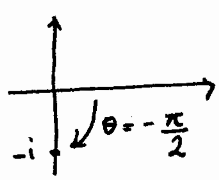
b)



$$\operatorname{Arg} z = \frac{\pi}{2}$$

$$\arg z = \{ \frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z} \}$$

c)



$$\operatorname{Arg} z = -\frac{\pi}{2}$$

$$\arg z = \{ -\frac{\pi}{2} + 2\pi n, \ n \in \mathbb{Z} \}$$

Using the symbol $e^{i\theta}$ or $\exp(i\theta)$, which is defined by Euler's formula for any real value of θ as

$$e^{i\theta} = \cos\theta + i\sin\theta$$

we can write the polar form

$$z = r(\cos\theta + i\sin\theta)$$

more compactly in <u>exponential form</u> as

$$z = re^{i\theta}$$

The choice of the symbol $e^{i\theta}$ will be motivated later.

<u>Example</u>. Write $z = -\sqrt{3} - i$ in its exponential form.

<u>Solution</u>. $r = \sqrt{3+1} = 2$ $\quad \theta = -\frac{5\pi}{6}$ $\quad z = 2e^{-\frac{5\pi}{6}i}$

<u>Remark</u>. ① For any real θ, $e^{i\theta}$ is located on the unit circle centered at the origin since
$|e^{i\theta}| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$.

② Complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi n$ for some $n \in \mathbb{Z}$.

<u>Properties</u>

① $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$ for any real θ_1 and θ_2.

② $(e^{i\theta})^n = e^{in\theta}$ for any real θ

<u>Proof of ①</u> $e^{i\theta_1} e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= \cos\theta_1\cos\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) - \sin\theta_1\sin\theta_2$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}.$$

② Exercise. Hint: use part ① and the induction argument.

Remark: If $z_1 \neq 0$, $z_2 \neq 0$, then $\arg z_1 + \arg z_2 = \arg(z_1 z_2)$ (Proof: Exercise!) whereas $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 = \operatorname{Arg}(z_1 z_2)$ is not always true (give a counter example)

Example. Let $z = 1+i$. Write z^{-1} in its exponential form, polar form and cartesian form.

Solution. Remember that $\operatorname{Arg} z = \frac{\pi}{4}$ and $r = |z| = \sqrt{1+1} = \sqrt{2}$

so $z = \sqrt{2}\, e^{i\frac{\pi}{4}}$ then

$$z^{-1} = \frac{1}{z} = \frac{1}{\sqrt{2}\, e^{i\frac{\pi}{4}}} = \underbrace{\frac{1}{\sqrt{2}}\, e^{-i\frac{\pi}{4}}}_{\text{exponential form}} \quad \text{by (2)}$$

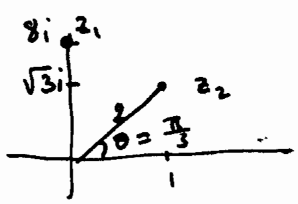
$$= \underbrace{\frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right)}_{\text{polar form}}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{4} - i \sin\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \underbrace{\frac{1}{2} - \frac{1}{2} i}_{\text{cartesian form.}}$$

Example. Let $z_1 = 8i$, $z_2 = 1 + i\sqrt{3}$ find $\frac{z_1}{z_2}$.

Solution $z_1 = 8 e^{i\frac{\pi}{2}}$ $z_2 = 2 e^{i\frac{\pi}{3}}$



$$\frac{z_1}{z_2} = \frac{8 e^{i\frac{\pi}{2}}}{2 e^{i\frac{\pi}{3}}} = 4 e^{i\frac{\pi}{6}}$$

$$= 4\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right)$$

$$= 4\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$$= 2\sqrt{3} + 2i.$$

The Algebra of Complex Numbers, Revisited.

Using the exponential form, we can easily conclude that

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n\,(\cos(n\theta) + i\sin(n\theta))$$

and

$$z^{-n} = (re^{i\theta})^{-n} = r^{-n} e^{-in\theta} = r^{-n}\,(\cos(-n\theta) + i\sin(-n\theta)) .$$

Example. Find $(-\sqrt{3}-i)^3$.

Solution 1st way

$$(-\sqrt{3}-i)^3 = (-\sqrt{3})^3 + 3(-\sqrt{3})^2(-i) + 3(-\sqrt{3})(-i)^2 + (-i)^3$$

$$= -3\sqrt{3} - 9i + 3\sqrt{3} - i = -8i$$

2nd way

$$(-\sqrt{3}-i) = 2e^{-i\frac{5\pi}{6}} \Rightarrow$$

$$(-\sqrt{3}-i)^3 = 8\,e^{-i\frac{5\pi}{2}}$$

$$= 8\left(\cos\left(-\frac{5\pi}{2}\right) + i\sin\left(-\frac{5\pi}{2}\right)\right) = -8i.$$

Exercise. Find $(-\sqrt{3}-i)^{30}$.

When we wish to write the identity $(e^{i\theta})^n = e^{in\theta}$ in its polar form, we get

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \qquad (*)$$

which is known as de Moivre's formula.

Example. Use de Moivre's formula to show that

a) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

b) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Solution a) By $(*)$ $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta \Rightarrow$

$$\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta = \cos 2\theta + i\sin 2\theta \Rightarrow$$
$$\mathrm{Re}\,(\cos^2\theta + 2i\cos\theta\sin\theta - \sin^2\theta) = \mathrm{Re}\,(\cos 2\theta + i\sin 2\theta) \Rightarrow$$
$$\cos^2\theta - \sin^2\theta = \cos 2\theta .$$

b) Exercise!

Example. Suppose that $z \neq 1$.

a) Show that $1 + z + z^2 + \cdots + z^n = \dfrac{1 - z^{n+1}}{1 - z}$. (*)

b) Use part (a) and de Moivre's formula to derive Lagrange's identity:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}} \text{ where } 0<\theta<2\pi.$$

Solution. a) Exercise!

b) Put $z = e^{i\theta}$ in (*), take real parts of the both sides;

$$Re(1 + e^{i\theta} + e^{2i\theta} + \cdots + e^{in\theta}) = Re\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}}\right)$$

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = Re\left(\frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} \cdot \frac{e^{-i\frac{\theta}{2}}}{e^{-i\frac{\theta}{2}}}\right)$$

$$= Re\left(\frac{e^{-i\frac{\theta}{2}} - e^{i(n+\frac{1}{2})\theta}}{e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}}\right)$$

$$= Re\left(\frac{\cos(-\frac{\theta}{2}) + i\sin(-\frac{\theta}{2}) - \cos((n+\frac{1}{2})\theta) - i\sin((n+\frac{1}{2})\theta)}{\cos(-\frac{\theta}{2}) + i\sin(-\frac{\theta}{2}) - \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})}\right)$$

$$= Re\left(\frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos((n+\frac{1}{2})\theta) - i\sin((n+\frac{1}{2})\theta)}{\cancel{\cos\frac{\theta}{2}} - i\sin\frac{\theta}{2} - \cancel{\cos\frac{\theta}{2}} - i\sin\frac{\theta}{2}}\right)$$

$$= \frac{1}{2} + \frac{\sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}.$$

The expression $z^n = r^n e^{in\theta}$ for integer powers of complex numbers $z = re^{i\theta}$ is useful in finding the nth roots of any nonzero complex number $c = \rho e^{i\phi}$, (where $n = 2, 3, \ldots$) that is the solution set for $z^n = c$.

We begin with the simpler equation

$$z^n = 1 . \qquad (*)$$

Suppose that $z = re^{i\theta}$ is a solution to $(*)$. Writing $(*)$ in exponential form, we get

$$r^n e^{in\theta} = 1 . e^{i0} .$$

So we must have $r^n = 1$ and $n\theta = 0 + 2\pi k$ for some $k \in \mathbb{Z}$.

So, $r = 1$ and $\theta = \frac{2\pi k}{n}$ for some $k \in \mathbb{Z}$.

That is any solution of $(*)$ should be of the form $z = re^{i\theta} = e^{i\frac{2\pi k}{n}}$ for some $k \in \mathbb{Z}$.

Conversely, if $r = 1$, and $\theta = \frac{2\pi k}{n}$ where $k \in \mathbb{Z}$, then $z = re^{i\theta} = e^{i\frac{2\pi k}{n}}$ is indeed a solution to $(*)$, because $z^n = (e^{i\frac{2\pi k}{n}})^n = e^{2\pi k i} = 1$.

Furthermore, it is easy to verify that we get n distinct solutions to $z^n = 1$ by setting $k = 0, 1, \ldots, n-1$ the solutions for $k = n, n+1, \ldots\ldots$ merely repeat those for $k = 0, 1, 2, \ldots, n-1$ (Similarly, $k = \ldots, -n, -n+1, \ldots -1$ repeat those for $k = 0, 1, \ldots, n-1$).

Therefore, n distinct solutions of $(*)$ can be expressed as $z_k = e^{i\frac{2\pi k}{n}} = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$, $k = 0, 1, \ldots, n-1$ and they are called the n^{th} roots of unity.

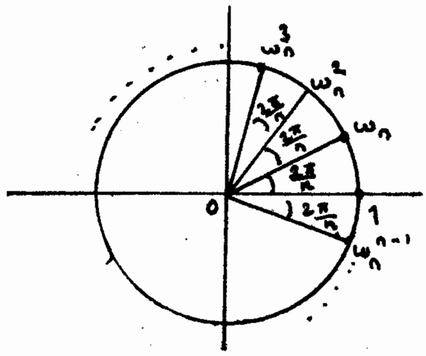
When $k = 0$ we get $z_0 = e^{i0} = 1$, that is the trivial solution to $(*)$. First nontrivial solution occurs when $k = 1$ and it's called the primitive n^{th} root of unity and denoted by w_n, that is

$$w_n = z_1 = e^{i\frac{2\pi}{n}} = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} .$$

Thus, the n^{th} roots of unity can be expressed as

$$1, \omega_n, \omega_n^2, \omega_n^3, \omega_n^4, \ldots\ldots, \omega_n^{n-1}$$

Geometrically, the n^{th} roots of unity are equally distributed n points that lie on the unit circle $\{z : |z| = 1\}$:

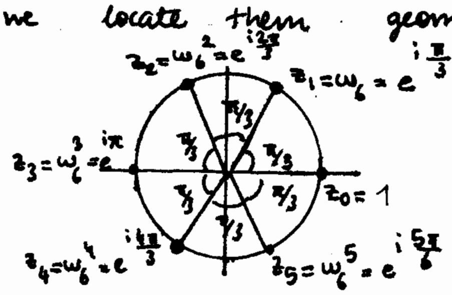


<u>Example.</u> Find all the complex numbers satisfying $z^6 = 1$.

<u>Solution.</u> the primitive sixth root is $\omega_6 = e^{i\frac{2\pi}{6}} = e^{i\frac{\pi}{3}}$.

So all roots are: $1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, e^{i\pi}, e^{i\frac{4\pi}{3}}, e^{i\frac{5\pi}{3}}$.

If we locate them geometrically, we obtain



In Cartesian coordinates:

$z_0 = 1$ $\quad z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ $\quad z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_3 = -1$ $\quad z_4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ $\quad z_5 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Solving the equation $z^n = c$ is similar.

If $c = \rho e^{i\phi}$ and $z = re^{i\theta}$ then

$$z^n = c \quad \text{implies} \quad r^n = \rho \quad \text{and} \quad n\theta = \phi + 2k\pi \quad \text{for}$$

some $k \in \mathbb{Z}$.

As before, n distinct solutions are

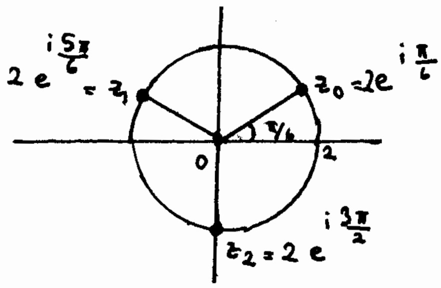
$$z_k = \rho^{\frac{1}{n}} e^{i\frac{\phi + 2k\pi}{n}}, \quad k = 0, 1, 2, \ldots, n-1.$$

<u>Example</u> Find all cube roots of $8i$.

<u>Solution.</u> $c = 8i = 8e^{i\frac{\pi}{2}} \Rightarrow z_k = 8^{\frac{1}{3}} e^{\frac{i(\frac{\pi}{2} + 2k\pi)}{3}}$, $k = 0, 1, 2$.

$z_0 = 2e^{i\frac{\pi}{6}}$, $z_1 = e^{i\frac{5\pi}{6}}$, $z_3 = e^{i\frac{9\pi}{6}} = e^{i\frac{3\pi}{2}}$.

they are located on the circle $\{z : |z| = 2\}$ and they are equally distant from each other.



In cartesian coordinates:

$$z_0 = 2e^{i\frac{\pi}{6}} = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$$
$$= \sqrt{3} + i$$
$$z_1 = 2e^{i\frac{5\pi}{6}} = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$$
$$= -2\sqrt{3} + 2i$$
$$z_2 = 2e^{i\frac{3\pi}{2}} = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = 2(0 - i) = -2i$$

<u>Remarks</u> ① If ζ is any particular solution to the equation $z^n = c$, then all solutions can be generated by multiplying ζ by the various n^{th} roots of unity. That is, the solution set is $\zeta, \zeta\omega_n, \zeta\omega_n^2, \zeta\omega_n^3, \ldots, \zeta\omega_n^{n-1}$. (Proof: Exercise)

② Quadratic formula is valid in $\mathbb{C}$, that is, if $az^2 + bc + c = 0$, then the solution set for z is

$$\left\{ \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a} \right\}$$

where by $(b^2 - 4ac)^{\frac{1}{2}}$ we mean all distinct square roots of the number inside the parenthesis. (Proof: Exercise)

<u>Example</u> Solve the equation

$$z^4 - 4z^3 + 6z^2 - 4z + 5 = 0 \qquad \text{if } z = i \text{ is a root.}$$

<u>Solution</u> Firstly, let us prove the following:

"If z_0 is a root of the polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ where $a_j \in \mathbb{R}$, $j = 0, 1, 2, \ldots, n$, (that is P is a polynomial with real coefficients) then $\bar{z}_0$ is also a root of $P(z)$."

If z_0 is a root of P, then $P(z_0) = 0$. Thus $\overline{P(z_0)} = \bar{0} = 0$ that is, $\bar{a}_n \overline{z_0^n} + \bar{a}_{n-1} \overline{z_0^{n-1}} + \cdots + \bar{a}_0 = a_n \bar{z}_0^n + a_{n-1} \bar{z}_0^{n-1} + \cdots + a_0 = 0$ (since $a_j \in \mathbb{R}$, $\bar{a}_j = a_j$ and $\overline{z^n} = \bar{z}^n$) that is $P(\bar{z}_0) = 0$ and so $\bar{z}_0$ is a root of P. ■

So $(z-i)(z+i) = z^2+1$ is a factor of $P(z)$, that is, $P(z) = (z^2+1)Q(z)$ for some polynomial Q. To find Q we divide $P(z)$ by z^2+1:

$$\begin{array}{rl}
z^4-4z^3+6z^2-4z+5 & \big|\ z^2+1 \\
-\ z^4+z^2 \qquad\qquad & \big|\ z^2-4z+5 \\
\hline
-4z^3+5z^2-4z+5 & \\
-\ -4z^3-4z \qquad & \\
\hline
5z^2+5 & \\
-\ 5z^2+5 & \\
\hline
0 &
\end{array}$$

So $P(z) = (z^2+1)(z^2-4z+5)$.

If we find the roots of z^2-4z+5 we can complete the solution. By quadratic formula roots are given by

$$\frac{4+(16-4.1.5)^{1/2}}{2} = \frac{4+(-4)^{1/2}}{2}$$

So, we need to find all square roots of -4: that is the set of all z satisfying $z^2 = -4 = 4e^{i\pi}$ by previous discussion $z_0 = 2e^{i\frac{\pi}{2}} = 2i$ and $z_1 = 2e^{i\frac{(\pi+2\pi)}{2}} = 2e^{i\frac{3\pi}{2}} = -2i$

Thus, roots of z^2-4z+5 are $\frac{4+2i}{2} = 2+i$ and $\frac{4-2i}{2} = 2-i$. and the solution set for $z^4-4z^3+6z^2-4z+5$ is

$\{i, -i, 2+i, 2-i\}$.

<u>Example.</u> Solve the equation $(z+1)^3 = z^3$.

<u>Solution.</u> $(z+1)^3 = z^3 \Rightarrow (z+1)^3 - z^3 = 0 \Rightarrow z^3+3z^2+3z+1-z^3 = 0 \Rightarrow$ $3z^2+3z+1 = 0$. So $z = \frac{-3+(9-4.3.1)^{1/2}}{6} = \frac{-3+(-3)^{1/2}}{6} = \frac{-3\mp\sqrt{3}i}{6}$

$\Rightarrow z_0 = -\frac{1}{2}+\frac{\sqrt{3}}{6}i$ and $z_1 = -\frac{1}{2}-\frac{\sqrt{3}}{6}i$ satisfy $(z+1)^3 = z^3$

<u>Exercise</u> Find all four roots of $z^4+4=0$, and use them to demonstrate that z^4+4 can be factored into two quadratics with real coefficients.

The Topology of Complex Numbers

A *curve* C is the range of a function given by $z(t) = (x(t), y(t)) = x(t) + iy(t)$, for $a \le t \le b$, where both $x(t)$ and $y(t)$ are continuous real valued functions. If $x(t)$ and $y(t)$ are differentiable, we say that the curve is *smooth*. A curve for which $x(t)$ and $y(t)$ are differentiable except for a finite number of points is called *piecewise smooth*.

We specify a curve C as

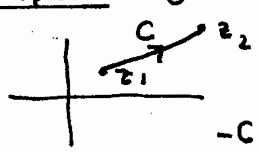
$$C: \; z(t) = x(t) + iy(t), \quad a \le t \le b,$$

and say that $z(t)$ is a *parametrization* for the curve C. We say that C is a curve that goes from the *initial point* $z(a) = x(a) + iy(a)$ to the *terminal point* $z(b) = x(b) + iy(b)$. If we had another function whose range was the same set of points as $z(t)$ but whose initial and final points were reversed, we would indicate the curve that this function defines by $-C$.

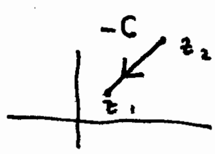
A curve C having the property that $z(a) = z(b)$ is said to be a *closed curve*.

The curve C is called *simple* if it does not cross itself, except possibly at its initial and terminal points.

Examples. ① Line segment joining $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

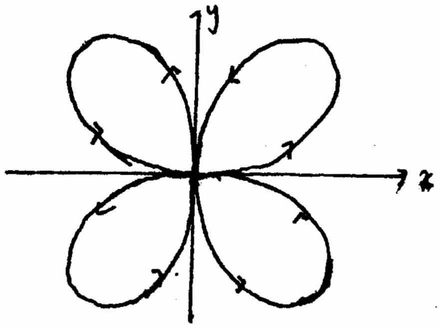


$$C: \; (x_1 + (x_2 - x_1)t) + i(y_1 + (y_2 - y_1)t), \quad 0 \le t \le 1$$



$$-C: \; (x_2 + (x_1 - x_2)t) + i(y_2 + (y_1 - y_2)t), \quad 0 \le t \le 1$$

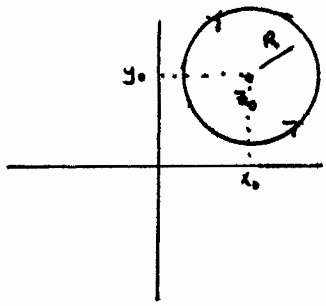
C and $-C$ are not closed.



$C: \sin 2t \cos t + i \sin 2t \sin t$ for $0 \le t \le 2\pi$

C is closed but not simple.

(four-leaved rose)



$C: (x_0 + R\cos t) + i(y_0 + R\sin t) = z_0 + Re^{it}$, $0 \le t \le 2\pi$

C is closed and simple.

An ε-neighborhood of the point z_0 is the open disk of radius $\varepsilon > 0$ about z_0, and denoted by $B_\varepsilon(z_0)$, namely

$$B_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$

and $\overline{B_\varepsilon}(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \le \varepsilon\}$ and $B_\varepsilon^*(z_0) = \{z \in \mathbb{C} \mid 0 < |z - z_0| < \varepsilon\}$ are the closed disk of radius ε centered at z_0 and the punctured disk of radius ε centered at z_0, respectively.

The point z_0 is said to be an interior point of the set S provided that there exists an ε-neighborhood of z_0 that contains only points of S, z_0 is called an exterior point of the set S if there exists an ε neighborhood of z_0 that contains no points of S. If z_0 is neither an interior point nor an exterior point, then it is called a boundary point of S.

The point z_0 is called an accumulation point of the set S if, for each ε, the punctured disk $B_\varepsilon^*(z_0)$ contains at least one point of S.

A set S is called an open set if every point of S is an interior point of S.

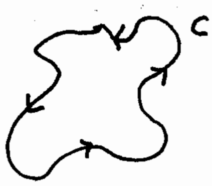
A set S is called a <u>closed set</u> if it contains all its boundary points. It can be easily shown that a set is closed if and only if its complement is open.

A set S is said to be <u>connected</u> (path-connected) if every pair of points z_1 and z_2 contained in S can be joined by a curve that lies entirely in S.

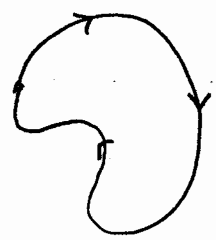
We call a connected open set a <u>domain</u>. A domain, together with some, none or all its boundary points, is called a <u>region</u>.

A set S is said to be a <u>bounded</u> set if it can be completely contained in some closed disk and said to be an <u>unbounded</u> set otherwise.

A simple closed curve is said to be <u>positively oriented</u> if its interior is on the left when the curve is traversed.



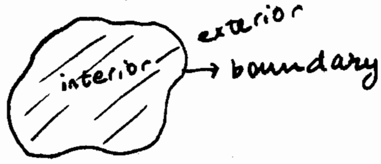
positively oriented



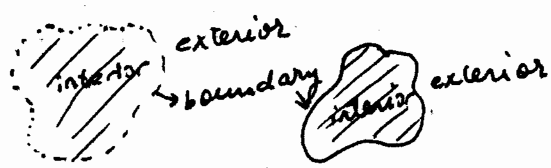
negatively oriented

counter clockwise = positive direction
clockwise = negative direction.

<u>Some Examples</u>



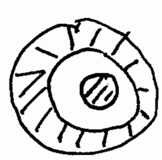
open set closed set



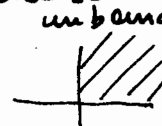
neither open nor closed set

connected set

disconnected set



unbounded



$\{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0, \operatorname{Re} z \geq 0\}$

Complex Functions

Functions and linear mappings

A <u>complex-valued function</u> f of the complex variable z is a rule that assigns to each complex number z in a set $\mathcal{D}$ one and only one complex number w. We write

$$w = f(z)$$

w is called the <u>image</u> of z under f, $\mathcal{D}$ is called the <u>domain</u> of f, and the set $\{w = f(z) \mid z \in \mathcal{D}\}$ is called the <u>range</u> of f.

Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that $u + iv = f(x+iy)$. Each of the real numbers u and v depends on the real variables x and y and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x,y) + iv(x,y), \qquad z = x + iy \in \mathbb{C}.$$

<u>Example.</u> Express the following functions in the form $u(x,y) + iv(x,y)$.

a) $f(z) = z^3$ b) $f(z) = \bar{z}^2 + (2-3i)z$.

<u>Solution.</u> a) $f(z) = (x+iy)^3 = x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$$= \underbrace{x^3 - 3xy^2}_{u(x,y)} + i\underbrace{(3x^2y - y^3)}_{v(x,y)}$$

b) $f(z) = f(x+iy) = (x-iy)^2 + (2-3i)(x+iy)$

$$= x^2 - 2ixy - y^2 + 2x + 2iy - 3ix + 3y$$
$$= \underbrace{x^2 - y^2 + 2x + 3y}_{u(x,y)} + i\underbrace{(-2xy + 2y - 3x)}_{v(x,y)}$$

Conversely, if $f(z) = u(x,y) + iv(x,y)$, $z = x+iy$ is given, we may use the formulas $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ to find a formula for f involving the variables z and $\bar{z}$.

Example. Express $f(z) = x^2 - y^2 + 2ixy$ by a formula involving the variables z and $\bar{z}$.

Solution.
$$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 + 2i\left(\frac{z+\bar{z}}{2}\right)\left(\frac{z-\bar{z}}{2i}\right)$$
$$= \frac{z^2 + 2z\bar{z} + \bar{z}^2}{4} - \frac{z^2 - 2z\bar{z} + \bar{z}^2}{4i^2} + \frac{z^2 - z\bar{z} + \bar{z}z - \bar{z}^2}{2}$$
$$= \frac{z^2 + \cancel{2|z|^2} + \cancel{\bar{z}^2} + z^2 - \cancel{2|z|^2} + \cancel{\bar{z}^2} + 2z^2 - \cancel{2\bar{z}^2}}{4} = z^2.$$

If we write $z = re^{i\theta}$ instead of $z = x+iy$, we have the following polar representation
$$w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$$
where u and v are real functions of the real variables r and θ.

Example. Write the function $f(z) = z + \frac{1}{z}$, $z \neq 0$ in the polar coordinate form.

Solution.
$$f(re^{i\theta}) = re^{i\theta} + \frac{1}{re^{i\theta}} = re^{i\theta} + \frac{e^{-i\theta}}{r}$$
$$= r\cos\theta + ir\sin\theta + \frac{1}{r}\cos\theta - \frac{1}{r}i\sin\theta$$
$$= \underbrace{\left(r + \frac{1}{r}\right)\cos\theta}_{u(r,\theta)} + i\underbrace{\left(r - \frac{1}{r}\right)\sin\theta}_{v(r,\theta)}$$

Exercise. Express $f(z) = z^5 + 4z^2 - 6$ in polar form.

Properties of a real-valued function of a real variable are often exhibited by the graph of the function. But when $w = f(z)$, where z and w are complex, no such convenient graphical representation of the function f

is available, because each of the numbers z and w is located in a plane rather than on a line. One can, however, display some information about the function by indicating pairs corresponding points $z = x+iy$ and $w = u+iv$. To do this, it is generally simpler to draw z and w planes separately. When a function is thought of in this way it is often referred to as a <u>mapping</u> or <u>transformation</u>.

If A is a subset of the domain $\mathcal{D}$ of f, the set $B = \{ f(z) \mid z \in \mathcal{A} \}$ is called the image of the set $\mathcal{A}$, and f is said to map $\mathcal{A}$ <u>onto</u> B. The mapping $w = f(z)$ is said to be from $\mathcal{A}$ <u>into</u> S if the image of A is contained in S (and we use the notation $f : \mathcal{A} \to S$)

The <u>inverse image</u> of a point w is the set of all points z in $\mathcal{D}$ such that $w = f(z)$. The inverse image of a set of points, denoted by $f^{-1}(S)$, is the collection of all points in the domain that map into S.

If f maps $\mathcal{D}$ onto R, it is possible for the inverse image of R to be function as well, but the original function must be <u>one-to-one</u>, that is, it maps distinct points $z_1 \neq z_2$ onto distinct points $f(z_1) \neq f(z_2)$. We usually indicate the inverse of f by the symbol f^{-1}.

<u>Example</u>. Find $f^{-1}(w)$ if $w = f(z) = iz$, $z \in \mathbb{C}$.

<u>Solution</u>. If $w = f(z)$ then $f^{-1}(w) = z$,

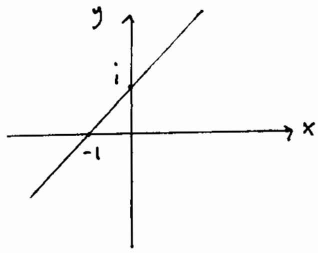
$$w = iz \Rightarrow (-i)w = (-i)iz \Rightarrow -iw = -i^2 z = z \Rightarrow f^{-1}(w) = -iw.$$

(or, equivalently we write $f^{-1}(z) = -iz$)

<u>Question.</u> How to find the image of a specified set under a given mapping?

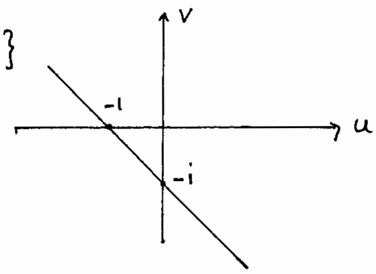
<u>Example.</u> Find the image of the line $y = x+1$ under the mapping $f(z) = iz$.

<u>Solution.</u> Let $A = \{z : x+iy \mid y = x+1\}$, we want to find $B = f(A)$.



$$u+iv = w = f(z) \in B \Leftrightarrow f^{-1}(w) = z = x+iy \in A$$
$$\Leftrightarrow -iw \in A$$
$$\Leftrightarrow -i(u+iv) = v - iu \in A$$
$$\Leftrightarrow -u = v+1 \Leftrightarrow v = -u-1$$

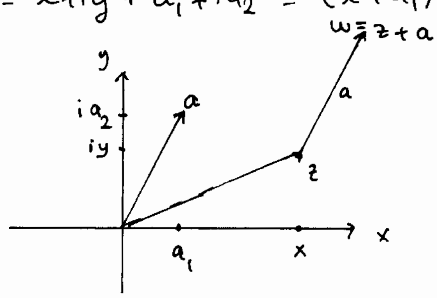
$\Rightarrow \quad f(A) = \{w : u+iv \mid v = -u-1\}$



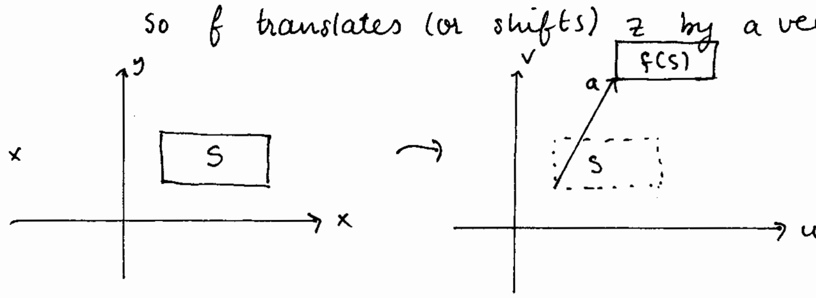
<u>Example.</u> Let $f(z) = z + a$, where $a = a_1 + ia_2 \in \mathbb{C}$. Describe the image of $S \subset \mathbb{C}$ under f.

<u>Solution.</u> Let $z = x+iy$ and $w = u+iv = f(z)$ then

$u+iv = x+iy + a_1 + ia_2 = (x+a_1) + i(y+a_2)$.

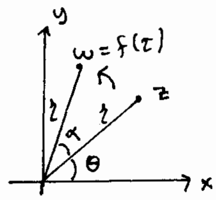


So f translates (or shifts) z by a vector a.

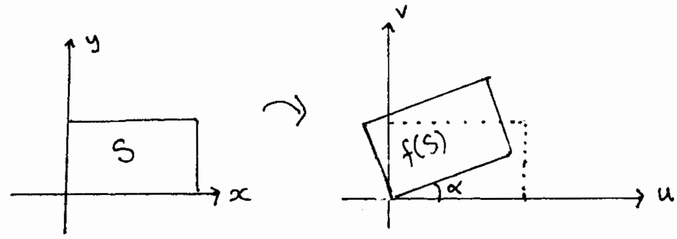


<u>Example.</u> Let $f(z) = e^{i\alpha} z$ where $\alpha \in \mathbb{R}$. Find the image of $S \subset \mathbb{C}$ under f.

Solution let $z = re^{i\theta}$ then $w = f(z) = e^{i\alpha} re^{i\theta} = re^{i(\alpha+\theta)}$

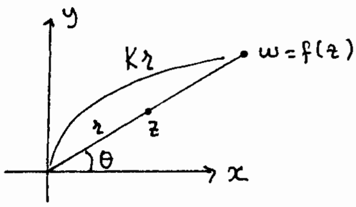


rotation by θ

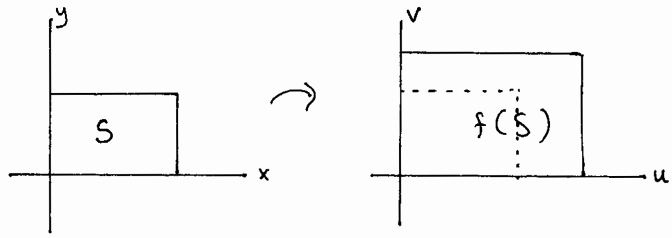


Example. Let $f(z) = Kz$ where $K > 0$. Describe the image of $S \subset \mathbb{C}$ under f.

Solution If $z = re^{i\theta}$ $w = f(z) = Kre^{i\theta}$

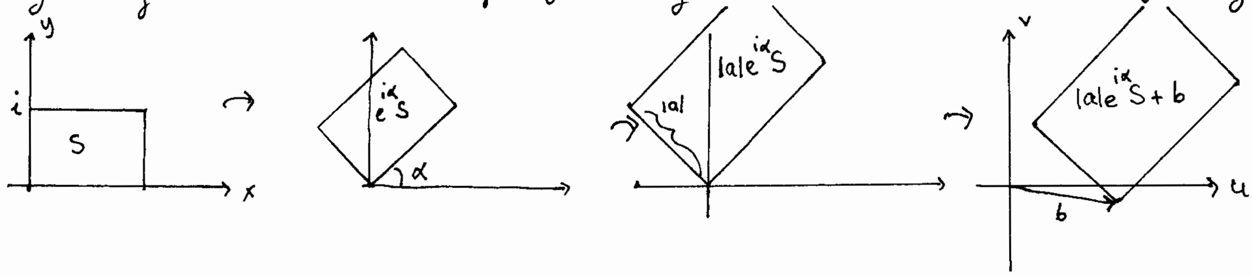


magnification by factor K

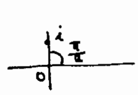


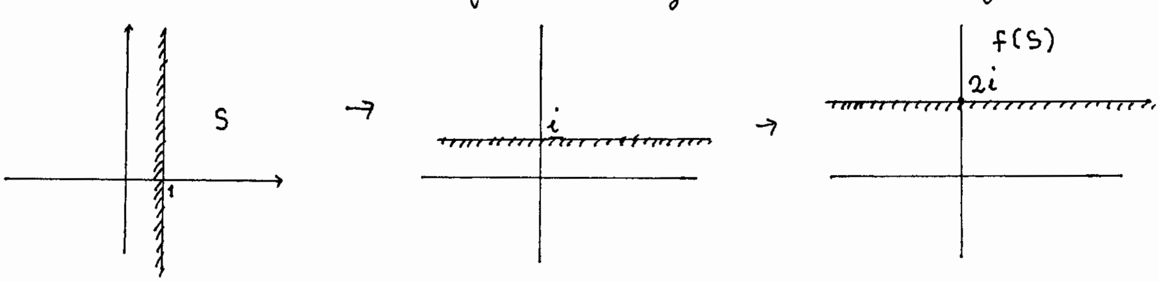
Example. Let $f(z) = az + b$ where $a, b \in \mathbb{C}$. Describe $f(S)$ for $S \subset \mathbb{C}$.

Solution Let $a = |a|e^{i\alpha}$ then $f(z) = |a| e^{i\alpha} z + b$, so f rotates z by angle θ then magnifies by $|a|$ and then shifts by b.



Example. Let $S = \{ z \mid \operatorname{Re} z \geq 1 \}$. Find the image of S under $f(z) = iz + i$

Solution $i = |i|e^{i\frac{\pi}{2}} = e^{i\frac{\pi}{2}}$  $\Rightarrow f(z) = e^{i\frac{\pi}{2}} z + i$ (rotation by $\frac{\pi}{2}$ in counter-clockwise direction followed by translation by 1 unit vertically)



Remark. Functions of the form $f(z) = az+b$, $a, b \in \mathbb{C}$ are called <u>linear functions</u>.

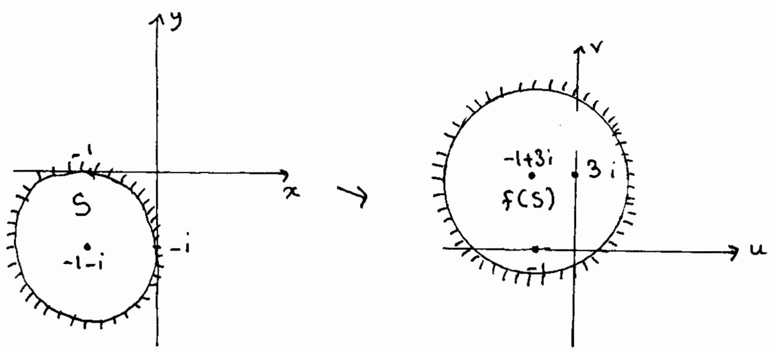
Example. Let $S = \{ z \mid |z+1+i| < 1 \}$ and $f(z) = (3-4i)z + 6+2i$. Find $f(S)$.

Solution $w = (3-4i)z + 6 + 2i \Rightarrow z = f^{-1}(w) = \dfrac{w-6-2i}{3-4i}$ and $w \in f(S)$

$$\Leftrightarrow \quad f^{-1}(w) \in S \Leftrightarrow \left| \frac{w-6-2i}{3-4i} + 1 + i \right| < 1$$

$$\Leftrightarrow \quad |w-6-2i+3-4i+3i+4| < |3-4i| = \sqrt{9+16} = 5$$

$$\Leftrightarrow \quad |w+1-3i| < 5 \Leftrightarrow |w-(-1+3i)| < 5.$$



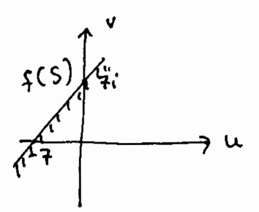
Example. Find $f(S)$ if $S = \{ z : \operatorname{Re} z \geq 1 \}$ and $f(z) = (-1+i)z - 2 + 3i$.

solution $f^{-1}(w) = \dfrac{w+2-3i}{-1+i} = \dfrac{u+iv+2-3i}{-1+i} \cdot (-1-i) = \dfrac{-u-iu-iv+v-2-2i+3i-3}{2}$

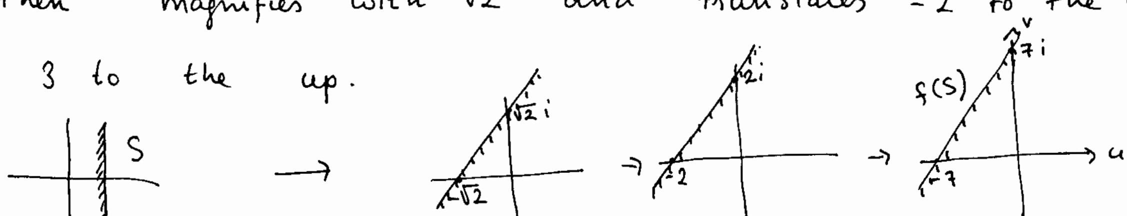
$$= \frac{-u+v-5}{2} + i\,\frac{-u-v+1}{2}.$$

$w \in f(S) \Leftrightarrow f^{-1}(w) \in S \Leftrightarrow \dfrac{-u+v-5}{2} \geq 1$

$\Leftrightarrow -u+v-5 \geq 2 \Leftrightarrow v \geq u+7$

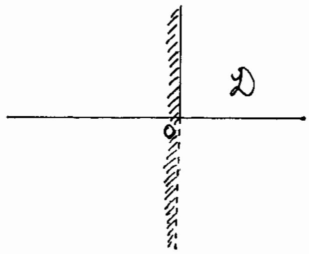


Alternatively, $f(z) = \underbrace{\sqrt{2}\, e^{i\frac{3\pi}{4}}}_{(-1+i)} z + (-2+3i)$ and f rotates S by $\frac{3\pi}{4}$ then magnifies with $\sqrt{2}$ and translates -2 to the left and 3 to the up.



The mappings $w = z^n$ and $w = z^{\frac{1}{n}}$

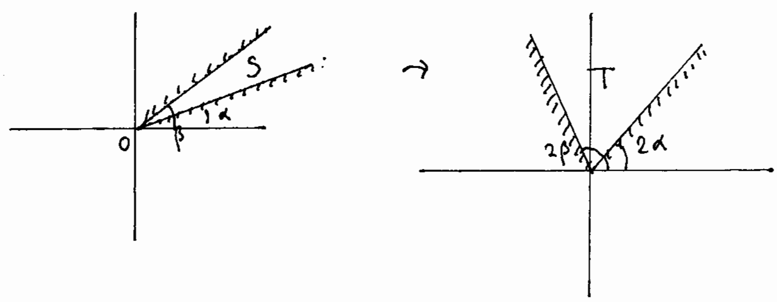
Consider $f(z) = z^2$, $z \in \mathbb{C}$. Clearly, f is not one-to-one on $\mathbb{C}$ because $f(i) = f(-i) = -1$ while $i \neq -1$. But we can restrict the domain of f to make it one-to-one. For example, f is one-to one on $\mathcal{D} = \{ re^{i\theta} \mid r>0,\ -\frac{\pi}{2} < \theta \leq \frac{\theta}{2} \}$. Indeed if $f(r_1 e^{i\theta_1}) = f(r_2 e^{i\theta_2})$,



then $r_1^2 e^{2i\theta_1} = r_2^2 e^{2i\theta_2}$ and so $r_1^2 = r_2^2$ and $2\theta_1 = 2\theta_2 + 2\pi k$ for some $k \in \mathbb{Z}$. Since $r_1 > 0$, $r_2 > 0$ we must have $r_1 = r_2$ and $\theta_1 = \theta_2 + \pi k$.

If k is even then $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ (In this case, $f(z_1) = f(z_2) \Rightarrow z_1$ and z_2). If k is odd then $r_1 e^{i\theta_1}$ and $r_2 e^{i\theta_2}$ are symmetric points with respect to the origin and so if one is in $\mathcal{D}$ the other can't be in $\mathcal{D}$.

Since $f(re^{i\theta}) = r^2 e^{2i\theta}$, f doubles the argument. So points that lie on the ray $r>0$, $\theta = \alpha$ are mapped onto points that lie on the ray $\rho > 0$, $\phi = 2\alpha$. Similarly, the sector $S = \{ re^{i\theta} \mid r>0,\ \alpha < \theta < \beta \}$ is mapped onto the sector $T = \{ \rho e^{i\phi} \mid \rho > 0,\ 2\alpha < \phi < 2\beta \}$.



Since f is one-to-one on $\mathcal{D}$, f^{-1} is defined on $f(\mathcal{D}) = \{ \rho e^{i\phi} \mid \rho > 0,\ -\pi < \theta \leq \pi \} = \mathbb{C} \setminus \{0\}$ by the formula

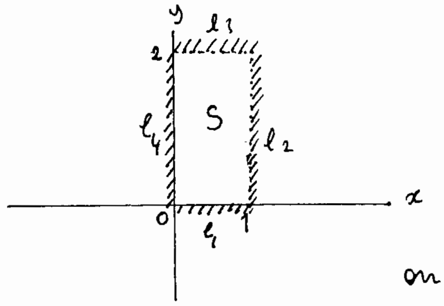
$$f^{-1}(w) = |w|^{\frac{1}{2}} e^{i \frac{\operatorname{Arg} w}{2}}$$

and called the <u>principle square root function</u>.

Now, let us see some examples on the image sets under f and f^{-1}!

Example. Find the image of $S = \{(x,y) \mid 0<x<1,\ 0<x<2\}$ under f.

Solution. $f(x+iy) = (x+iy)^2 = x^2-y^2+i2xy = u(x,y)+iv(x,y)$

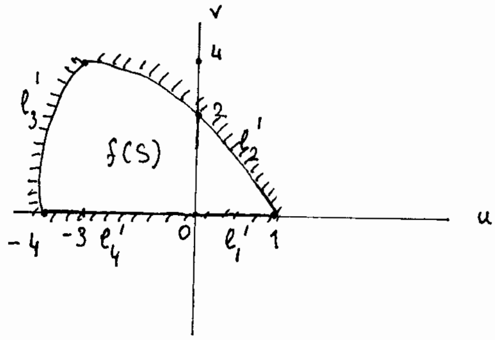


on l_1: $y=0$, $0<x<1$ so $u=x^2$, $v=0$ and $0<u<1$

on l_2: $x=1$, $0<y<2$ so $u=1-y^2$ $v=2y$ and this implies $0<v<4$ and $u = 1-\frac{v^2}{4}$.

on l_3: $y=2$, $0<x<1 \Rightarrow u=x^2-4$, $v=4x$, $0<v<4$ and $u = \frac{v^2}{16}-4$

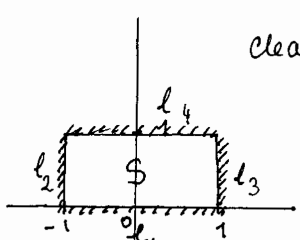
on l_4: $x=0$, $0<y<2 \Rightarrow u=-y^2$, $v=0$ and $-4<u<0$. Thus the image set T is.



Exercise. Show that the transformation $w=f(z)=z^2$, for $z \neq 0$, usually maps vertical and horizontal lines onto parabolas and use this fact to find the image of the rectangle $\{(x,y) \mid 0<x<a, 0<y<b\}$.

Example. Find the image $S = \{(x,y) \mid -1<x<1,\ 0<y<1\}$ under $g(z) = |z|^{\frac{1}{2}} e^{i\frac{Arg z}{2}}$.

Solution. If $w=u+iv$ is the image of $z=x+iy$ then we know that $z=w^2$ so $x+iy = u^2-v^2+2iuv$ and so $x=u^2-v^2$ and $y=2uv$.



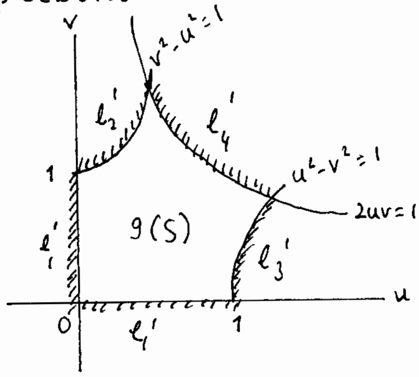
Clearly, right half of l_1 is mapped to $v=0$, $0<u<1$ and left half of l_1 is mapped to $u=0$, $0<v<1$ ($\theta=0$ on the right half, $\theta=\pi$ on the left half)

on l_2: $x=-1$ $0<y<1$, so $u^2-v^2=-1$, $0<2uv<1$

Since image set must be contained in $\{(u,v)\in\mathbb{C} \mid u>0\}$ $u>0$ so $v>0$. Thus l_2' is a part of the hyperbola $v^2-u^2=1$ which stays to left of the hyperbola $2uv=1$ and to the right of the imaginary axis.

on l_3: $x=1$, $0<y<1$ $\Rightarrow$ $u^2-v^2=1$, $0<2uv<1$ and so l_3' is the part of the hyperbola $u^2-v^2=1$ which lies below the hyperbola $2uv=1$ and above the real axis.

on l_4: $-1<x<1$ and $y=1$ $\Rightarrow$ $-1<u^2-v^2<1$ and $1=2uv$ so l_4' is the part of the hyperbola $2uv=1$ which stays between the hyperbolas $u^2-v^2=1$ and $v^2-u^2=1$.



Exercise. Show that the transformation $w=f(z)=z^{\frac{1}{2}}$ usually maps vertical and horizontal lines onto portions of hyperbolas.

Similarly, $\rho e^{i\phi}=w=f(z)=z^n=r^n e^{in\theta}$ is one-to-one on $\{re^{i\theta} \mid r>0, \frac{-\pi}{n}<\theta\leqslant\frac{\pi}{n}\}$ and it maps the ray $r>0$, $\theta=\alpha$ to the ray $\rho>0$, $\phi=n\alpha$ and it has the inverse

$$f^{-1}(w)=w^{\frac{1}{n}}=|w|^{\frac{1}{n}}e^{i\frac{Arg(w)}{n}} \quad \text{for } w\neq 0$$

which is called the <u>principal n^{th} root function.</u>

Exercise. Find the image of $\{re^{i\theta} \mid r>2$ and $\frac{\pi}{4}<\theta<\frac{\pi}{3}\}$ under $w=z^3$.

Exercise. Find the image of the sector $\{re^{i\theta} \mid r>0, -\pi<\theta<\frac{2\pi}{3}\}$ under $w=z^{\frac{1}{4}}$.

Limits and Continuity

Let f be a function defined on $\{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$ for some $z_0 \in \mathbb{C}$ and $r > 0$. We say that the limit of $f(z)$ as z approaches z_0 is w_0, if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Remark When a limit of a function $f(z)$ exists at a point z_0, it is unique. (Think why!)

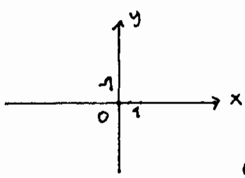
Example Show that $\lim\limits_{z \to i} z^2 = -1$.

Solution. Let $\varepsilon > 0$ be given. Choose $\delta = \min\{1, \frac{\varepsilon}{3}\}$. Suppose that $0 < |z - i| < \delta$, then $|z - i| < 1$ and $|z - i| < \frac{\varepsilon}{3}$.

Note that
$$\begin{aligned} |z^2 - (-1)| &= |z^2 + 1| = |(z+i)(z-i)| \\ &= |z+i||z-i| = |z - i + 2i||z-i| \\ &\le (|z-i| + |2i|)|z-i| \\ &< (1+2)\frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Example. Show that $\lim\limits_{z \to 0} \frac{z}{\bar z}$ does not exist.

Solution. On real axis $\frac{z}{\bar z} = \frac{x}{x} = 1$ while on the imaginary axis $\frac{z}{\bar z} = \frac{iy}{-iy} = -1$.



So $\frac{z}{\bar z}$ approaches 1 if z approaches 0 along the real axis, and $\frac{z}{\bar z}$ approaches -1 if z approaches 0 along the imaginary axis, since we can't have two different limits, $\lim\limits_{z \to z_0} \frac{z}{z_0}$ does not exist.

Theorem. Suppose that $f(z) = u(x+iy) + iv(x,y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$. Then $\lim_{z \to z_0} f(z) = w_0$ if and only if

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \quad \text{and} \quad \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0.$$

Proof. ($\Rightarrow$) Suppose $\lim_{z \to z_0} f(z) = w_0$. Let $\varepsilon > 0$ be given, choose $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \varepsilon$.

Since
$$|u(x,y) - u_0| = |Re(f(z)) - Re(w_0)| = |Re(f(z) - w_0)| \leq |f(z) - w_0|,$$
$$|u(x,y) - u_0| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta.$$
This proves that $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$. Similarly, one can prove $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$.

($\Leftarrow$) Suppose $\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0$ and $\lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$.

Let $\varepsilon > 0$ be arbitrary. Choose $\delta_1, \delta_2 > 0$ so that

$$|u(x,y) - u_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |(x,y) - (x_0,y_0)| < \delta_1 \quad \text{and}$$
$$|v(x,y) - v_0| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |(x,y) - (x_0,y_0)| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |z - z_0| < \delta$, then

$$|f(z) - w_0| = |u(x,y) + iv(x,y) - (u_0 + iv_0)| = |u(x,y) - u_0 + i(v(x,y) - v_0)|$$
$$\leq |u(x,y) - u_0| + |v(x,y) - v_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
This proves that $\lim_{z \to z_0} f(z) = w_0$. ∎

Theorem Suppose that $\lim_{z \to z_0} f(z) = A$ and $\lim_{z \to z_0} g(z) = B$. Then

$\lim_{z \to z_0} f(z) \pm g(z) = A \pm B$, $\lim_{z \to z_0} f(z)g(z) = AB$, and $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$,

provided that $B \neq 0$.

Proof. Exercise!

It is easy to show that

$$\lim_{z\to z_0} C = C \quad \text{and} \quad \lim_{z\to z_0} z = z_0$$

Then, it follows from the repeated use of the previous theorem that $\lim_{z\to z_0} cz^n = cz_0^n$ for $c\in\mathbb{C}$ and $n\in\mathbb{N}$,

and finally $\lim_{z\to z_0}(a_n z^n + a_{n-1}z^{n-1}+\cdots+a_0) = a_n z_0^n + a_{n-1}z_0^{n-1}+\cdots+a_0$.

Thus, we have proved:

If $P(z)$ is a complex polynomial in z then

$$\lim_{z\to z_0} P(z) = P(z_0).$$

<u>Example</u>. We have $\lim_{z\to 1+i}(z^2-2z+1) = \lim_{z\to 1+i}(1+i)^2-2(1+i)+1 = -1$.

Let $f(z)$ be a complex function of the complex variable z defined for all values of z in some neighborhood of z_0. We say that f is <u>continuous at z_0</u> if

$\lim_{z\to z_0} f(z)$ exists and $\lim_{z\to z_0} f(z) = f(z_0)$

<u>Example</u>. Any polynomial is continuous for all $z\in\mathbb{C}$.

<u>Theorem</u>. Let $f(z) = u(x,y)+iv(x,y)$ be defined in some neighborhood of z_0. Then f is continuous at $z_0 = x_0+iy_0$ if and only if u and v are continuous at (x_0,y_0)

<u>Proof</u>. Exercise!

<u>Theorem</u> Suppose that f and g are continuous at the point z_0. Then the following functions are continuous at z_0:

$f(z)+g(z)$, $f(z)-g(z)$, $f(z)g(z)$, $\dfrac{f(z)}{g(z)}$ provided $g(z_0)\neq 0$,

$f(g(z))$ provided f is continuous in a neighborhood of $g(z_0)$.

<u>Proof</u>. Exercise!

Example. Evaluate a) $\lim\limits_{z\to i} \dfrac{z^2+4z+2}{z+1}$ b) $\lim\limits_{z\to i} \dfrac{z^4-1}{z-i}$

Solution a) z^2+4z+2 and $z+1$ are polynomials, so

$$\lim_{z\to i} z^2+4z+2 = i^2+4i+2 = 1+4i \quad \text{and} \quad \lim_{z\to i} z+1 = i+1$$

Since $\lim\limits_{z\to i} z+1 = i+1 \neq 0$, $\lim\limits_{z\to i} \dfrac{z^2+4z+2}{z+1} = \dfrac{1+4i}{1+i} = \dfrac{(1+4i)(1-i)}{2} = \dfrac{5}{2}+\dfrac{3}{2}i$.

b) $$\frac{z^4-1}{z-i} = \frac{(z^2-1)(z^2+1)}{(z-i)} = \frac{(z^2-1)(z+i)(z-i)}{(z-i)} = (z^2-1)(z+i) \quad \text{if } z\neq i$$

Then $$\lim_{z\to i} \frac{z^4-1}{z-i} = \lim_{z\to i} (z^2-1)(z+i) = (i^2-1)(i+i) = -4i.$$

Example. Show that a) $\lim\limits_{z\to 0} \dfrac{|z|^2}{z} = 0$ b) $\lim\limits_{z\to 0} \dfrac{x^2}{z} = 0$.

Solution a) let $z = re^{i\theta} \neq 0$ then $\dfrac{|z|^2}{z} = \dfrac{r^2}{re^{i\theta}} = re^{-i\theta}$

and $0 \leq |re^{-i\theta}| = r \to 0$ as $r \to 0$ so

$$\lim_{z\to 0} \frac{|z|^2}{z} = 0.$$

b) similarly $x = r\cos\theta \Rightarrow 0 \leq \left|\dfrac{x^2}{z}\right| = \dfrac{r^2\cos^2\theta}{r} \leq r \to 0$ as $r\to 0$

So $$\lim_{z\to 0} \frac{x^2}{z} = 0.$$

Example. Let $|g(z)| \leq M$ and $\lim\limits_{z\to z_0} f(z) = 0$. Show that $\lim\limits_{z\to z_0} f(z)g(z) = 0$.

Solution. Let $\varepsilon > 0$ be given. Choose $\delta > 0$ so that

$$|f(z)| = |f(z)-0| < \frac{\varepsilon}{M} \quad \text{whenever} \quad |z-z_0| < \delta.$$

Now $|f(z)g(z) - 0| = |f(z)g(z)| < \dfrac{\varepsilon}{M}\cdot M = \varepsilon$ whenever $|z-z_0| < \delta$.

This implies that $\lim\limits_{z\to z_0} f(z)g(z) = 0$.

Branches of Functions

If for any $z \in \mathcal{D}$ there corresponds a set $F(z) \subset \mathbb{C}$ we say that we have a multivalued function $F(z)$ defined on $\mathcal{D}$.

Example. $F(z) = \arg z, \quad z \in \mathbb{C} \setminus \{0\}$

$F(z) = z^{\frac{1}{2}}$.

$\arg(1+i) = \{ \frac{\pi}{4}, \frac{\pi}{4} \pm 2\pi, \frac{\pi}{4} \pm 4\pi, \ldots\ldots, \frac{\pi}{4} + 2k\pi, \ldots \}$

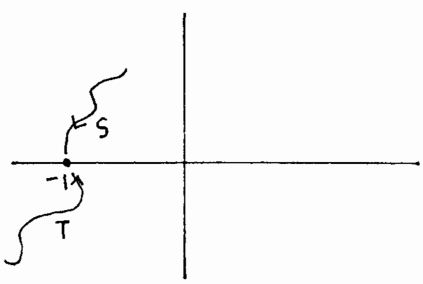
$(1)^{1/2} = \{1, -1\} \qquad (-1)^{1/2} = \{i, -i\} \qquad (i)^{\frac{1}{2}} = \{ \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \}$

Let $F(z)$ be a multivalued function defined in a region G. A (usual) complex valued function $f(z)$ is called a branch of $F(z)$ if

i) $\forall z \in G, \quad f(z) \in F(z)$

ii) $f(z)$ is continuous in G.

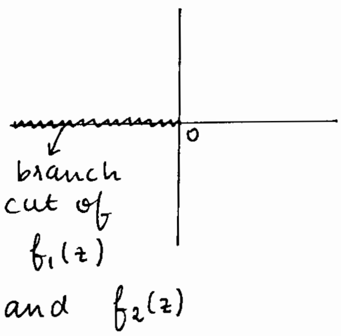
Example. Let $F(z) = z^{\frac{1}{2}}$, and $f_1(z) = |z|^{\frac{1}{2}} e^{i\frac{Arg(z)}{2}}$, $z \in \mathbb{C} \setminus \{0\}$.
$f_1(z)$ is a single valued function but f_1 is not continuous on the negative real axis, for example



$f_1(z)$ approaches i as z approaches -1 along any curve S completely contained in the upper half plane and $f_1(z)$ approaches $-i$ a z approaches -1 along any curve T contained in the lower half plane So $\lim_{z \to -1} f_1(z)$ does not exist. Similarly, it can be shown that $\lim_{z \to a} f_1(z)$ does not exist for any $a \leqslant 0$.

If we remove the points of discontinuities f_1 from $\mathbb{C}$, that is the half line $(-\infty, 0]$, from the domain of f_1

we see that f_1 is continuous on $\mathbb{C}\setminus(-\infty,0]$.
and so f_1 is a branch of $z^{\frac{1}{2}}$ on $\mathbb{C}\setminus(-\infty,0]$. (The line $(-\infty,0]$ is called a branch cut for f_1.)



Similarly, $f_2(z)=|z|^{\frac{1}{2}}e^{i\frac{\operatorname{Arg}z+2\pi}{2}}$, $z\in\mathbb{C}\setminus(-\infty,0]$ is a branch of $z^{\frac{1}{2}}$.
(Note that $f_2(z)=|z|^{\frac{1}{2}}e^{i\frac{\operatorname{Arg}z}{2}}\cdot\underbrace{e^{i\pi}}_{-1}=-f_1(z)$)

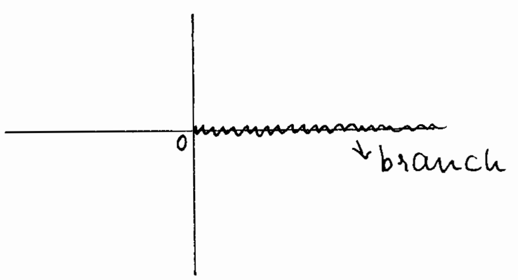
The nonpositive real axis is called a branch cut for these functions. Each point of the branch cut is a point of discontinuity for both functions f_1 and f_2.

The multivalued function $z^{\frac{1}{2}}$ may have other branches, for example,

$$h_1(z)=|z|^{\frac{1}{2}}e^{i\frac{\theta}{2}},\quad 0<\theta<2\pi$$

$$h_2(z)=|z|^{\frac{1}{2}}e^{i\frac{\theta+2\pi}{2}},\quad 0<\theta<2\pi$$

are also branches of $z^{\frac{1}{2}}$, now nonnegative real axis is the branch cut.



One can easily show that

$$f_1(i)=\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}},\quad f_1(-i)=\frac{1}{2}-i\frac{\sqrt{2}}{2}$$

$$f_2(i)=-\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\quad f_2(-i)=-\frac{1}{2}+i\frac{\sqrt{2}}{2}$$

$$h_1(i)=\frac{1}{2}+\frac{i}{\sqrt{2}},\quad h_1(-i)=-\frac{1}{2}+\frac{i\sqrt{2}}{2}$$

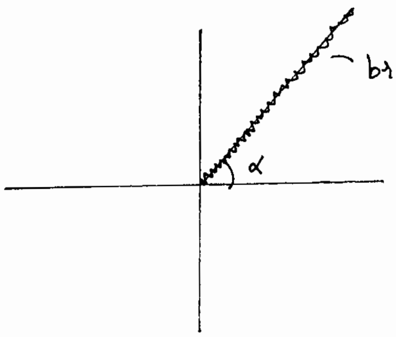
$$h_2(i)=-\frac{1}{2}-\frac{i}{\sqrt{2}},\quad h_2(-i)=\frac{1}{2}-i\frac{\sqrt{2}}{2}$$

These shows that all these functions are different from each other.

Similarly, $f_\alpha(z) = |z|^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, $\alpha < \theta < 2\pi$

is a branch of $z^{\frac{1}{2}}$ and now the ray

$\{z : \arg z = \alpha\} \cup \{0\}$ is the branch cut

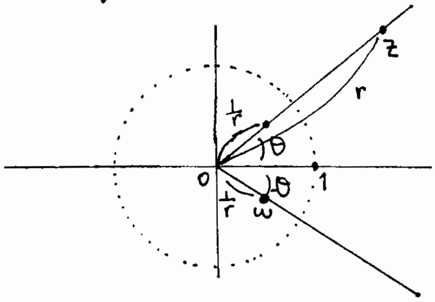


The point $z=0$, common to all branch cuts for the multivalued square root function, is called a branch point.

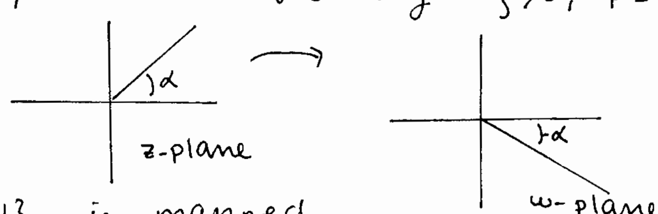
The Reciprocal transformation

The mapping $f(z) = \frac{1}{z}$, called the reciprocal transformation, maps $\mathbb{C}\setminus\{0\}$ one-to-one and onto $\mathbb{C}\setminus\{0\}$.

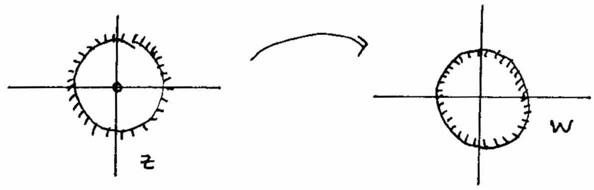
If $z = re^{i\theta}$ then $w = \rho e^{i\phi} = f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$ and so $\rho = \frac{1}{r}$ and $\phi = -\theta$, that is an inversion is followed by a reflection through the x-axis.



For example, the ray $r>0$, $\theta = \alpha$ is mapped onto the ray $\rho > 0$, $\phi = -\alpha$



The punctured disk $\{z : 0 < |z| < 1\}$ is mapped onto the region



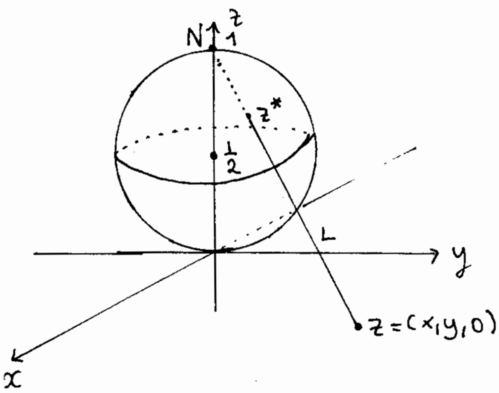
and vice versa.

The system of complex numbers can be extended by joining to it an ideal "point" denoted by ∞ and called <u>the point at infinity</u> and the new set is called <u>the extended complex plane</u> and denoted by $\overline{\mathbb{C}}$.

An ε-neighborhood of the point at infinity is the set $\{z : |z| > \frac{1}{\varepsilon}\}$.

We can visualize the point at infinity using "stereographic projection":

Let Ω be the sphere $\{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}\}$ (Ω is called the Riemann sphere)



The point $N=(0,0,1)$ on Ω is called the north pole of Ω.

Let $z = x+iy$ be a complex number (we may think of z as the point $(x,y,0)$ on the xy-plane) and L be the line segment that joins z to the north pole. L intersects Ω in exactly one point z^*.

The correspondence $z \leftrightarrow z^*$ is called the stereographic projection of the complex z-plane on the Riemann sphere.

Note that, if $|z|=1$ then z^* lies on the equator $\left(z^* = (\frac{x}{2}, \frac{y}{2}, \frac{1}{2})\right)$ if $|z|<1$ then z^* lies on the lower hemisphere and if $|z|>1$ then z^* lies on the upper hemisphere, the complex number 0 corresponds to the south pole $S=(0,0,0)$.

You can see that $z \to \infty$ $(|z| \to \infty)$ if and only if $z^* \to N$. Hence N corresponds to the ideal point at infinity.

Now we can define $f(z)=\frac{1}{z}$ on the extended complex plane $\overline{\mathbb{C}}$ as follows:

$$w=f(z)=\begin{cases}\frac{1}{z} & \text{when } z\in\mathbb{C}\setminus\{0\}\\ 0 & \text{when } z=\infty\\ \infty & \text{when } z=0\end{cases}$$

Note that $f(z):\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ is one-to-one and onto.

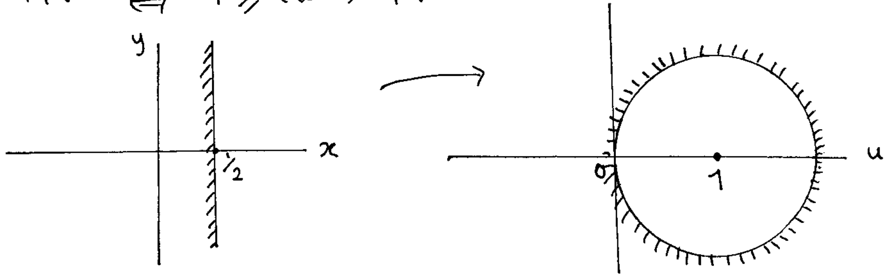
<u>Exercise</u> Show that $f(z)$ is continuous on $\overline{\mathbb{C}}$.

<u>Example.</u> Find the image of $A=\{z: \operatorname{Re} z \geqslant \frac{1}{2}\}$ under $f(z)=\frac{1}{z}$.

<u>Solution.</u> $u+iv=w\in f(A) \Leftrightarrow f^{-1}(w)\in A \Leftrightarrow \frac{1}{w}=\frac{1}{u+iv}\in A \Leftrightarrow$

$\operatorname{Re}\left\{\frac{1}{u+iv}\right\}=\operatorname{Re}\left\{\frac{u-iv}{u^2+v^2}\right\}=\frac{u}{u^2+v^2}\geqslant\frac{1}{2} \Rightarrow 2u\geqslant u^2+v^2 \Leftrightarrow 0\geqslant u^2-2u+v^2 \Leftrightarrow$

$1\geqslant u^2-2u+1+v^2 \Leftrightarrow 1\geqslant (u-1)^2+v^2$



A <u>generalized circle</u> is either a usual circle or a straight line and so it has the equation

$$A(x^2+y^2)+Bx+Cy+D=0 \quad \text{for} \quad A,B,C,D\in\mathbb{R} \qquad (*)$$

<u>Theorem.</u> $f(z)=\frac{1}{z}$ maps generalized circles to generalized circles.

<u>Proof.</u> $x=r\cos\theta,\ y=r\sin\theta \Rightarrow \rho=\frac{1}{r}\quad \phi=-\theta \quad u=\rho\cos\phi,\ v=\rho\sin\phi$

(*) implies $A(r^2\cos^2\theta+r^2\sin^2\theta)+Br\cos\theta+Cr\sin\theta+D=0 \Rightarrow$

$Ar^2+Br\cos\theta+Cr\sin\theta+D=0 \Rightarrow \quad A+B\frac{\cos\theta}{r}+C\frac{\sin\theta}{r}+\frac{D}{r^2}=0 \Rightarrow$

$A+B\rho\cos\phi-C\rho\sin\phi+D\rho^2=0 \Rightarrow$

$D(\rho^2\cos^2\phi+\rho^2\sin^2\phi)+B\rho\cos\phi+(-C)\rho\sin\phi+A=0 \Rightarrow$

$D(u^2+v^2)+Bu+(-C)v+A=0$ and this defines a generalized circle in the uv-plane.

Analytic and harmonic functions

Differentiable and analytic functions

we say that a function f is <u>differentiable</u> at the point z_0 if

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{or equivalently}$$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \text{exists.}$$

($f'(z_0)$ is called the derivative of f at z_0)

<u>Example.</u> Find $f'(z)$ if $f(z) = z^3$.

<u>Solution.</u>

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

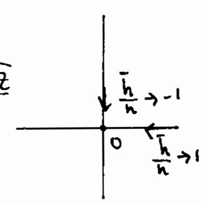
$$= \lim_{h \to 0} \frac{(z+h)^3 - z^3}{h} = \lim_{h \to 0} \frac{\cancel{z^3} + 3z^2h + 3zh^2 + h^3 - \cancel{z^3}}{h}$$

$$= \lim_{h \to 0} \underbrace{3z^2 + 3zh + h^2}_{\text{polynomial in } h} = 3z^2 + 3z \cdot 0 + 0^2 = 3z^2.$$

<u>Example.</u> show that the function $f(z) = \bar{z}$ is nowhere differentiable.

<u>Solution</u> First, note that $f(x+iy) = x - iy = u(x,y) + iv(x,y)$ where $u(x,y) = x$ and $v(x,y) = -y$. Since u and v are continuous for all values of x and y f is continuous for all values of z. On the other hand

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h \to 0} \frac{\overline{z+h} - \bar{z}}{h} = \lim_{h \to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h}$$



$$= \lim_{h \to 0} \frac{\bar{h}}{h} \quad \text{does not exist.}$$

we say that the complex function f is <u>analytic at the point z_0</u>, provided there is some $\varepsilon > 0$ such that $f'(z)$ exists for all z, $|z - z_0| < \varepsilon$. That is f is analytic at z_0 if f is differentiable in some neighborhood of z_0.

f is said to be <u>analytic on a region</u> R if f is analytic at each point of R.

f is said to be <u>entire</u> if it is analytic on $\mathbb{C}$.

Points of nonanalyticity for a function are called <u>singular</u> points.

<u>Theorem</u> If f is differentiable at z_0, then f is continuous at z_0.

<u>Proof</u>. f is continuous at z_0 if and only if $\lim_{z \to z_0} f(z) = f(z_0)$, or equivalently $\lim_{z \to z_0} (f(z) - f(z_0)) = 0$.

Note that
$$\lim_{z \to z_0} (f(z) - f(z_0)) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \cdot z - z_0 \right)$$
$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \lim_{z \to z_0} (z - z_0)$$
$$= f'(z_0) \cdot 0 = 0. \qquad \blacksquare$$

<u>Properties</u>. If f and g are differentiable functions and c is a constant, then

(i) $c' = 0$

(ii) $(z^n)' = n z^{n-1}$

(iii) $(cf)' = c f'$

(iv) $(f \pm g)' = f' \pm g'$

(v) $(fg)' = f'g + fg'$

(vi) $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$ provided that $g \neq 0$

(vii) $(f(g(z)))' = f'(g(z)) g'(z_0)$.

<u>Proof of (v)</u>: $(f(z)g(z))' = \lim_{h \to 0} \frac{f(z+h)g(z+h) - f(z)g(z)}{h}$
$$= \lim_{h \to 0} \frac{f(z+h)g(z+h) - f(z)g(z+h) + f(z)g(z+h) - f(z)g(z)}{h}$$

$$= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \lim_{h\to 0} g(z+h) + f(z) \lim_{h\to 0} \frac{g(z+h)-g(z)}{h}$$

$$= f'(z) \lim_{w\to z} g(w) + f(z)\, g'(z) \qquad \text{(since } g \text{ is diff'ble, it is continuous)}$$

$$= f'(z)\, g(z) + f(z)\, g'(z)$$

Proof of the other items are left as exercises.

Example. Find $f'(z)$ if $f(z) = (z^2+2iz+3)^4$.

Solution. $f'(z) = 4(z^2+2iz+3)^3(2z+2i) = 8(z^2+2iz+3)^3(z+i)$.

Example. Show that the function $f(z) = |z|^2$ is differentiable only at the point $z=0$.

Solution. $$\lim_{h\to 0} \frac{|z+h|^2-|z|^2}{h} = \lim_{h\to 0} \frac{(z+h)\overline{(z+h)} - z\bar{z}}{h}$$

$$= \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h} = \lim_{h\to 0} \frac{\cancel{z\bar{z}} + z\bar{h} + h\bar{z} + h\bar{h} - \cancel{z\bar{z}}}{h}$$

$$= \lim_{h\to 0} \Big(z \underbrace{\frac{\bar{h}}{h}}_{\text{d.n.e.}} + \underbrace{\bar{z}}_{\to \bar{z}} + \underbrace{\bar{h}}_{\to 0} \Big) \quad \text{does not exist unless } z=0.$$

Example. Consider the differentiable function $f(z) = z^3$ and the two points $z_1 = 1$ and $z_2 = i$. Show that there does not exist a point c on the line $y = 1-x$ between 1 and i such that

$$\frac{f(z_2)-f(z_1)}{z_2-z_1} = f'(c).$$

This result shows that the mean value theorem for derivatives does not extend to complex functions.

Solution. $$\frac{f(z_2)-f(z_1)}{z_2-z_1} = \frac{i^3-1^3}{i-1} = \frac{-i-1}{\underset{(-i-1)}{i-1}} = \frac{-1+2i+1}{2} = i,$$ and if c

is of the form $c = x+i(1-x)$, $0<x<1$, then $f'(c) = 3c^2 = (-3+6x) + i(6x-6x^2)$.

If $i = 3c^2$ then $-3+6x=0$ and $6x-6x^2 = 1$. But this is not possible because $x = \frac{1}{2}$, which the solution of the first equation is not a solution of the second equation.

Theorem (L'Hôpital's Rule). Assume that f and g are analytic at z_0. If $f(z_0)=0$, $g(z_0)=0$, and $g'(z_0)\neq 0$, then

$$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\lim_{z\to z_0}\frac{f'(z)}{g'(z)}.$$

Proof. Postponed.

Example. Find $\lim_{z\to 1+i}\frac{z^2-iz-1-i}{z^2-2z+2}$.

Solution. $(1+i)^2-i(1+i)-1-i=\cancel{1}+\cancel{2i}\cancel{-1}\cancel{-i}\cancel{+1}\cancel{-1}\cancel{-i}=0$ and

$(1+i)^2-2(1+i)+2=\cancel{1}+\cancel{2i}\cancel{-1}\cancel{-2}-\cancel{2i}+\cancel{2}=0$.

$(z^2-2z+2)'=2z-2$ and $2(1+i)-2=\cancel{2}+2i-\cancel{2}=2i\neq 0$. Then

$$\lim_{z\to 1+i}\frac{z^2-iz-1-i}{z^2-2z+2}=\lim_{z\to 1+i}\frac{2z-i}{2z-2}=\frac{2(1+i)-i}{2(1+i)-2}=\frac{2+2i-i}{2i}=\frac{2+i}{2i}$$

$$=\frac{1}{2}-i.$$

The Cauchy - Riemann Equations

Theorem. Suppose that $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is differentiable at the point $z_0 = x_0 + iy_0$. Then the partial derivatives of u and v exist at the point (x_0, y_0) and

$$f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0), \qquad (1)$$

and also

$$f'(z_0) = v_y(x_0,y_0) - iu_y(x_0,y_0). \qquad (2)$$

Equating the real and imaginary parts of the equations (1) and (2) gives $u_x(x_0,y_0) = v_y(x_0,y_0)$ (3)

and $u_y(x_0,y_0) = -v_x(x_0,y_0)$. (4)

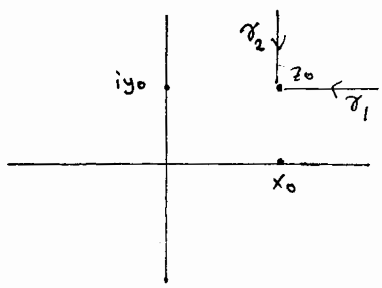
(Equations (3) and (4) are called Cauchy - Riemann equations.)

Proof If f is differentiable at z_0

$f'(z_0) = \lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists and independent of the path on which z approaches to z_0.

let $\gamma_1 := \{ z = (x_0+h) + iy_0 : 0 \le h < r \text{ for some } r > 0 \}$ and

$\gamma_2 := \{ z = x_0 + i(y_0+h) : 0 \le h < r \text{ for some } r > 0 \}$



$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in \gamma_1}} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{h \to 0} \frac{f((x_0+h) + iy_0) - f(x_0+iy_0)}{(x_0+h) + iy_0 - (x_0+iy_0)}$$

$$= \lim_{h \to 0} \frac{u(x_0+h, y_0) + iv(x_0+h, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{h}$$

$$= \lim_{h \to 0} \frac{u(x_0+h, y_0) - u(x_0,y_0)}{h} + i \lim_{h \to 0} \frac{v(x_0+h,y_0) - v(x_0,y_0)}{h}$$

$= u_x(x_0,y_0) + iv_x(x_0,y_0)$. This proves (1).

Similarly, $f'(z_0) = \lim\limits_{\substack{z \to z_0 \\ z \in \gamma_2}} \dfrac{f(z) - f(z_0)}{z - z_0} = \lim\limits_{h \to 0} \dfrac{f(x_0 + i(y_0+h)) - f(x_0+iy_0)}{x_0 + i(y_0+h) - (x_0+iy_0)}$

$$= \lim_{h\to 0} \frac{u(x_0, y_0+h) + iv(x_0, y_0+h) - u(x_0, y_0) - iv(x_0, y_0)}{ih}$$

$$= \lim_{h\to 0} \frac{-iu(x_0, y_0+h) + v(x_0, y_0+h) + iu(x_0, y_0) - v(x_0, y_0)}{h}$$

$$= \lim_{h\to 0} \frac{v(x_0, y_0+h) - v(x_0, y_0)}{h} - i \lim_{h\to 0} \frac{u(x_0, y_0+h) - u(x_0, y_0)}{h}$$

$$= v_y(x_0, y_0) - i u_y(x_0, y_0) . \quad ▨$$

Equations (3) and (4) are called Cauchy-Riemann equations and simply written as $u_x = v_y$, $u_y = -v_x$.

Remarks. 1) Let $f = u + iv$. If Cauchy-Riemann equations are not satisfied at z_0, then f is not differentiable at z_0.

2) If Cauchy-Riemann equations are satisfied at z_0, we can't say that the function is differentiable at z_0.

Example. verify the theorem for $f(z) = z^2$.

Solution. $f(x+iy) = (x+iy)^2 = x^2 - y^2 + 2ixy \Rightarrow u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$. $\quad u_x = 2x$, $u_y = -2y$, $v_x = 2y$, $v_y = 2x$

C.R. equations: $\quad u_x = v_y$ ✓ $\quad u_y = -v_x$ ✓

We know that

$$f'(z) = 2z = 2(x+iy) = 2x + 2iy$$
$$= u_x + iv_x$$
$$= v_y - iu_y .$$

Example. Show that $f(z) = \bar{z}$ is nowhere differentiable.

Solution. $f(x+iy) = x - iy \qquad u = x \qquad v = -y$

$u_x = 1$, $u_y = 0$, $v_x = 0$, $v_y = -1$

$u_x \neq v_y \qquad u_y = -v_x$. One of the Cauchy-Riemann equations does not hold. So f is not differentiable at any point.

Example. Let $f(z) = \begin{cases} \dfrac{\bar{z}^2}{z} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$

Show that f is not differentiable at $z_0 = 0$ even though Cauchy-Riemann equations are satisfied.

Solution

$$f(x,y) = \frac{(x-iy)^2}{x+iy} = \frac{x^2-y^2-2ixy}{x+iy} \quad (x-iy)$$

$$= \frac{x^3-3xy^2}{x^2+y^2} + i\,\frac{y^3-3x^2y}{x^2+y^2} \quad \text{if } (x,y) \neq (0,0)$$

So
$$u(x,y) = \begin{cases} \dfrac{x^3-3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

and
$$v(x,y) = \begin{cases} \dfrac{y^3-3x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$u_x(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

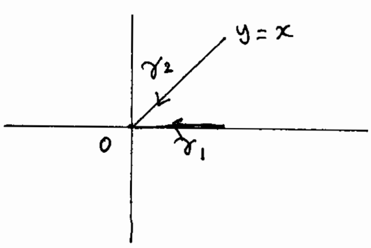
$$u_y(0,0) = \lim_{h\to 0} \frac{u(0,h)-u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$v_x(0,0) = \lim_{h\to 0} \frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

$$v_y(0,0) = \lim_{h\to 0} \frac{v(0,h)-v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1$$

So $u_x(0,0) = v_y(0,0) = 1$ and $u_y(0,0) = -v_y(0,0) = 0$.

Cauchy Riemann equations are satisfied at $(0,0)$.



$$\lim_{\substack{z\to 0 \\ z\in\gamma_1}} \frac{f(z)-f(0)}{z-0} = \lim_{x\to 0} \frac{\frac{x^2}{x} - 0}{x-0} = 1$$

while
$$\lim_{\substack{z\to 0 \\ z\in\gamma_2}} \frac{f(z)-f(0)}{z-0} = \lim_{x\to 0} \frac{\frac{(x-ix)^2}{x+ix} - 0}{x+ix}$$

$$= \lim_{x\to 0} \frac{(x-ix)^2}{(x+ix)^2} = \lim_{x\to 0} \frac{\cancel{x^2}-2ix^2-\cancel{x^2}}{\cancel{x^2}+2ix^2-\cancel{x^2}} = -1$$

So $\displaystyle\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist. That is f is not differentiable at $z=0$.

Theorem (Cauchy-Riemann Conditions for differentiability).

Let $f(z) = u(x,y) + iv(x,y)$ be a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + iy_0$. If all the partial derivatives u_x, u_y, v_x and v_y are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations $u_x(x_0,y_0) = v_y(x_0,y_0)$ and $v_x(x_0,y_0) = -u_y(x_0,y_0)$ hold at (x_0,y_0), then f is differentiable at z_0 and the derivative $f'(z_0)$ can be computed with either equation (1) or (2).

Proof. It is well known (from the course of Advanced Calculus) that:

" Let F be a real valued function of two variables, suppose that $F \in C^1$ in an open set S (that is, F and all its partial derivatives are continuous in S). If

$$R = F(x,y) - F(x_0,y_0) - \left(F_x(x_0,y_0)(x-x_0) + F_y(x_0,y_0)(y-y_0)\right)$$

for $(x,y), (x_0,y_0) \in S$, then $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{|R|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$. "

If we apply the theorem for u and v for $x = x_0 + h_1$ and $y = y_0 + h_2$, we obtain

$$\Delta u = u(x_0+h_1, y_0+h_2) - u(x_0,y_0) = u_x(x_0,y_0)h_1 + u_y(x_0,y_0)h_2 + R_1 \quad \text{and}$$

$$\Delta v = v(x_0+h_1, y_0+h_2) - v(x_0,y_0) = v_x(x_0,y_0)h_1 + v_y(x_0,y_0)h_2 + R_2$$

where $\dfrac{|R|}{\sqrt{h_1^2+h_2^2}}$ as $(h_1,h_2)\to 0$ and $\dfrac{|R_2|}{\sqrt{h_1^2+h_2^2}} \to 0$ as $(h_1,h_2)\to 0$

Now let $h = h_1 + ih_2$ and consider

$$\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} = \lim_{h\to 0} \frac{u(x_0+h_1, y_0+h_2) + iv(x_0+h_1,y_0+h_2) - u(x_0,y_0) - iv(x_0,y_0)}{h_1+ih_2}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{\Delta u + i\Delta v}{h_1+ih_2}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{u_x(x_0,y_0)h_1 + u_y(x_0,y_0)h_2 + R_1 + iv_x(x_0,y_0)h_1 + iv_y(x_0,y_0)h_2 + iR_2}{h_1+ih_2}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{u_x(x_0,y_0)h_1 - v_x(x_0,y_0)h_2 + R_1 \quad iv_x(x_0,y_0)h_1 + iu_x(x_0,y_0)h_2 + iR_2}{h_1+ih_2}$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{u_x(x_0,y_0)(h_1+ih_2) + iv_x(x_0,y_0)(h_1+ih_2) + R_1 + iR_2}{h_1+ih_2}$$

$$= u_x(x_0,y_0) + iv_x(x_0,y_0) + \underbrace{\lim_{(h_1,h_2)\to(0,0)} \frac{R_1}{h_1+ih_2}}_{=0} + i\underbrace{\lim_{(h_1,h_2)} \frac{R_2}{(h_1,h_2)}}_{=0}$$

$= u_x(x_0,y_0) + iv_x(x_0,y_0)$. Thus $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists and equal to $u_x(x_0,y_0) + iv_x(x_0,y_0)$. By Cauchy-Riemann equations $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0) = v_y(x_0,y_0) - iu_y(x_0,y_0)$ ■

Example. Show that $f(z) = e^{-y}\cos x + ie^{-y}\sin x$ is entire.

Solution $u(x,y) = e^{-y}\cos x$ and $v(x,y) = e^{-y}\sin x$

u and v are continuously differentiable everywhere and

$u_x = -e^{-y}\sin x \qquad u_y = -e^{-y}\cos x$

$v_y = -e^{-y}\sin x \qquad v_x = e^{-y}\cos x \qquad \Rightarrow u_x = v_y$ & $v_x = -u_y$ for all x & y

So f is differentiable for all $z = x+iy$ and so it is an entire function.

Example. Let $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$.

a) where is f differentiable.

b) " " " analytic.

Solution.

a) $u(x,y) = x^3 + 3xy^2 \qquad v(x,y) = y^3 + 3x^2y$

$u_x = 3x^2 + 3y^2 \qquad u_y = 6xy \qquad v_x = 6xy \qquad v_y = 3y^2 + 3x^2$

$u_x = v_y$ always but $v_x = -u_y$ if $x=0$ or $y=0$

So f is differentiable on the real line and imaginary line. b) No where.

Polar form of the Cauchy-Riemann equations.

Assuming $z_0 \neq 0$ and using the coordinate transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

we may write the Cauchy-Riemann equations in polar form and restate the last theorem in polar coordinates.

Suppose that the first-order partial derivatives of u and v with respect to x and y exist everywhere in some neighborhood of a given nonzero point z_0 and are continuous at that point. The first order partial derivatives with respect to r and θ also have these properties, and the chain rule for differentiating real-valued functions of two real variables can be used to write them in terms of the ones with respect to x and y. Moreover,

$$u_r = u_x \cdot x_r + u_y \cdot y_r = u_x \cos\theta + u_y \sin\theta \qquad (5)$$

$$u_\theta = u_x \cdot x_\theta + u_y \cdot y_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

$$v_r = v_x\cos\theta + v_y\sin\theta \quad \text{and} \quad v_\theta = -v_x r\sin\theta + v_y r\cos\theta \qquad (6)$$

If the partial derivatives with respect to x and y also satisfy the Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$ at z_0, equations (6) become

$$v_r = -u_y\cos\theta + u_x\sin\theta, \quad v_\theta = u_y r\sin\theta + u_x r\cos\theta$$

at that point. It is then clear from equations (5) and (6) that $v_\theta = r u_r$ and $u_\theta = -r v_r$ at the point z_0.

Equations $v_\theta = r u_r$ and $u_\theta = -r v_r$ are known as Cauchy-Riemann equations in polar form. Now we are ready to restate the last theorem in polar coordinates

Theorem. Let the function

$$f(z) = u(r,\theta) + iv(r,\theta)$$

be defined throughout some ε neighborhood of a nonzero point $z_0 = r_0 e^{i\theta_0}$, and suppose that the first order partial derivatives of the functions u and v with respect to r and θ exist everywhere in that neighborhood. If those partial derivatives are continuous at (r_0,θ_0) and satisfy the polar form

$$ru_r = v_\theta, \qquad u_\theta = -rv_r$$

of the Cauchy-Riemann equations at (r_0,θ_0), then $f'(z_0)$ exists and can be computed by using either

$$f'(z_0) = e^{-i\theta_0}(u_r(r_0,\theta_0) + iv_r(r_0,\theta_0))$$

or

$$f'(z_0) = \frac{1}{r_0} e^{-i\theta_0}(v_\theta(r_0,\theta_0) - iu_\theta(r_0,\theta_0)).$$

Proof. Exercise.

Example. Let $f(re^{i\theta}) = r^{\frac{1}{2}} e^{i\frac{\theta}{2}}$, $-\pi < \theta < \pi$ (the principal square root function). Show that f is differentiable on $\{re^{i\theta} \mid r>0, -\pi<\theta<\pi\}$ and find its derivative

Solution. $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ where

$u(r,\theta) = \sqrt{r}\cos\frac{\theta}{2}$ and $v(r,\theta) = \sqrt{r}\sin\frac{\theta}{2}$. Clearly, u and θ has continuous partial derivatives

$$u_r = \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} \qquad u_\theta = -\frac{1}{2}\sqrt{r}\sin\frac{\theta}{2} \qquad v_r = \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2} \text{ and } v_\theta = \frac{1}{2}\sqrt{r}\cos\frac{\theta}{2}.$$

Since $ru_r = \frac{1}{2}\sqrt{r}\cos\frac{\theta}{2} = v_\theta$ and $u_\theta = -\frac{1}{2}\sqrt{r}\sin\frac{\theta}{2} = -rv_r$, f is differentiable at any $re^{i\theta}$ with $r>0$ and $-\pi<\theta<\pi$, and

$$f'(re^{i\theta}) = e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} + \frac{i}{2\sqrt{r}}\sin\frac{\theta}{2}\right) = \frac{e^{-i\theta}}{2\sqrt{r}} e^{i\frac{\theta}{2}} = \frac{1}{2\sqrt{r}} e^{-i\frac{\theta}{2}}.$$

Theorem Let $f = u + iv$ be an analytic function on the domain $\mathcal{D}$. If $|f(z)| = C$ for all $z \in \mathcal{D}$. Then f is constant on $\mathcal{D}$.

Proof. Then $u^2+v^2 = C^2$ for all $x+iy \in \mathcal{D}$ and so

$$2uu_x + 2vv_x = 0 \quad \text{and} \quad 2uu_y + 2vv_y = 0$$

Since $v_x = -u_y$ and $v_y = v_x$, we have

$$uu_x - vu_y = 0 \quad \text{and} \quad uu_y + vu_x = 0$$

wlog $u^2+v^2 = C^2 \neq 0$ (otherwise, $u = v = 0$ for all values of x and y, and the problem is trivial).

So the solution of the system $\begin{matrix} uu_x - vu_y = 0 \\ vu_x + uu_y = 0 \end{matrix}$ is

$$u_x = \frac{0}{u^2+v^2} = 0 \quad \text{and} \quad u_y = \frac{0}{u^2+v^2} = 0 \Rightarrow u \text{ is constant}$$

and $v_x = -u_y = 0$ and $v_y = u_x = 0 \Rightarrow v$ is constant

and hence f is constant. ■

Theorem If $f'(z) = 0$ $\forall z \in \mathbb{C}$, then f is constant.

Proof. Exercise!

Example Let f be an analytic function in the domain $\mathcal{D}$. Show that if $Re(f(z)) = 0$ at all points in $\mathcal{D}$, then f is constant.

Solution $Re f(z) = u(x,y) = 0 \Rightarrow u_x = 0,\ u_y = 0 \Rightarrow v_x = 0,\ v_y = 0 \Rightarrow v$ is constant $\Rightarrow f$ is constant.

Exercise Let f be a nonconstant analytic function in the domain $\mathcal{D}$. Show that the function $\overline{f(z)}$ is not analytic in $\mathcal{D}$.

Harmonic Functions

A real-valued function u of two real variables x and y is said to be harmonic in a given domain of the xy-plane if it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$u_{xx} + u_{yy} = 0$$

known as Laplace's equation.

Exercise. Let u and v be harmonic on $\mathcal{D}$ and c be a complex constant. Show that $u+v$, $u-v$, and cu are also harmonic on $\mathcal{D}$.

Example. Is $u(x,y) = e^{-y}\sin x$ harmonic in $\mathbb{C}$?

Solution. Clearly, $u_x = e^{-y}\cos x$, $u_{xx} = -e^{-y}\sin x$, $u_y = -e^{-y}\sin x$ and $u_{yy} = e^{-y}\sin x$, and $u_{xx} + u_{yy} = -e^{-y}\sin x + e^{-y}\sin x = 0$. This shows that u is harmonic in $\mathbb{C}$.

Theorem. If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain $\mathcal{D}$, then its component functions u and v are harmonic in $\mathcal{D}$.

Proof. We need the following result which will be proved later:

"If a function of a complex variable is analytic at a point, then its real and imaginary components have continuous partial derivatives of all orders at that point."

Since f is analytic, its components satisfy the Cauchy-Riemann equations, namely

$$u_x = v_y, \qquad u_y = -v_x \qquad (*)$$

So differentiating both equations with respect to x we obtain

$u_{xx} = v_{yx}$ and $u_{yx} = -v_{xx}$, and differentiating both equations in (*), we obtain $u_{xy} = v_{yy}$ and $u_{yy} = -v_{xy}$. We know from "Advanced Calculus" that the continuity of the partial derivatives of u and v ensures that $u_{yx} = u_{xy}$ and $v_{yx} = v_{xy}$. Therefore, we have

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0 \quad \text{and} \quad v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0 .$$

This proves the harmonicity of u and v. ∎

<u>Example.</u> $f(z) = \frac{i}{z^2}$ is analytic on $\mathbb{C}\setminus\{0\}$, so its real and imaginary parts are harmonic on $\mathbb{C}\setminus\{0\}$. Since

$$f(z) = \frac{i}{(x+iy)^2} = \frac{i}{(x^2-y^2)+2ixy} \cdot \frac{(x^2-y^2)-2ixy}{} = \frac{2xy}{(x^2-y^2)^2+(2xy)^2} + i\,\frac{x^2-y^2}{(x^2-y^2)^2+(2xy)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2} + i\,\frac{x^2-y^2}{(x^2+y^2)^2}\,,$$

functions

$u(x,y) = \frac{2xy}{(x^2+y^2)^2}$ and $v(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ are harmonic on $\mathbb{R}^2\setminus\{(0,0)\}$.

If two given functions u and v are harmonic in a domain $\mathcal{D}$ and their first-order partial derivatives satisfy the Cauchy-Riemann equations through $\mathcal{D}$, v is said to be a <u>harmonic conjugate</u> of u.

<u>Example.</u> The function $v(x,y) = 2xy$ is a harmonic conjugate of $u(x,y) = x^2-y^2$, because

$$f(z) = u(x,y) + iv(x,y) = x^2-y^2+2ixy = (x+iy)^2 = z^2$$

is analytic in $\mathbb{C}$.

<u>Question.</u> How to find a harmonic conjugate of a given function?

Example. Find a harmonic conjugate for $u(x,y) = y^3 - 3x^2y$.

Solution. Suppose $v(x,y)$ is a harmonic conjugate of u, then v should satisfy

$$v_y = u_x = -6xy \quad (**) \quad \text{and} \quad v_x = -u_y = -3y^2 + 3x^2 \quad (***)$$

Integrating $(**)$ with respect to y we get

$$v(x,y) = -6x\frac{y^2}{2} + \phi(x) = -3xy^2 + \phi(x)$$

where ϕ is a function depending only on x.

By $(***)$ we must have

$$-3y^2 + \phi'(x) = -3y^2 + 3x^2 \quad \text{and so} \quad \phi'(x) = 3x^2.$$

Integrating it with respect to x we get

$$\phi(x) = 3\frac{x^3}{3} + C = x^3 + C$$

where C is an arbitrary real constant.

So $\quad v(x,y) = -3xy^2 + x^3 + C$ for some real constant C.

Example. If v is a harmonic conjugate of u, show that their product uv is a harmonic function.

Solution. If v is a harmonic conjugate of u, then $f = u + iv$ is analytic and so $f^2 = (u+iv)^2 = (u^2 - v^2) + i(2uv)$ is. Therefore $2uv$ is harmonic as the imaginary part of an analytic function. So uv is also harmonic.

Remark. If u and v are harmonic, it is not true in general that uv is harmonic.

Example. Show that $u(x,y) = x^2 - y^2$ and $v(x,y) = x^3 - 3xy^2$ are harmonic while $(x^2-y^2)(x^3-3xy^2)$ is not.

Solution. $u(x,y) = Re(z^2)$ and $v(x,y) = Re(z^3) \Rightarrow u$ and v are harmonic on $\mathbb{C}$. $u(x,y)v(x,y) = x^5 - 4x^3y^2 + 3xy^4 \Rightarrow (uv)_{xx} = 20x^3 - 24xy^2$ and $(uv)_{yy} = -8x^3 + 36xy^2$, and hence $(uv)_{xx} + (uv)_{yy} = 12x^3 + 12xy^2 \neq 0$.

Example. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be analytic in a domain D that does not include the origin. Using the polar form of the Cauchy-Riemann equations $u_\theta = -rv_r$ and $v_\theta = ru_r$, show that the function $u(r,\theta)$ satisfies the partial differential equation

$$r^2 u_{rr}(r,\theta) + r u_r(r,\theta) + u_{\theta\theta}(r,\theta) = 0,$$

which is the polar form of Laplace's equation. Show that the same is true of the function $v(r,\theta)$.

Solution $u_{\theta\theta} = -rv_{r\theta}$, $u_{\theta r} = -v_r - rv_{rr}$, $v_{\theta\theta} = ru_{r\theta}$, $v_{\theta r} = u_r + ru_{rr}$.

$\Rightarrow$

$$r^2 u_{rr} = r v_{\theta r} - ru_r \quad \text{and}$$

$$r^2 u_{rr} + ru_r + u_{\theta\theta} = (rv_{\theta r} - ru_r) + ru_r + (-rv_{r\theta}) = 0 \quad (\text{since } v_{r\theta} = v_{\theta r})$$

Similarly,

$$r^2 v_{rr} = -rv_r - ru_{\theta r} \quad \text{and}$$

$$r^2 v_{rr} + rv_r + v_{\theta\theta} = (-rv_r - ru_{\theta r}) + rv_r + ru_{r\theta} = 0 \quad (\text{since } u_{\theta r} = u_{r\theta})$$

Sequences and Power Series

Sequences and series

A complex sequence f is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers. ($f: \mathbb{N} \to \mathbb{C}$)

Examples

1. $f(n) = \left(2 - \frac{1}{n}\right) + \left(5 + \frac{1}{n}\right)i$, $n \in \mathbb{N}$
2. $f(n) = e^{\frac{i\pi n}{4}}$, $n \in \mathbb{N}$
3. $f(n) = 5 + 3i + \left(\frac{1}{1+i}\right)^n$, $n \in \mathbb{N}$
4. $f(n) = \left(\frac{1}{4} + \frac{i}{2}\right)^n$, $n \in \mathbb{N}$.

The values $z_1 = f(1)$, $z_2 = f(2)$, $z_3 = f(3)$, ... are called the terms of a sequence f. Mostly, we refer to $z_1, z_2, z_3, \ldots$ as the sequence itself.

We say that the limit of the sequence $z_1, z_2, \ldots$ is z if $\forall \varepsilon > 0$ $\exists N_\varepsilon$ such that $|z_n - z| < \varepsilon$ whenever $n > N_\varepsilon$, and write $\lim_{n\to\infty} z_n = z$ (or, $z_n \to z$ as $n \to \infty$).

Theorem Let $z_n = x_n + iy_n$ be a sequence and $z = x + iy$ be a complex number. Then $z_n \to z$ if and only if $x_n \to x$ and $y_n \to y$.

Proof. Exercise.

Example. Find $\lim_{n\to\infty} z_n$, if $z_n = \dfrac{\sqrt{n} + i(n+1)}{n}$.

Solution. Clearly, $z_n = x_n + iy_n$ where $x_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$ and $y_n = \frac{n+1}{n}$. Since $x_n \to 0$, $y_n \to 1$ as $n \to \infty$; $z_n \to 0 + i.1 = i$ as $n \to \infty$. That is $\lim_{n\to\infty} \frac{\sqrt{n} + i(n+1)}{n} = i$.

Example. Show that $(1+i)^n$ diverges as $n \to \infty$.

Solution. $(1+i)^n = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^n = (\sqrt{2})^n \cos\frac{n\pi}{4} + i(\sqrt{2})^n \sin\frac{n\pi}{4}$.

Clearly, $(\sqrt{2})^n \cos\frac{n\pi}{4}$ is divergent as an unbounded sequence. (Similarly, $(\sqrt{2})^n \sin\frac{n\pi}{4}$ is also divergent)

Therefore $(1+i)^n$ is divergent.

A sequence $\{z_n\}$ is bounded if there exists a positive real number M such that $|z_n| < M$ for all $n \in \mathbb{N}$.

Theorem. If $\{z_n\}$ is convergent, then it is bounded.

Proof. Exercise!

The sequence $\{z_n\}$ is a Cauchy sequence if $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$ such that $|z_n - z_m| < \varepsilon$ whenever $n, m > N_\varepsilon$.

Theorem. $\{z_n\}$ is convergent if and only if $\{z_n\}$ is a Cauchy sequence.

Proof. Exercise!

An infinite series $\sum_{n=1}^{\infty} z_n = z_1 + z_2 + \cdots + z_n + \cdots$ of complex numbers converges to the sum S if the sequence

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N \qquad (N = 1, 2, 3, \ldots)$$ of partial sums converges to S; we then write $\sum_{n=1}^{\infty} z_n = S$.

The series $\sum_{n=1}^{\infty} z_n$ is said to be <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |z_n|$ converges.

If a series does not converge, we say that it <u>diverges</u>.

<u>Theorem</u> Let $z_n = x_n + iy_n$ and $S = X + iY$. Then

$$S = \sum_{n=1}^{\infty} z_n \quad \text{if and only if} \quad X = \sum_{n=1}^{\infty} x_n \quad \text{and} \quad Y = \sum_{n=1}^{\infty} y_n.$$

<u>Proof.</u> Exercise.

<u>Theorem</u> If $\sum_{n=1}^{\infty} z_n$ converges, then $\lim_{n\to\infty} z_n = 0$.

<u>Proof</u> Exercise.

<u>Example.</u> Determine whether the following series converge or diverge.

(a) $\sum_{n=1}^{\infty} \dfrac{1+in(-1)^n}{n^2}$

(b) $\sum_{n=1}^{\infty} \dfrac{(-1)^n+i}{n}$

(c) $\sum_{n=1}^{\infty} (1+i)^n$.

<u>Solution.</u>

(a) Note $\dfrac{1+in(-1)^n}{n^2} = \dfrac{1}{n^2} + i\,\dfrac{(-1)^n}{n}$.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Alternating series test. Therefore $\sum_{n=1}^{\infty} \frac{1+in(-1)^n}{n^2}$ is convergent.

(b) Note that $\dfrac{(-1)^n+i}{n} = \dfrac{(-1)^n}{n} + i\,\dfrac{1}{n}$. $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-test. Hence $\sum_{n=1}^{\infty} \frac{(-1)^n+i}{n}$ is divergent.

(c) Since $|z_n| = |1+i|^n = (\sqrt{2})^n \to \infty$ as $n\to\infty$, $z_n \not\to 0$ as $n\to\infty \Rightarrow \sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} (1+i)^n$ diverges.

Theorem. Let $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ be convergent series and a, b be complex constants. Then

$$\sum_{n=1}^{\infty} (az_n + bw_n) = a\sum_{n=1}^{\infty} z_n + b\sum_{n=1}^{\infty} w_n.$$

Proof. Exercise!

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be convergent series, where a_n and b_n are complex numbers. The Cauchy product of the two series given above is defined to be the series

$$\sum_{n=0}^{\infty} c_n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Theorem. If the Cauchy product converges, then

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

Proof. Omitted.

Theorem (Comparison Test) Let $\sum_{n=1}^{\infty} M_n$ be convergent series of nonnegative real numbers. If $\{z_n\}$ is a sequence of complex numbers and $|z_n| \le M_n$ for all n, then $\sum_{n=1}^{\infty} z_n$ converges.

Proof. Exercise!

Corollary If $\sum_{n=1}^{\infty} |z_n|$ converges, then $\sum_{n=0}^{\infty} z_n$ converges

Proof. Exercise!

Example. Show that $\sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ converges.

Solution. Since $|z_n| = \frac{|3+4i|^n}{5^n n^2} = \frac{5^n}{5^n n^2} = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \frac{(3+4i)^n}{5^n n^2}$ converges.

Examples 1) Evaluate, if exists

(a) $\lim\limits_{n\to\infty} \dfrac{n+(i)^n}{n}$

(b) $\lim\limits_{n\to\infty} \dfrac{(n+i)(1+ni)}{n^2}$

(c) $\lim\limits_{n\to\infty} (i)^{\frac{1}{n}}$, where $(i)^{\frac{1}{n}}$ is the principal value of the n^{th} root of i

(d) $\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{n+1+i} - \dfrac{1}{n+i}\right)$

(e) $\lim\limits_{n\to\infty} \left(\dfrac{1+i}{\sqrt{2}}\right)^n$

2) Determine whether the following series converge or diverge.

(a) $\sum\limits_{i=1}^{\infty} \dfrac{(i)^n}{n}$

(b) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n} + \dfrac{i}{2^n}$.

Solutions. 1) (a) $\frac{n}{n} = 1 \to 1$ $\qquad \dfrac{i^n}{n} = \begin{cases} \frac{1}{n} & \text{if } n = 0 \bmod 4 \\ \frac{i}{n} & \text{"} \quad n=1 \text{ " "} \\ \frac{-1}{n} & \text{"} \quad n=2 \text{ " "} \\ \frac{-i}{n} & \text{"} \quad n=3 \text{ " "}. \end{cases}$

In each case $\frac{i^n}{n} \to 0$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty} \frac{n+(i)^n}{n} = 1$$

(b) $\dfrac{(n+i)(1+ni)}{n^2} = \dfrac{\cancel{n} + n^2 i + i - \cancel{n}}{n^2} = i\,\dfrac{n^2+1}{n^2}$

since $\dfrac{n^2+1}{n^2} \to 1$ as $n\to\infty$, $\lim\limits_{n\to\infty} \dfrac{(n+i)(1+ni)}{n^2} = i$.

(c) $(i)^{\frac{1}{n}} = 1^{\frac{1}{n}} e^{i\frac{Arg(i)}{n}}$, $Arg(i) = \frac{\pi}{2} \Rightarrow (i)^{\frac{1}{n}} = e^{i\frac{\pi}{2n}} = \cos\frac{\pi}{2n} + i\sin\frac{\pi}{2n}$.

$\cos\frac{\pi}{2n} \to 1$ as $n\to\infty$, $\sin\frac{\pi}{2n} \to 0$ as $n\to\infty$ $\Rightarrow \lim\limits_{n\to\infty} (i)^{\frac{1}{n}} = 1$.

(d) $$S_N = \sum_{n=0}^{N} \left(\frac{1}{n+1+i} - \frac{1}{n+i} \right)$$

$$= \left(\cancel{\frac{1}{1+i}} - \frac{1}{i}\right) + \left(\cancel{\frac{1}{2+i}} - \cancel{\frac{1}{1+i}}\right) + \left(\frac{1}{3+i} - \cancel{\frac{1}{2+i}}\right) + \cdots + \left(\frac{1}{N+1+i} - \cancel{\frac{1}{N+i}}\right)$$

$$= -\underset{(-i)}{\frac{1}{i}} + \underset{(N+1)-i}{\frac{1}{N+1+i}} = i + \frac{N+1-i}{(N+1)^2+1} = i + \underbrace{\frac{N+1}{(N+1)^2+1}}_{\to 0} - i\,\underbrace{\frac{1}{(N+1)^2+1}}_{\to 0} \to i, \ n \to \infty$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1+i} - \frac{1}{n+i} \right) = i$$

e) $$\left(\frac{1+i}{\sqrt{2}}\right)^n = \left(e^{i\frac{\pi}{4}}\right)^n = e^{i\frac{n\pi}{4}} = \cos\frac{n\pi}{4} + i \sin\frac{n\pi}{4}$$

$$\left\{\cos\frac{n\pi}{4}\right\} = \{\underbrace{1, \tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}, -1, -\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}}_{\text{Repeats this pattern}}, 1, \tfrac{1}{\sqrt{2}}, 0, \tfrac{-1}{\sqrt{2}}, -1, -\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}, 1 \ldots\}$$

Hence no limit of the real part. So $\lim_{n\to\infty} \left(\frac{1+i}{\sqrt{2}}\right)^n$ does not exist.

2) (a) $$\frac{i^n}{n} = \frac{\left(e^{i\frac{\pi}{2}}\right)^n}{n} = \frac{e^{i\frac{n\pi}{2}}}{n} = \frac{\cos\frac{n\pi}{2}}{n} + i\,\frac{\sin\frac{n\pi}{2}}{n}$$

$$\sum_{n=1}^{\infty} \frac{\cos\frac{n\pi}{2}}{n} = 0 + \frac{(-1)}{2} + 0 + \frac{1}{4} + 0 - \frac{1}{6} + 0 + \frac{1}{8} + \cdots$$

$$= -\frac{1}{2}\Big(\underbrace{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}_{\text{harmonic series} \to \text{convergent}}\Big)$$

$$\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{n} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$ converges by alternating series test.

So $\sum \frac{(i)^n}{n}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, So $\sum_{n=1}^{\infty} \left(\frac{1}{n} + \frac{i}{2^n}\right)$ diverges.

Geometric Series and Convergence Theorems

The series $\sum_{n=0}^{\infty} z^n$ is called a geometric series.

Theorem If $|z|<1$, the series $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$. If $|z|\geq 1$, the series diverges.

Proof. $S_n = \sum_{k=0}^{n} z^k = 1+z+z^2+\cdots+z^n$. Clearly

$zS_n = z+z^2+\cdots+z^{n+1}$ and

$S_n - zS_n = 1-z^{n+1}$. Thus $S_n = \frac{1}{1-z} - \frac{z^{n+1}}{1-z}$.

Since $|z|<1$, $\lim_{n\to\infty} |z|^{n+1} = 0 \Rightarrow \lim_{n\to\infty} S_n = \frac{1}{1-z}$.

If $|z|\geq 1$, then $\lim_{n\to\infty} |z^n| \neq 0 \Rightarrow \lim_{n\to\infty} z^n \neq 0 \Rightarrow \sum_{n=1}^{\infty} z^n$ diverges. ∎

Corollary. If $|z|>1$, then $\sum_{n=1}^{\infty} z^{-n}$ converges to $\frac{1}{z-1}$. If $|z|\leq 1$, then $\sum_{n=1}^{\infty} z^{-n}$ diverges.

Proof. put $w=\frac{1}{z}$. Then $\sum_{n=1}^{\infty} z^{-n} = \sum_{n=1}^{\infty} w^n$ converges if $|w|<1$ (that is $|z|>1$) and diverges $|w|\geq 1$ (that is $|z|\leq 1$)

and $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = 1+\sum_{n=1}^{\infty} w^n \Rightarrow \sum_{n=1}^{\infty} w^n = \frac{1}{1-w} - 1 = \frac{w}{1-w}$

$$= \frac{\frac{1}{z}}{1-\frac{1}{z}} = \frac{1}{z-1}. \ ∎$$

Example. Find a) $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n}$

b) $\sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n$

Solution. a) geometric series with $z=\frac{1-i}{2}$. Clearly, $|z| = \frac{1}{\sqrt{2}} < 1$.

Thus $\sum_{n=0}^{\infty} \frac{(1-i)^n}{2^n} = \frac{1}{1-\frac{1-i}{2}} = \frac{2}{1+i}_{(1-i)} = \frac{2-2i}{2} = 1-i$.

b) Put $k=n-3$, then $\sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n = \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^{k+3} = \left(\frac{i}{2}\right)^3 \sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k$.

Since $\left|\frac{i}{2}\right| = \frac{1}{2} < 1$, $\sum_{k=0}^{\infty} \left(\frac{i}{2}\right)^k = \frac{1}{1-\frac{i}{2}} = \frac{2}{2-i}_{(2+i)} = \frac{4+2i}{5}$, and $\sum_{n=3}^{\infty} \left(\frac{i}{2}\right)^n = \frac{-i}{8}\cdot\frac{4+2i}{5} = \frac{1-2i}{20}$.

Theorem. (d'Alembert's ratio test)

If $\sum_{n=0}^{\infty} z_n$ is a complex series with the property that $\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = L$, then the series is absolutely convergent if $L<1$ and divergent if $L>1$.

Proof. Exercise!

Example. Show that $\sum_{n=0}^{\infty} \frac{(1-i)^n}{n!}$ converges.

Solution. $z_n = \frac{(1-i)^n}{n!}$, $\frac{|z_{n+1}|}{|z_n|} = \left|\frac{1-i}{n+1}\right| = \frac{\sqrt{2}}{n+1} \to 0$ as $n\to\infty$.

That is, $L=0$. Since $L<1$, series converges absolutely.

Example. For what values of z, $\sum_{n=0}^{\infty} \frac{(z-i)^n}{2^n}$ converges.

Solution. $\frac{|z_{n+1}|}{|z_n|} = \frac{|z-i|}{2} \to \frac{|z-i|}{2}$, as $n\to\infty$. Therefore, if $|z-i|<2$, series converges absolutely, and if $|z-i|>2$, series diverges. If $|z-i|=2$, Ratio test says nothing. But if $|z-i|=2$, $|z_n| = \left|\frac{z-i}{2}\right|^n = 1 \not\to 0$ as $n\to\infty$, hence $\lim_{n\to\infty} z_n \neq 0 \Rightarrow \sum_{n=0}^{\infty} z_n$ is divergent.

Let $\{a_n\}$ be a sequence of real numbers. The limit supremum of $\{a_n\}$, denoted by $\limsup_{n\to\infty} a_n$ is the smallest real number L having the property that for any $\varepsilon>0$, there are at most finitely many terms in the sequence that are larger than $L+\varepsilon$.

If there is no such number L, then $\limsup_{n\to\infty} a_n = \infty$

Example. Find limsup of the following sequences

(a) $a_n = \{1,1,2,3,5,8,13,21,\ldots\}$ $\quad (a_n = a_{n-2}+a_{n-1})$

(b) $a_n = \sin\frac{(n+1)\pi}{2}$

Solution (a) $\limsup a_n = \infty$, the sequence is unbounded.

(b) $a_n = \{\underbrace{1, 0, -1, 0}_{\text{repeated}}, 1, 0, -1, 0, \ldots\}$

$\limsup\limits_{n\to\infty} a_n = 1$.

Theorem (Root Test) Suppose the series $\sum\limits_{n=0}^{\infty} z_n$ has $\limsup\limits_{n\to\infty} |z_n|^{\frac{1}{n}} = L$. Then the series absolutely convergent if $L<1$ and divergent if $L>1$.

Proof. Exercise!

Examples. 1. Evaluate $\sum\limits_{n=0}^{\infty} \frac{(1+i)^n}{2^n}$

2. Determine whether the series is convergent.

a) $\sum\limits_{n=0}^{\infty} \frac{(4i)^n}{n!}$

(b) $\sum\limits_{n=0}^{\infty} \frac{(1+i)^{2n}}{(2n+1)!}$

3. For what values of z, the following series converges.

(a) $\sum\limits_{n=0}^{\infty} \frac{z^n}{(3+4i)^n}$

(b) $\sum\limits_{n=0}^{\infty} \frac{(z-3-4i)^n}{2^n}$.

Solutions. 1. Geometric series with $|z| = \left|\frac{1+i}{2}\right| = \frac{1}{\sqrt{2}} < 1$, so

$$\sum_{n=0}^{\infty} \frac{(1+i)^n}{2^n} = \frac{1}{1-\frac{1+i}{2}} = \frac{2}{1-i} = \frac{2+2i}{2} = 1+i.$$

(1+i)

2. a) $\frac{|z_{n+1}|}{|z_n|} = \left|\frac{4i}{n+1}\right| = \frac{4}{n+1} \to 0 \Rightarrow$ Convergent.

b) $\frac{|z_{n+1}|}{|z_n|} = \left|\frac{(1+i)^{2(n+1)}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(1+i)^{2n}}\right| = \frac{|1+i|^2}{(2n+3)(2n+1)} \to 0 \Rightarrow$ convergent.

3. a) Let $\zeta_n = \dfrac{z^n}{(3+4i)^n}$. Then

$$\frac{|\zeta_{n+1}|}{|\zeta_n|} = \frac{|z|}{|3+4i|} = \frac{|z|}{5} \to \frac{|z|}{5} \quad \text{as } n\to\infty .$$

Therefore series converges absolutely if $|z|<5$ and diverges if $|z|>5$ by ratio test.

If $|z|=5$ ratio test fails. But $|\zeta_n| = 1 \not\to 0$, $n\to\infty$

$\Rightarrow$ $\displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(3+4i)^n}$ is divergent.

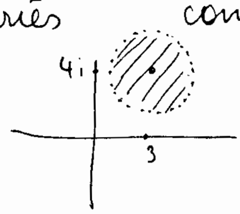
b) Let $\zeta_n = \dfrac{(z-3-4i)^n}{2^n}$. Then

$$\frac{|\zeta_{n+1}|}{|\zeta_n|} = \frac{|z-3-4i|}{2} \to \frac{|z-3-4i|}{2} \quad \text{as} \quad n\to\infty$$

Therefore series converges absolutely if $|z-3-4i|<2$ and diverges if $|z-3-4i|>2$.

If $|z-3-4i|=2$, then $|\zeta_n| = 1 \not\to 0 \Rightarrow$ series diverges.

Thus series converges on $\{z\in\mathbb{C} \mid |z-3-4i|<2\}$



Power Series Functions

A series of the form

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n, \qquad \alpha, c_n \in \mathbb{C} \quad n=0,1,2,\ldots$$

is called a <u>power series</u>, and defines a function on the set of points for which the series converges.

<u>Theorem</u>. Suppose that $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$. Then the set of points z for which the series converges is one of the following:

(i.) the single point $z=\alpha$.
(ii) the disk $D_\rho(\alpha) = \{z : |z-\alpha| < \rho\}$ along with part (either none, some or all) of the circle $C_\rho(\alpha) = \{z : |z-\alpha| = \rho\}$.
(iii) the entire complex plane.

<u>Proof.</u> The series converges absolutely if

$$\limsup_{n\to\infty} |c_n (z-\alpha)^n|^{\frac{1}{n}} < 1,$$

or equivalently

$$|z-\alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < 1. \qquad (*)$$

There are three possibilities for $L = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$:

$$L=0, \quad L=\infty, \quad \text{or} \quad 0<L<\infty.$$

if $L=0$ $(*)$ always holds and (iii) occurs.

If $L=\infty$ $|z-\alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \infty > 1$ except for $z=\alpha$, and series diverges. If $z=\alpha$, $\sum_{n=0}^{\infty} c_n (z-\alpha)^n = c_0$ and series is convergent. This is case (i).

If $0<L<\infty$, series converges if $|z-\alpha| < \frac{1}{L} =: \rho$, diverges if $|z-\alpha| > \rho$. But there are severeal possibilities if $|z-\alpha| = \rho$. This corresponds to case (ii). ∎

we call the number ρ the <u>radius of convergence</u> of the power series. In case (i) $\rho = 0$, in case (iii) $\rho = \infty$.

<u>Remark.</u> For the power series function
$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n,$$
we can find ρ, its radius of convergence, by any of the following methods

i) Cauchy's root test: $\rho = \dfrac{1}{\lim\limits_{n\to\infty} |c_n|^{\frac{1}{n}}}$ provided the limit exists.

ii) Cauchy-Hadamard formula:
$$\rho = \frac{1}{\limsup\limits_{n\to\infty} |c_n|^{\frac{1}{n}}}$$

iii) d'Alembert's ratio test:
$$\rho = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$$ provided the limit exists.

<u>Example.</u> Find the radius of convergence of the following series.

a) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{n+2}{3n+1}\right)^n (z-4)^n$

b) $\displaystyle\sum_{n=1}^{\infty} c_n z^n = 4z + 5^2 z^2 + 4^3 z^3 + 5^4 z^4 + 4^5 z^5 + 5^6 z^6 + 4^7 z^7 + \cdots$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n.$

<u>Solution.</u> a) $|c_n|^{\frac{1}{n}} = \dfrac{n+2}{3n+1} \to \dfrac{1}{3}$ as $n\to\infty \Rightarrow \rho = \dfrac{1}{\frac{1}{3}} = 3$ by Cauchy's root test.

b) $\{c_n\} = \{4, 5^2, 4^3, 5^4, 4^5, 5^6, 4^7, \ldots\} \Rightarrow \{|c_n|^{\frac{1}{n}}\} = \{4, 5, 4, 5, 4, 5, \ldots\}$

$\limsup\limits_{n\to\infty} |c_n|^{\frac{1}{n}} = 5 \Rightarrow \rho = \dfrac{1}{5}$ by Cauchy-Hadamard formula.

c) $\left|\dfrac{c_{n+1}}{c_n}\right| = \dfrac{1}{n+1} \to 0$ as $n\to\infty \Rightarrow \rho = \dfrac{1}{0} = \infty$ by D'Alembert's ratio test.

Examples. 1. Consider the series

$$\sum_{n=0}^{\infty} z^n, \quad \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{z^n}{n}$$

(a) Show that each series has radius of convergence 1.

(b) Show that the first series converges nowhere on $C_1(0)=\{z:|z|=1\}$.

(c) Show that the second series converges everywhere on $C_1(0)=\{z:|z|=1\}$

(d) It turns out that the third series converges everywhere on $C_1(0)$ except at the point $z=1$.

2. Find the radius of convergence of the series

(a) $\sum_{n=0}^{\infty} n!\, z^n$

(b) $\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} z^n$

(c) $\sum_{n=0}^{\infty} \frac{n(n-1)}{(3+4i)^n} z^n$

(d) $\sum_{n=0}^{\infty} \frac{2^n}{1+3^n} z^n$

(e) $\sum_{n=0}^{\infty} \frac{n^n}{n!} z^n$

3. Show that for $|z-i|<\sqrt{2}$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

Solutions. 1. (a) First series: $c_n=1 \Rightarrow \lim_{n\to\infty} |c_n|^{\frac{1}{n}}=1 \Rightarrow \rho=\frac{1}{1}=1.$

Second series: $c_n=\frac{1}{n^2} \Rightarrow \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}=1 \Rightarrow \rho=\frac{1}{1}=1.$

Third series: $c_n=\frac{1}{n} \Rightarrow \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \frac{n}{n+1}=1 \Rightarrow \rho=\frac{1}{1}=1.$

(b) If $|z|=1$, then $|z^n|=|z|^n=1 \to 1 \neq 0$ as $n\to\infty$. So $z^n \not\to 0$ as $n\to\infty \Rightarrow \sum_{n=0}^{\infty} z^n$ diverges (by n^{th} term test) since z is an arbitrary point on $C_1(0)$, this proves that the first converges nowhere on $C_1(0)$.

(c) If $|z|=1$, then $\left|\frac{z^n}{n^2}\right| = \frac{|z|^n}{n^2} = \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent (by p-test), series converges absolutely. Since z is an arbitrary point of $C_1(0)$. The second series converges everywhere on $C_1(0)$.

(d) If $z=1$, the series becomes $\sum_{n=1}^{\infty}\frac{1}{n}$ and clearly diverges (by p-test) if $|z|=1$, $z\neq 1$, then z is of the form $z=e^{i\theta}$, $0<\theta<2\pi$, and
$$\sum_{n=1}^{\infty}\frac{z^n}{n} = \sum_{n=1}^{\infty}\frac{e^{in\theta}}{n} = \sum_{n=1}^{\infty}\frac{\cos n\theta}{n} + i\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}.$$

At this stage we recall the Dirichlet test for series from Advanced Calculus.

Dirichlet test: Let $\{a_n\}$ and $\{b_n\}$ be sequences which obey the following:
$$\lim_{n\to\infty} a_n = 0$$
$$\sum |a_{n+1}-a_n| \quad \text{converges}$$
The series $\sum b_n$ is such that its partial sums are uniformly bounded.

Then $\sum a_n b_n$ converges. ■

Using Dirichlet test we may prove $\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$ is convergent: let $a_n=\frac{1}{n}$ and $b_n=\sin(n\theta)$. Clearly $\lim_{n\to\infty} a_n = 0$ and
$$\sum_{n=1}^{\infty}|a_{n+1}-a_n| = \sum_{n=1}^{\infty}\frac{1}{n(n+1)} \text{ is convergent (Comparison with } \sum_{n=1}^{\infty}\frac{1}{n^2}\text{)}.$$

Note that $\sum_{n=1}^{N}\sin(n\theta) = \sin\theta + \sin(2\theta) + \sin(3\theta) + \cdots + \sin(N\theta)$ and so
$$\sin\left(\tfrac{\theta}{2}\right)\sum_{n=1}^{N}\sin(n\theta) = \sin\left(\tfrac{\theta}{2}\right)\sin\theta + \sin\left(\tfrac{\theta}{2}\right)\sin(2\theta) + \cdots + \sin\left(\tfrac{\theta}{2}\right)\sin(N\theta).$$
Using the identity $2\sin A\sin B = \cos(B-A) - \cos(B+A)$, we get
$$2\sin\left(\tfrac{\theta}{2}\right)\sum_{n=1}^{N}\sin(n\theta) = \left(\cos\left(\tfrac{\theta}{2}\right) - \cos\left(\tfrac{3\theta}{2}\right)\right) + \left(\cos\left(\tfrac{3\theta}{2}\right) - \cos\left(\tfrac{5\theta}{2}\right)\right) + \cdots + \left(\cos\left(N-\tfrac{1}{2}\right)\theta - \cos\left(N+\tfrac{1}{2}\right)\theta\right)$$

and so $\quad 2\sin\left(\frac{\theta}{2}\right)\sum_{n=1}^{N}\sin(n\theta) = \cos\left(\frac{\theta}{2}\right) - \cos\left(n+\frac{1}{2}\right)\theta$, or equivalently,

$$\sum_{n=1}^{N}\sin(n\theta) = \frac{\cos\left(\frac{\theta}{2}\right) - \left(n+\frac{1}{2}\right)\theta}{2\sin\left(\frac{\theta}{2}\right)}$$

It is now clear that the partial sums of $\sum_{n=1}^{\infty}\sin(n\theta)$ are bounded by $\frac{1}{\sin\left(\frac{\theta}{2}\right)}$. Thus $\sum_{n=1}^{\infty}\frac{\sin(n\theta)}{n}$ converges by Dirichlet's test.

Similarly, one can prove that if $0<\theta<2\pi$, $\sum_{n=1}^{\infty}\frac{\cos(n\theta)}{n}$ converges and hence $\sum_{n=1}^{\infty}\frac{e^{in\theta}}{n}$ converges.

2. (a) $\quad c_n = n! \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \frac{(n+1)!}{n!} = (n+1) \to \infty$ as $n\to\infty \Rightarrow \rho = \frac{1}{\infty} = 0$ by D'Alembert's ratio test.

(b) $\quad c_n = \frac{(n!)^2}{(2n)!} \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \frac{((n+1)!)^2}{(2(n+1))!}\cdot\frac{(2n)!}{(n!)^2} = \frac{(n+1)^2\cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{\cancel{(n!)^2}}$

$\Rightarrow \quad \left|\frac{c_{n+1}}{c_n}\right| = \frac{n^2+2n+1}{4n^2+6n+2} \to \frac{1}{4}$ as $n\to\infty \Rightarrow \rho = \frac{1}{\frac{1}{4}} = 4$ by D'Alembert's ratio test.

(c) $\quad c_n = \frac{n(n-1)}{(3+4i)^n} \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(n+1)n}{(3+4i)^{n+1}}\cdot\frac{(3+4i)^n}{n(n-1)}\right| = \frac{n+1}{n-1}\frac{1}{|3+4i|} = \frac{n+1}{5(n-1)} \to \frac{1}{5}$

as $n\to\infty \Rightarrow \rho = \frac{1}{\frac{1}{5}} = 5$ by D'Alembert's ratio test.

(d) $\quad c_n = \frac{2^n}{1+3^n} \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \frac{2^{n+1}}{1+3^{n+1}}\cdot\frac{1+3^n}{2^n} = 2\cdot\frac{\cancel{3^n}\left(1+\overbrace{\frac{1}{3^n}}^{\to 0}\right)}{\cancel{3^n}\left(3+\underbrace{\frac{1}{3^n}}_{\to 0}\right)} \to \frac{2}{3}$

as $n\to\infty \Rightarrow \rho = \frac{1}{\frac{2}{3}} = \frac{3}{2}$.

(e) $\quad c_n = \frac{n^n}{n!} \Rightarrow \left|\frac{c_{n+1}}{c_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} = \frac{\cancel{(n+1)}(n+1)^n\cancel{n!}}{\cancel{(n+1)}\cancel{n!}\,n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$

So $\quad \left|\frac{c_{n+1}}{c_n}\right| \to e$ as $n\to\infty$ and $\rho = \frac{1}{e}$.

3. Let $c_n = \frac{1}{(1-i)^{n+1}}$. Then $\left|\frac{c_{n+1}}{c_n}\right| = \left|\frac{(1-i)^n}{(1-i)^{n+1}}\right| = \frac{1}{|1-i|} = \frac{1}{\sqrt{2}} \to \frac{1}{\sqrt{2}}$

as $n \to \infty \Rightarrow \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$ converges if $|z-i| < \rho = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$.

And

$$\frac{1}{1-z} = \frac{1}{1-i+i-z} = \frac{1}{1-i-(z-i)} = \frac{1}{(1-i)\left(1-\frac{z-i}{1-i}\right)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}}$$

Remind that $\sum_{n=0}^{\infty} w^n = \frac{1}{1-w}$ if $|w|<1$.

Putting $w = \frac{z-i}{1-i}$, we obtain $|w|<1$ whenever $|z-i|<\sqrt{2}$

and so

$$\frac{1}{1-z} = \frac{1}{1-i}\,\frac{1}{1-\frac{z-i}{1-i}} = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n = \sum_{n=0}^{\infty}\frac{(z-i)^n}{(1-i)^{n+1}}.$$

<u>Theorem.</u> Suppose that the function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has radius of convergence $\rho > 0$. Then

(i) f is infinitely differentiable for all $z \in D_\rho(\alpha) = \{z : |z-\alpha| < \rho\}$.

In fact,

(ii) for all k, $f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (z-\alpha)^{n-k}$; and

(iii) $c_k = \frac{f^{(k)}(\alpha)}{k!}$, where $f^{(k)}$ denotes the k^{th} derivative of f. (When $k=0$, $f^{(k)}$ denotes the function f itself, so that $f^{(0)}(z) = f(z)$ for all z.)

<u>Lemma 1.</u> Suppose that the function $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has radius of convergence $\rho > 0$. Then $g(z) = \sum_{n=1}^{\infty} n c_n (z-\alpha)^{n-1}$ has also the same radius of convergence ρ.

<u>Proof.</u> Clearly g has the radius of convergence

$$\rho_1 = \frac{1}{\limsup_{k\to\infty} |(k+1)c_{k+1}|^{\frac{1}{k}}}.$$

Note that $\limsup_{k\to\infty} (k+1)^{\frac{1}{k}} = \lim_{k\to\infty} (k+1)^{\frac{1}{k}} = 1$ (Exercise!)

and so

$$\limsup_{k\to\infty}|(k+1)c_{k+1}|^{\frac{1}{k}} = \lim_{k\to\infty}(k+1)^{\frac{1}{k}} \limsup_{k\to\infty}|c_{k+1}|^{\frac{1}{k}} = \limsup_{k\to\infty}|c_{k+1}|^{\frac{1}{k}}$$

(Exercise: show why $\limsup_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \limsup_{n\to\infty} b_n$ whenever $\lim_{n\to\infty} a_n$ exists. It is not in general true that

$$\limsup_{n\to\infty}(a_n b_n) = \left(\limsup_{n\to\infty} a_n\right)\left(\limsup_{n\to\infty} b_n\right) \quad)$$

One can show that

$$\limsup_{k\to\infty}|c_{k+1}|^{\frac{1}{k}} = \limsup_{k\to\infty}|c_k|^{\frac{1}{k}} \quad \text{(Exercise!)}$$

and hence
$$\rho_1 = \frac{1}{\limsup_{k\to\infty}|(k+1)c_{k+1}|^{\frac{1}{k}}} = \frac{1}{\limsup_{k\to\infty}|c_k|^{\frac{1}{k}}} = \rho. \quad ∎$$

<u>Lemma 2.</u> If $|a|<r$, $|b|<r$ then $\left|\frac{a^n-b^n}{a-b}\right| < nr^{n-1}$.

<u>Proof.</u> Clearly, $$\frac{a^n-b^n}{a-b} = \frac{\cancel{(a-b)}(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\dots+ab^{n-2}+b^{n-1})}{\cancel{a-b}}$$

and so
$$\left|\frac{a^n-b^n}{a-b}\right| \le |a|^{n-1}+|a|^{n-2}|b|+|a|^{n-3}|b|^2+\dots+|a||b|^{n-2}+|b|^{n-1}$$
$$< \underbrace{r^{n-1}+r^{n-2}r+r^{n-3}r^2+\dots+rr^{n-2}+r^{n-1}}_{n \text{ terms}} < nr^{n-1} \quad ∎$$

<u>Proof of Theorem.</u>

<u>Step 1</u>: (f is differentiable on $D_\rho(\alpha)$ and $f'(z) = \sum_{n=1}^{\infty} nc_n(z-\alpha)^{n-1}$)

Let $g(z) = \sum_{n=1}^{\infty} nc_n(z-\alpha)^{n-1}$, $S_j(z) = \sum_{n=0}^{j} c_n(z-\alpha)^n$, $R_j(z) = \sum_{n=j+1}^{\infty} c_n(z-\alpha)^n$.

Note that $f(z) = S_j(z) + R_j(z)$.

Let $z_0 \in D_\rho(\alpha)$. We will show that $f'(z_0) = g(z_0)$.

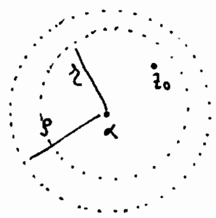
(To do this, we must show that
$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = g(z_0)$$

or equivalently,

$\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta. \quad)$$

Let $\varepsilon > 0$ be given. Choose $r < \rho$ so that $z_0 \in \mathcal{D}_r(\alpha) = \{z : |z - \alpha| < r\}$



For any $z \in \mathcal{D}_\rho(\alpha)$, and for any $j \in \mathbb{N}$,

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \frac{(S_j(z) + R_j(z)) - (S_j(z_0) + R_j(z_0))}{z - z_0} - g(z_0)$$

$$= \left\{ \frac{S_j(z) - S_j(z_0)}{z - z_0} - S_j'(z_0) \right\} + \left\{ S_j'(z_0) - g(z_0) \right\} + \left\{ \frac{R_j(z) - R_j(z_0)}{z - z_0} \right\}$$

$$=: A_j(z, z_0) + B_j(z_0) + C_j(z, z_0). \qquad (*)$$

Note that,

$$|C_j(z, z_0)| = \left| \frac{\sum_{n=j+1}^{\infty} c_n ((z-\alpha)^n - (z_0 - \alpha)^n)}{z - z_0} \right| \leq \sum_{n=j+1}^{\infty} |c_n| \left| \frac{(z-\alpha)^n - (z_0-\alpha)^n}{(z-\alpha) - (z_0 - \alpha)} \right|.$$

It follows from Lemma 2 that

$$|C_j(z, z_0)| \leq \sum_{n=j+1}^{\infty} |c_n| n r^{n-1} \quad \text{whenever} \quad |z - \alpha| < r.$$

Since $\sum_{n=1}^{\infty} |c_n| n r^{n-1}$ converges by Lemma 1, there exists $N_1 \in \mathbb{N}$

so that $\sum_{n=j+1}^{\infty} |c_n| n r^{n-1} < \frac{\varepsilon}{3}$ whenever $j \geq N_1$. Therefore

$\exists N_1 \in \mathbb{N}$ so that

$$|C_j(z, z_0)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad |z - \alpha| < r \ \text{and} \ j \geq N_1 \qquad (**)$$

Clearly, $S_j'(z_0) = \sum_{n=1}^{j} n c_n (z_0 - \alpha)^{n-1}$ and converges to $g(z_0)$ as $j \to \infty$

by Lemma 1. Therefore there exists $N_2 \in \mathbb{N}$ so that

$$|S_j'(z_0) - g(z_0)| < \frac{\varepsilon}{3} \quad \text{whenever} \quad j \geq N_2.$$ That is

$\exists N_2 \in \mathbb{N}$ so that $|B_j(z_0)| < \frac{\varepsilon}{3}$ whenever $j \geq N_2$ $\qquad (***)$

For fixed N, S_N is a polynomial and $S_N'(z_0)$ exists and

$$S_N'(z_0) = \lim_{z \to z_0} \frac{S_N(z) - S_N(z_0)}{z - z_0}$$

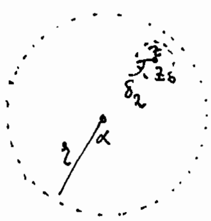
So, there exists $\delta_1 > 0$ such that

$$\left| \frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0) \right| < \frac{\varepsilon}{3} \quad \text{whenever} \quad 0 < |z - z_0| < \delta_1,$$

or equivalently

For fixed $N \in \mathbb{N}$, $\exists\, \delta_1 > 0$ so that $|A_N(z, z_0)| < \frac{\varepsilon}{3}$ whenever $0 < |z - z_0| < \delta$. (****)

Let $\delta_2 > 0$ be so that $\{z : |z - z_0| < \delta_2\} \subset \{z : |z - \alpha| < \varrho\}$.



Let $N = \max\{N_1, N_2\}$ choose δ_1 so that (****) holds and let $\delta = \min\{\delta_1, \delta_2\}$.

By (*) $\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| \leq |A_N(z, z_0)| + |B_N(z_0)| + |C_N(z, z_0)|$

If $0 < |z - z_0| < \delta$ then $|z - \alpha| < \varrho$ and by (**) and (***) (****), we have $|C_N(z, z_0)| < \frac{\varepsilon}{3}$, $|B_N(z_0)| < \frac{\varepsilon}{3}$ and $|A_N(z, z_0)| < \frac{\varepsilon}{3}$.

Thus
$$\left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

This completes Step 1.

<u>Step 2.</u> (f is k times differentiable on $D_\varrho(\alpha)$ and
$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)\, c_n (z - \alpha)^{n-k}.)$$

Apply Step 1 to $f'(z)$, then to $f''(z), \ldots$, to $f^{(k-1)}(z)$.

Thus (ii) of Theorem holds and so (i) as an easy consequence.
To obtain (iii) put $z = \alpha$ in (ii). ▨

Example. Show that $\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2}$ for all $z \in D_1(0) = \{z : |z|<1\}$.

Solution Since $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ for all $z \in D_1(0)$, it follows from the theorem that

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{k=0}^{\infty}(k+1)z^k. \quad \text{Since} \quad \left(\frac{1}{1-z}\right)' = \frac{-1(-1)}{(1-z)^2} = \frac{1}{(1-z)^2}$$

we get $\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2}$.

Example. Show that $\sum_{n=0}^{\infty}(n+1)^2 z^n = \frac{1+z}{(1-z)^3}$. For what values of z is this valid?

Solution Let $C_n = (n+1)^2$. Note that $\left|\frac{C_{n+1}}{C_n}\right| = \left|\frac{(n+2)^2}{(n+1)^2}\right| \to 1$ as $n \to \infty$ $\quad \rho = \frac{1}{\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right|} = 1 \Rightarrow$ series converges when $|z|<1$.

If $|z|=1$, $|(n+1)^2 z^n| = (n+1)^2 \to \infty \neq 0 \Rightarrow$ series diverges when $|z|=1$.

On the other hand, if $|z|<1$, by previous example we have

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2} \quad \text{and hence}$$

$$\sum_{n=0}^{\infty}(n+1)z^{n+1} = \frac{z}{(1-z)^2} \quad \text{and by theorem}$$

$$\sum_{n=0}^{\infty}(n+1)^2 z^n = \frac{d}{dz}\left(\frac{z}{(1-z)^2}\right) = \frac{1(1-z)^2 - 2(1-z)(-1)z}{(1-z)^4}$$

$$= \frac{(1-z)+2z}{(1-z)^3} = \frac{1+z}{(1-z)^3}.$$

Elementary Functions

The complex exponent function

e^z, $z \in \mathbb{C}$ (also written as $\exp(z)$, $z \in \mathbb{C}$) is defined by the power series:

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

Remark.
This definition agrees with that of real exponent function when z is a real number.

Properties

(i) e^z is an entire function and $\frac{d}{dz}(e^z) = e^z$ for all $z \in \mathbb{C}$.

(ii) $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

(iii) If θ is a real number, then
$e^{i\theta} = \cos\theta + i\sin\theta$ (promised to be established earlier)

Proof: (i) Let $c_n = \frac{1}{n!}$. Clearly $\frac{|c_{n+1}|}{|c_n|} = \frac{1}{n+1} \to 0$ as $n \to \infty$ and $\rho = \frac{1}{0} = \infty \Rightarrow \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ converges everywhere and defines a function on $\mathbb{C}$.

By the last theorem of the last chapter, f is infinitely differentiable on $\mathbb{C}$ and

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \quad (\text{put } n-1 = k)$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

(ii) Let $\zeta \in \mathbb{C}$ be arbitrary. Define $g(z) = e^z e^{\zeta - z}$.
By the chain rule $g'(z) = e^z e^{\zeta-z} + e^z e^{\zeta-z} \cdot (-1) = 0 \Rightarrow$
$g(z) = \text{constant}$, hence $g(z) = g(0) = e^0 e^{\zeta} = e^{\zeta}$.
Hence $e^z e^{\zeta - z} = e^{\zeta}$ for all $z, \zeta \in \mathbb{C}$.

putting $z = z_1$, $\zeta = z_1 + z_2$, we get

$$e^{z_1} e^{z_2} = e^{z_1+z_2}.$$

(iii) $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$ The general term $\frac{(i\theta)^n}{n!} = \begin{cases} \frac{\theta^n}{n!} & \text{if } n = 0 \bmod 4 \\ i\frac{\theta^n}{n!} & \text{if } n = 1 \bmod 4 \\ -\frac{\theta^n}{n!} & \text{if } n = 2 \bmod 4 \\ -i\frac{\theta^n}{n!} & \text{if } n = 3 \bmod 4 \end{cases}$

or equivalently $\frac{(i\theta)^n}{n!} = \begin{cases} (-1)^k \frac{\theta^{2k}}{(2k)!} & \text{if } n = 2k, \ k \in \mathbb{N} \cup \{0\} \\ (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} i & \text{if } n = 2k+1, \ k \in \mathbb{N} \cup \{0\} \end{cases}$

Therefore $e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!}$

$$= \cos\theta + i \sin\theta.$$

<u>More properties</u>

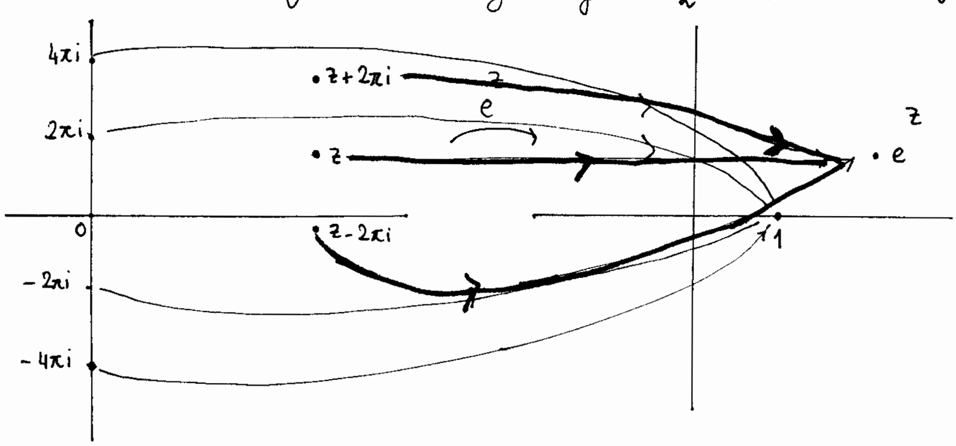
(i) $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$ for $z = x+iy \in \mathbb{C}$.

(ii) $|e^z| = e^x$, $z = x+iy \in \mathbb{C}$

(iii) $e^{z+i2n\pi} = e^z$, $z \in \mathbb{C}$, $n \in \mathbb{Z}$

(iv) $e^z = 1$ if and only if $z = i2n\pi$, $n \in \mathbb{Z}$

(v) $e^{z_1} = e^{z_2}$ if and only if $z_2 = z_1 + i2n\pi$ for some $n \in \mathbb{Z}$.



<u>Proofs</u> are left as exercises!

__Example__ Express e^z in the form $u+iv$ for the following values of z.

(a) $\frac{1}{2} - i\frac{\pi}{4}$

(b) $-1 + i\frac{3\pi}{2}$

(c) $\frac{\pi}{3} - 2i$

__Solution.__ (a) $e^{\frac{1}{2}-i\frac{\pi}{4}} = e^{\frac{1}{2}} e^{-i\frac{\pi}{4}} = e^{\frac{1}{2}} e^{i(-\frac{\pi}{4})} = e^{\frac{1}{2}}\left(\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})\right)$

$= e^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{e}{2}} - i\sqrt{\frac{e}{2}} \approx 1.17 - 1.17i.$

(b) $e^{-1+i\frac{3\pi}{2}} = e^{-1}\left(\underbrace{\cos(\tfrac{3\pi}{2})}_{=0} + i\underbrace{\sin(\tfrac{3\pi}{2})}_{-1}\right) = -\frac{i}{e} \approx 0.37i$

(c) $e^{\frac{\pi}{3}-2i} = e^{\frac{\pi}{3}} e^{-2i} = e^{\frac{\pi}{3}}(\cos(-2) + i\sin(-2))$

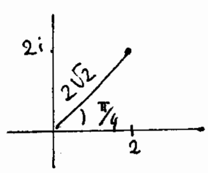
$= e^{\frac{\pi}{3}}(\cos 2 - i\sin 2) = e^{\frac{\pi}{3}}\cos 2 - ie^{\frac{\pi}{3}}\sin 2$

$\approx 2.85(-0.42 - i\,0.91) = -1.197 - 2.5935i.$

__Example.__ Solve (a) $e^z = 2+2i$

(b) $e^z = -1+i\sqrt{3}$

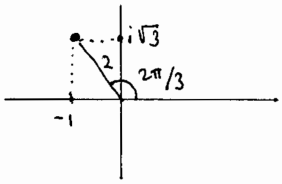
__Solution.__ (a) $e^z = 2+2i = 2\sqrt{2}\,e^{i\frac{\pi}{4}} \Rightarrow e^z = e^x e^{iy} = 2\sqrt{2}\,e^{i\frac{\pi}{4}} \Rightarrow$



$e^x = 2\sqrt{2}$ and $y = \frac{\pi}{4} + 2k\pi$, $k \in \mathbb{Z}$

$x = \ln(2\sqrt{2})$
$= \frac{3}{2}\ln 2$ $\Rightarrow$ $z = \frac{3}{2}\ln 2 + i\left(\frac{\pi}{4} + 2k\pi\right)$, $k \in \mathbb{Z}$.

(b) $e^x e^{iy} = 2e^{i\frac{2\pi}{3}} \Rightarrow e^x = 2$ $\quad y = \frac{2\pi}{3} + 2k\pi$, $k \in \mathbb{Z}$



$z = \ln 2 + i\left(\frac{2\pi}{3} + 2k\pi\right)$, $k \in \mathbb{Z}$.

__Example.__ Find $f'(z)$ if $f(z) = z^4 e^{z^3}$.

__Solution.__ $f'(z) = 4z^3 e^{z^3} + z^4 e^{z^3}\cdot 3z^2 = 4z^3 e^{z^3} + 3z^6 e^{z^3}.$

Example. Show that $\lim_{z \to i\pi} \frac{e^z+1}{z-i\pi} = -1$.

Solution. $e^{i\pi}+1 = (\underbrace{\cos\pi}_{-1} + i\underbrace{\sin\pi}_{0})+1 = -1+1=0$

$i\pi - i\pi = 0$

$(z-i\pi)' = 1 \neq 0$ L'Hospital's rule can be applied.

$$\lim_{z\to i\pi} \frac{e^z+1}{z-i\pi} = \lim_{z\to i\pi} \frac{e^z}{1} = e^{i\pi} = -1.$$

Example. Prove $|e^{z^2}| \leq e^{|z|^2}$.

Solution. Let $w = z^2$. we need to prove $|e^w| \leq e^{|w|}$.

$$|e^w| = |e^{Re(w)+i\,Im(w)}| = |e^{Re(w)} \cdot e^{i\,Im(w)}| = |e^{Re(w)}||e^{i\,Im(w)}|$$

$$= e^{Re(w)} \underbrace{|\cos(Im\,w) + i\sin(Im\,w)|}_{1} = e^{Re(w)}$$

Since $Re(w) \leq |Re(w)| \leq |w|$, we get $|e^w| \leq e^{|w|}$.

Example. Show that $e^{\bar{z}}$ is nowhere analytic.

Solution. $e^{\bar{z}} = e^{x-iy} = e^x e^{-iy} = e^x(\cos(-y) + i\sin(-y)) = e^x\cos y - ie^x \sin y$

$u(x,y) = e^x \cos y \qquad v(x,y) = -e^x \sin y.$

$u_x = e^x\cos y \qquad v_y = -e^x\cos y, \qquad u_y = -e^x \sin y, \qquad v_x = -e^x \sin y$

$u_x = v_y$ iff $\cos y = 0 \qquad u_y = -v_x$ iff $\sin y = 0$

$\cos y$ and $\sin y$ do not simultaneously vanish, hence f is nowhere differentiable and so nowhere analytic.

Example. Find the image of $\mathbb{C}$ under e^z.

Solution $e^z = e^{x+iy} = e^x\cos y + ie^x \sin y$, e^x is never zero, $\sin y$ and $\cos y$ can't be simultaneously 0, so $e^z \neq 0$ for any $z \in \mathbb{C}$.

That is, 0 is not in the image set of e^z.

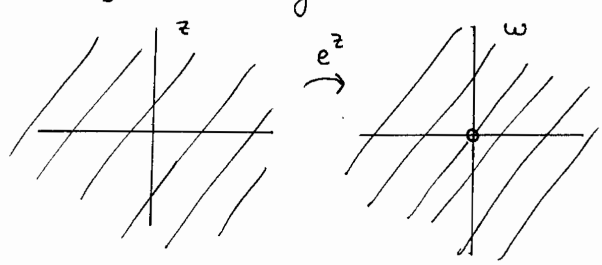
On the other hand, if $w \neq 0 \in \mathbb{C}$, we can show that there exists $z \in \mathbb{C}$ such that $e^z = w$ (and hence the image set is $\mathbb{C}\setminus\{0\}$)

Indeed, $0 \neq w = \rho e^{i\theta} = e^{z} = e^{x}e^{iy}$ if and only if

$$\rho = e^{x} \quad \text{and} \quad \theta = y + 2k\pi \quad \text{for some } k \in \mathbb{Z}.$$

or equivalently $\quad x = \ln\rho \qquad y = \theta + 2k\pi$ for some $k \in \mathbb{Z}$

Thus for any point $z = \ln\rho + i(\theta + 2k\pi) = \ln|w| + i(\mathrm{Arg}\,w + 2k\pi)$, we have $e^{z} = w$. Since $w \neq 0$ is arbitrary, e^{z} maps $\mathbb{C}$ onto $\mathbb{C}\setminus\{0\}$ (Clearly e^{z} is not one-to-one on $\mathbb{C}$)



Example. Show that e^{z} maps $\mathcal{D} = \{(x,y) : -\pi < y \leq \pi\}$ one-to-one and onto the set $S = \{w : w \neq 0\}$.

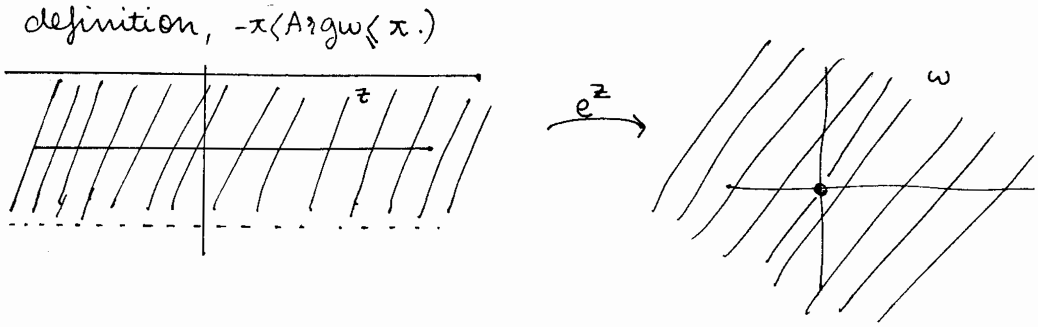
Solution. $e^{z_1} = e^{z_2} \iff z_2 = z_1 + i2n\pi$ for some $n \in \mathbb{Z}$.

$\iff \quad \mathrm{Re}\, z_1 = \mathrm{Re}\, z_2 \qquad \mathrm{Im}\, z_1 = \mathrm{Im}\, z_2 + 2n\pi \ (*)$ for some $n \in \mathbb{Z}$

But if $z_1, z_2 \in \mathcal{D}$ $\quad -\pi < \mathrm{Im}\, z_1 \leq \pi$ and $-\pi < \mathrm{Im}\, z_2 \leq \pi$, and so $(*)$ holds if and only if $n = 0$, that is $\mathrm{Im}\, z_1 = \mathrm{Im}\, z_2$.

Thus $z_1 = z_2$. Therefore f is one-to-one on $\mathcal{D}$.

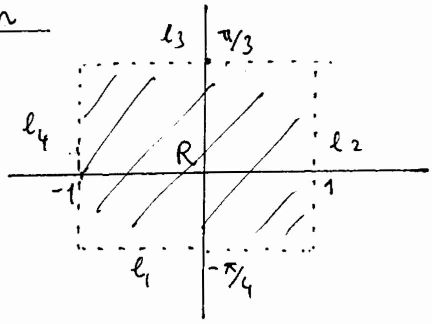
On the other hand, if $w \in \mathbb{C}\setminus\{0\}$, using the ideas in the solution of the previous example, we can show that for $z = \ln|w| + i\,\mathrm{Arg}\,w$, $e^{z} = w$, and clearly $z \in \mathcal{D}$. (by definition, $-\pi < \mathrm{Arg}\,w \leq \pi$.)



The strip $\{(x,y) : -\pi < y \leq \pi\}$ is called the <u>fundamental period strip</u> of the exponential function.

Example Find the image of $R = \{(x,y): -1<x<1,\ -\frac{\pi}{4}<y<\frac{\pi}{3}\}$ under e^z.

Solution

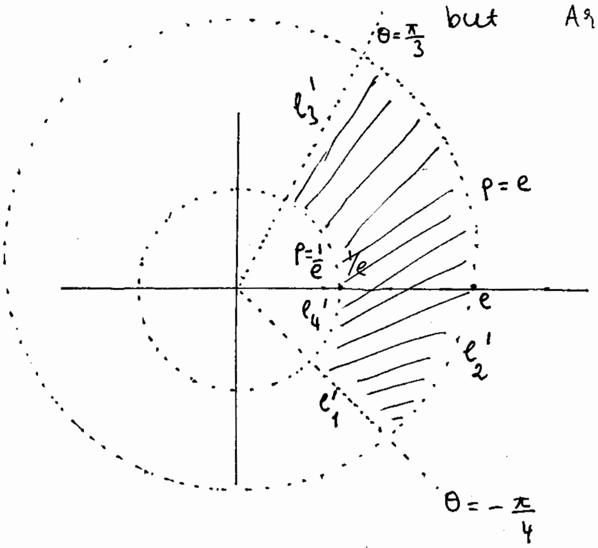


on $l_1: -1<x<1,\ y=-\frac{\pi}{4}$

on l_1' $e^z = e^x e^{iy} = e^x e^{-i\frac{\pi}{4}} = w$

$|w| = e^x$ changes from e^{-1} to e^1 (that is from $\frac{1}{e}$ to e)

but $\operatorname{Arg} w = -\frac{\pi}{4}$ is constant.



on l_2: $x=1,\ -\frac{\pi}{4}<y<\frac{\pi}{3}$

on l_2' $e^x e^{iy} = e\, e^{iy} = w$

$|w| = e$ is constant

$\operatorname{Arg} w = y$ changes from $-\frac{\pi}{4}$ to $\frac{\pi}{3}$.

on l_3': $|w|$ changes from $\frac{1}{e}$ to e.

$\operatorname{Arg} w = \frac{\pi}{3}$ is constant.

on l_4': $|w| = \frac{1}{e}$ is constant $\operatorname{Arg} w$ changes from $-\frac{\pi}{4}$ to $\frac{\pi}{3}$.

Exercise Show that (under e^z) (a) the horizontal line $z=t+ib$, for $-\infty<t<\infty$ in the z-plane, is mapped onto the ray $w=e^t e^{ib}$ that is inclined at an angle $\phi=b$.

(b) The vertical segment $z=a+i\theta$, $-\pi<\theta\le\pi$ in the z-plane, is mapped onto the circle centered at the origin with radius e^a in the w-plane.

The complex logarithm

Remind that the function e^z is a many-to-one function ($e^{z+2n\pi i} = e^z$, $\forall z \in \mathbb{C}$, $n \in \mathbb{Z}$) from $\mathbb{C}$ to $\mathbb{C}\setminus\{0\}$.

For $z \neq 0$, we define the multivalued function log as the inverse of the exponential function. That is,

$$\log z = w \text{ if and only if } e^w = z.$$

Remind that $z = e^w$ has infinitely many solutions (for w), namely

$$w = \ln|z| + i(\operatorname{Arg} z + 2n\pi), \quad n \in \mathbb{Z}.$$

Because $\arg z$ is the set $\{\operatorname{Arg} z + 2n\pi\}$ we can write

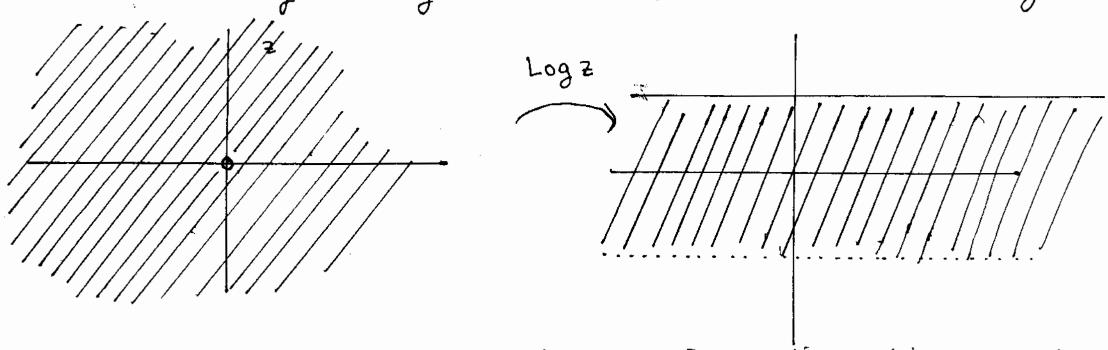
$$\log z = \ln|z| + i \arg z = \{\ln|z| + i(\operatorname{Arg} z + 2n\pi), n \in \mathbb{Z}\}.$$

We call any of the values belonging to $\log z$, a <u>logarithm</u> of z. When $n=0$, we have a special situation:

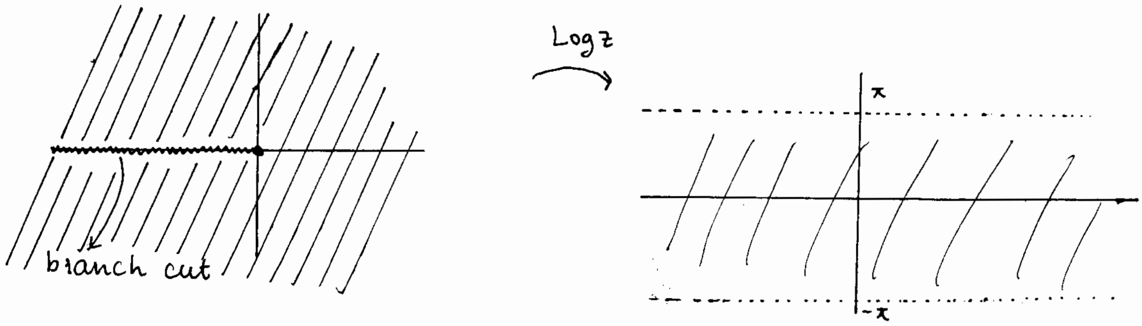
For $z \neq 0$, we define $\operatorname{Log} z$, the <u>principal value of the logarithm</u> by

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$$

The domain of Log is $\mathbb{C}\setminus\{0\}$ and its range is $\{w : -\pi < \operatorname{Im} w \leq \pi\}$



<u>Remark.</u> Remind that $\operatorname{Arg} z$ is discontinous at each point along the negative real axis, hence so is $\operatorname{Log} z$.
If we remove $\{z \in \mathbb{C} : z < 0\}$ from the domain of $\operatorname{Log} z$, we obtain a branch of $\log z$ which is called the <u>principal branch</u> of the logarithm function.

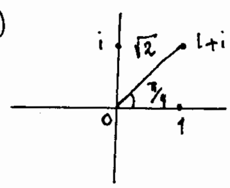


$$z=\rho e^{i\theta}\ ,\quad \rho>0\ ,\ -\pi<\theta<\pi \quad\Rightarrow\quad \operatorname{Log} z=\ln\rho+i\theta$$

Because any branch of the multivalued function arg is discontinuous along some ray, a corresponding branch of the logarithm will have a discontinuity along that same ray.

__Example.__ Find
(a) $\log(1+i)$, $\operatorname{Log}(1+i)$ (c) $\log(-e)$, $\operatorname{Log}(-e)$
(b) $\log(-i)$, $\operatorname{Log}(-i)$

__Solution.__ (a)

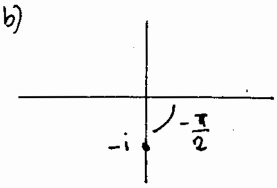


$|1+i|=\sqrt{2}$ $\quad \operatorname{Arg}(1+i)=\frac{\pi}{4}$.

$$\begin{aligned}\log(1+i)&=\ln|1+i|+i\,(\operatorname{Arg}(1+i)+2n\pi)\ ,\quad n\in\mathbb{Z}\\&=\ln\sqrt{2}+i\left(\frac{\pi}{4}+2n\pi\right)\ ,\quad n\in\mathbb{Z}\\&=\frac{\ln 2}{2}+i\,\frac{(8n+1)\pi}{4}\ ,\quad n\in\mathbb{Z}\end{aligned}$$

$\operatorname{Log}(1+i)=\frac{\ln 2}{2}+i\,\frac{\pi}{4}$.

(b)

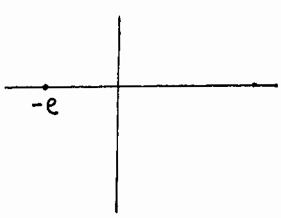


$|-i|=1$ $\quad \operatorname{Arg}(-i)=-\frac{\pi}{2}$

$$\begin{aligned}\log(-i)&=\ln 1+i\left(-\frac{\pi}{2}+2n\pi\right),\quad n\in\mathbb{Z}\\&=\frac{(4n-1)\pi}{2}\,i\ ,\quad n\in\mathbb{Z}\end{aligned}$$

$\operatorname{Log}(-i)=-\frac{\pi}{2}i$.

(c)



$|-e|=e$ $\quad \operatorname{Arg}(-e)=\pi$

$$\begin{aligned}\log(-e)&=\ln e+i\,(\pi+2n\pi)\ ,\quad n\in\mathbb{Z}\\&=1+(2n+1)\pi i\ ,\quad n\in\mathbb{Z}\ .\end{aligned}$$

$\operatorname{Log}(-e)=1+\pi i$.

Properties

i) $e^{\operatorname{Log} z} = z \qquad \forall z \neq 0$

ii) $\operatorname{Log}(e^z) = z \qquad$ if $\quad -\pi < \operatorname{Im} z \leq \pi$

Proof: Exercise.

Remark. When $z \in \mathbb{R}_+$, $z = x + i0$ for some $x > 0$ and so $z = xe^{i0} \Rightarrow \operatorname{Log} z = \ln x + i0 = \ln x$. Therefore Log is an extension of the real function ln to the complex case.

Example. Find the derivative of the principal branch of the logarithm on its domain, if it exists.

Solution. By using polar coordinates

$$\operatorname{Log} z = \operatorname{Log}(re^{i\theta}) = \ln r + i\theta \quad \text{for} \quad r > 0, \ -\pi < \theta < \pi$$

Thus $\qquad \operatorname{Log} z = u(r,\theta) + iv(r,\theta) \quad$ where

$$u(r,\theta) = \ln r, \ r > 0 \qquad v(r,\theta) = \theta, \qquad -\pi < \theta < \pi.$$

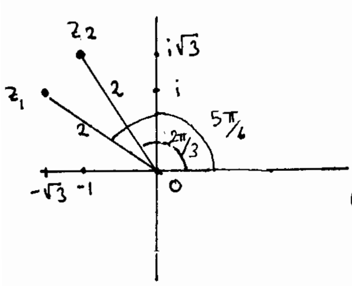
clearly u and v has continuous partial derivatives of all orders, and $\quad u_r = \frac{1}{r}, \ u_\theta = 0, \ v_r = 0,$ and $v_\theta = 1$.

Clearly, u and v satisfy the polar form of the Cauchy-Riemann equations: $v_\theta = r u_r$ and $u_\theta = -r v_r$. So

$$\frac{d}{dz} \operatorname{Log} z = e^{-i\theta}(u_r + iv_r) = \frac{e^{-i\theta}}{r}(v_\theta - iu_\theta) = e^{-i\theta}\frac{1}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

Example. Show that $\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$ is not always valid.

Solution. Take $z_1 = -\sqrt{3} + i \qquad z_2 = -1 + i\sqrt{3}$. Then



$$z_1 = 2e^{i\frac{5\pi}{6}}, \ z_2 = 2e^{i\frac{2\pi}{3}} \ \text{and} \ z_1 z_2 = 4e^{i\frac{9\pi}{6}} = 4e^{i\frac{3\pi}{2}} = 4e^{-i\frac{\pi}{2}}.$$

$$\Rightarrow \quad \operatorname{Log} z_1 = \ln 2 + i\frac{5\pi}{6}, \quad \operatorname{Log} z_2 = \ln 2 + i\frac{2\pi}{3}.$$

Clearly $\quad \operatorname{Log} z_1 + \operatorname{Log} z_2 = 2\ln 2 + i\frac{9\pi}{6} = \ln 4 + i\frac{3\pi}{2}$

$$\neq \ln 4 - i\frac{\pi}{2} = \operatorname{Log}(z_1 z_2).$$

__Exercise.__ Show that the identity

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log}(z_1) + \operatorname{Log}(z_2)$$

holds if and only if $-\pi < \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \leq \pi$.

__Theorem.__ Let z_1 and z_2 be nonzero complex numbers. The multivalued function $\log$ obeys the familiar properties of logarithms:

(i) $\log(z_1 z_2) = \log z_1 + \log z_2$

(ii) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

(iii) $\log\left(\frac{1}{z}\right) = -\log z$

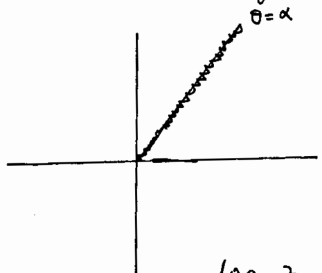
__Proof.__ Exercise!

__Remark.__ We can construct many different branches of the multivalued logarithm function that are continuous and differentiable except at points along any preassigned ray $\{re^{i\alpha}, r>0\}$:

Let $\alpha \in \mathbb{R}$, for any $z \neq 0$ we can choose the value of $\theta \in \arg z$ that lies in the range $\alpha < \theta \leq \alpha + 2\pi$, then the function $\log_\alpha z$ is defined by

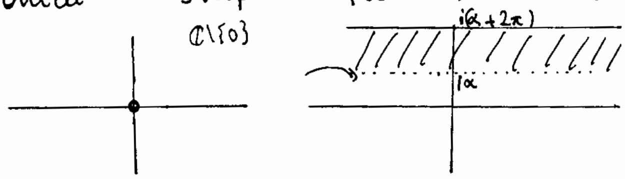
$$\log_\alpha(z) = \ln r + i\theta \quad \text{where } z = re^{i\theta}, \quad r>0, \ -\alpha < \theta \leq \alpha + 2\pi,$$

is a single-valued function.

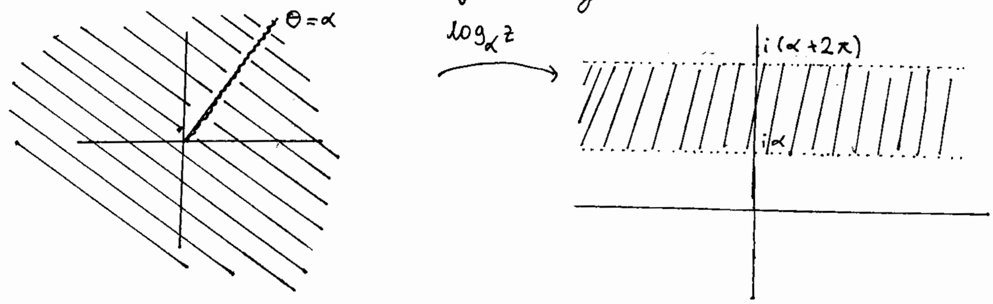


The branch cut for $\log_\alpha z$ is the ray $\{re^{i\alpha}, r \geq 0\}$ ($\log_\alpha z$ is discontinuous on this ray)

$\log_\alpha z$ maps $\mathbb{C}\setminus\{0\}$ one-to-one and onto the horizontal strip $\{w : \alpha < \operatorname{Im} w \leq \alpha + 2\pi\}$



If we remove the branch cut $\{z = re^{i\theta},\ \theta = \alpha\}$ from $\mathbb{C}\setminus\{0\}$ we obtain a branch of $\log z$.



$\log_\alpha z$ is analytic and satisfies $\frac{d}{dz}\log_\alpha z = \frac{1}{z}$.

<u>Example</u> Find $f(3)$, $f(i)$ and $f(-4)$, if f is the branch of the multivalued logarithm function defined by

$$f(z) = f(re^{i\theta}) = \ln r + i\theta, \quad \frac{9\pi}{2} < \theta \leq \frac{13\pi}{2}. \quad (\text{i.e., } f(z) = \log_{\frac{9\pi}{2}} z)$$

<u>Solution</u>

$$3 = 3e^{i6\pi}, \quad \frac{9\pi}{2} < 6\pi \leq \frac{13\pi}{2} \Rightarrow f(3) = \ln 3 + i6\pi$$

$$i = e^{i\frac{13\pi}{2}}, \quad \frac{9\pi}{2} < \frac{13\pi}{2} \leq \frac{13\pi}{2} \Rightarrow f(i) = i\frac{13\pi}{2}.$$

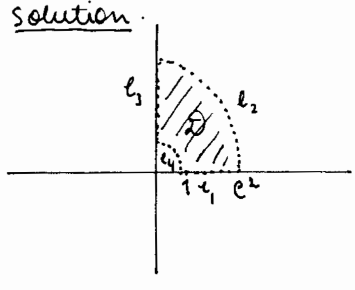
$$-4 = 4e^{i5\pi}, \quad \frac{9\pi}{2} < 5\pi \leq \frac{13\pi}{2} \Rightarrow f(-4) = \ln 4 + i5\pi.$$

<u>Example.</u> Let D be the part of the annulus $\{z = re^{i\theta} : 1 < r < e^2\}$ in the open first quadrant $(= \{x+iy \in \mathbb{C} : x>0, y>0\})$. Find the image of D under

(a) the principal logarithm function.

(b) the branch of $\log z$ defined by $\ln r + i\theta$, $\frac{7\pi}{4} < \theta < \frac{15\pi}{4}$.

<u>Solution</u>.



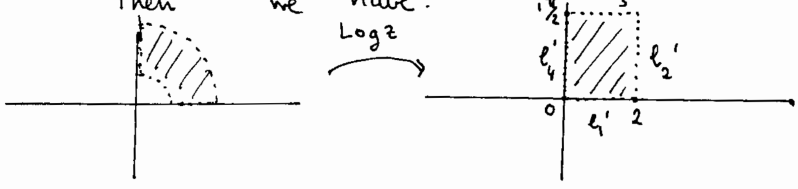
a) on ℓ_1: $z = re^{i0} \Rightarrow \operatorname{Log} z = \ln r + i0 = \ln r$, $1 < r < e^2$

on ℓ_2: $z = e^2 e^{i\theta}$, $0 < \theta < \frac{\pi}{2} \Rightarrow \operatorname{Log} z = \ln e^2 + i\theta$, $0 < \theta < \frac{\pi}{2}$
$= 2 + i\theta$, $0 < \theta < \frac{\pi}{2}$

on ℓ_3: $z = re^{i\frac{\pi}{2}}$, $1 < r < e^2 \Rightarrow \operatorname{Log} z = \ln r + i\frac{\pi}{2}$, $1 < r < e^2$

on ℓ_4: $z = e^{i\theta}$, $0 < \theta < \frac{\pi}{2} \Rightarrow \operatorname{Log} z = i\theta$, $0 < \theta < \frac{\pi}{2}$

Then we have:



(b) Let f be the branch of logarithm defined by

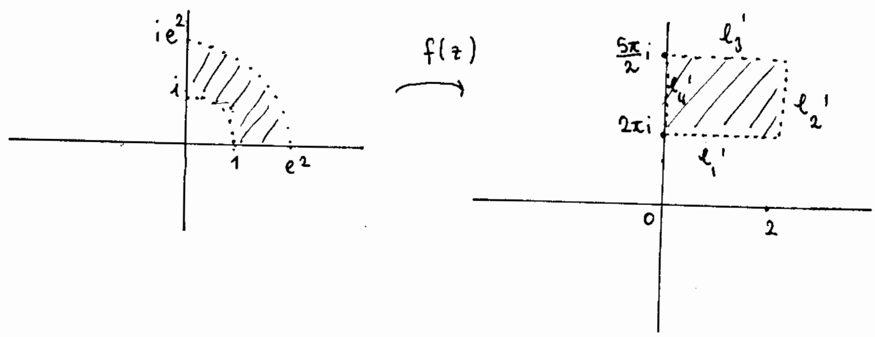
$$f(re^{i\theta}) = \ln r + i\theta, \qquad \frac{7\pi}{4} < \theta < \frac{15\pi}{4}$$

on ℓ_1: $\quad z = re^{i2\pi}, \quad 1 < r < e^2 \Rightarrow f(z) = \ln r + i2\pi, \quad 0 < \ln r < 2$

on ℓ_2: $\quad z = e^2 e^{i\theta}, \quad 2\pi < \theta < \frac{5\pi}{2} \Rightarrow f(z) = 2 + i\theta, \quad 2\pi < \theta < \frac{5\pi}{2}$

on ℓ_3: $\quad z = re^{i\frac{5\pi}{2}}, \quad 1 < r < e^2 \Rightarrow f(z) = \ln r + i\frac{5\pi}{2}, \quad 0 < \ln r < 2$

on ℓ_4: $\quad z = e^{i\theta}, \quad 2\pi < \theta < \frac{5\pi}{2} \Rightarrow f(z) = i\theta, \quad 2\pi < \theta < \frac{5\pi}{2} \Rightarrow$



<u>Exercise.</u> Find the image of $D = \{re^{i\theta},\ a < r < b,\ \alpha < c < \theta < d < \alpha + 2\pi\}$ under $\log_\alpha z$.

<u>Example.</u> Construct a branch of $f(z) = \log(z+4)$ that is analytic at the point $z = -5$ and take on the value $7\pi i$ there.

<u>Solution</u> $\quad f(z) = \log_\alpha(z+4) = \ln|z+4| + i\theta \quad$ such that $\quad \theta \in \arg(z+4)$ and $\alpha < \theta < \alpha + 2\pi$

f is analytic at $z = -5 \Rightarrow \log_\alpha z$ is analytic at $z = -1 \Rightarrow$ negative real line can't be the branch cut $\Rightarrow \alpha \neq (2k+1)\pi,\ k \in \mathbb{Z}$

$$f(-5) = \ln|-5+4| + i\theta = \ln|-1| + i\theta = i\theta = 7\pi i \quad \Rightarrow \quad \alpha < 7\pi < \alpha + 2\pi.$$

Thus any function $f(z) = \log_\alpha(z+4)$ with $5\pi < \alpha < 7\pi$ satisfies the desired properties, e.g. $f(z) = \log_{6\pi}(z+4)$, $\log_{\frac{13\pi}{2}}(z+1)$ or $f(z) = \log_{\frac{11\pi}{2}}(z+4)$ can be taken.

Complex Exponents

Let c be a complex number. We define z^c as

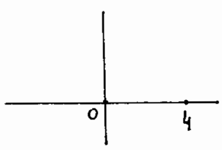
$$z^c = e^{c \log z}, \quad z \neq 0 \in \mathbb{C}.$$

Remark. Note that $\log z$ is a multivalued function, so z^c is also multivalued in general. We will consider several possibilities later.

Example. Evaluate $4^{\frac{1}{2}}$.

Solution. $4^{\frac{1}{2}} = e^{\frac{1}{2}\log 4}$ and $\log 4 = \ln|4| + i(\mathrm{Arg}(4) + 2n\pi)$, $n \in \mathbb{Z}$

Note that $\mathrm{Arg}(4) = 0$ and so



$$\frac{1}{2}\log 4 = \frac{1}{2}\ln 4 + in\pi, \quad n \in \mathbb{Z}$$
$$= \ln 2 + in\pi, \quad n \in \mathbb{Z}$$

Therefore $4^{\frac{1}{2}} = e^{\ln 2 + in\pi} = e^{\ln 2} e^{in\pi} = 2e^{in\pi} = 2(\cos n\pi + i\underbrace{\sin n\pi}_{0})$
$= 2\cos n\pi, \quad n \in \mathbb{Z}$

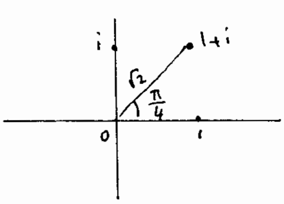
If n is even clearly $\cos n\pi = 1$ and if n is odd then $\cos n\pi = -1$. Hence $4^{\frac{1}{2}} = \{2, -2\}$.

We can specify a branch of $\log z$ to obtain a single valued z^c.

For example, $f(z) = e^{c \,\mathrm{Log} z}$ is a branch of z^c and called the <u>principal branch of</u> the multivalued function <u>z^c</u>.

Example. Find the principal values of $(1+i)^{\frac{1}{2}}$ and i^i.

Solution a) $(1+i)^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}(1+i)} = e^{\frac{1}{2}(\ln|1+i| + i\mathrm{Arg}(1+i))}$
$= e^{\frac{1}{2}(\ln\sqrt{2} + i\frac{\pi}{4})} = e^{\ln 2^{\frac{1}{4}} + i\frac{\pi}{8}} = 2^{\frac{1}{4}} e^{i\frac{\pi}{8}} = 2^{\frac{1}{4}}(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8})$



b) $i^i = e^{i\mathrm{Log}(i)} = e^{i(\ln|i| + i\mathrm{Arg}(i))} = e^{i(\ln 1 + i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$

$= \dfrac{1}{\sqrt{e^{\pi}}}$ (a real number!)

Now we will consider the possibilities that arise when we apply $z^c = e^{c\log z}$.

<u>Case i</u> $\quad c = k \in \mathbb{Z}$

if $\quad z = re^{i\theta} \neq 0, \quad k\log z = k(\ln r + i(\theta + 2n\pi))\ , \ n \in \mathbb{Z} \quad \Rightarrow$

$$z^k = e^{k\log z} = e^{k\ln r + ik\theta + i2nk\pi} = r^k e^{ik\theta} \cdot \underbrace{e^{i2nk\pi}}_{=1 \text{ because } nk \in \mathbb{Z}} = (re^{i\theta})^k$$

the result does not depend on n, hence z^c is single-valued and gives the same result obtained by the usual power function z^k.
$\left(z^k = \underbrace{z \cdot z \cdot z \cdot z \cdots z}_{k \text{ times}} \right.$ if k is positive and $z^k = \dfrac{1}{\underbrace{z \cdot z \cdots z}_{|k| \text{ times}}}$ if k is negative)

<u>case ii</u> $\quad c = \dfrac{1}{k}\ , \ k \in \mathbb{Z}$.

if $\quad z = re^{i\theta} \neq 0$, then $\quad \dfrac{1}{k}\log z = \dfrac{1}{k}(\ln r + i(\theta + 2n\pi))$

$$= \frac{1}{k}\ln r + i\left(\frac{\theta}{k} + \frac{2n\pi}{k}\right) \ , \ n \in \mathbb{Z} \text{ , and}$$

$$z^{\frac{1}{k}} = e^{\frac{1}{k}\ln r} e^{i\left(\frac{\theta}{k} + \frac{2n\pi}{k}\right)} = r^{\frac{1}{k}} e^{i\left(\frac{\theta}{k} + \frac{2n\pi}{k}\right)} \ , n \in \mathbb{Z}$$

gives k different values $n = 0, 1, 2, \cdots, k-1$ and repeats this values for other n's. Thus z^c corresponds to usual multivalued k^{th} root function.

<u>Case iii</u> $\quad c = \dfrac{j}{k} \in \mathbb{Q}$ (for simplicity suppose $j, k \in \mathbb{Z}_+$ and j and k has no common factor, that is greatest common divisor = gcd of j and k is 1 , $\gcd(j,k) = 1$)

For $z = re^{i\theta} \neq 0$, we have $\quad \dfrac{j}{k}\log z = \dfrac{j}{k}(\ln r + i(\theta + 2n\pi)) = \dfrac{j}{k}\ln r + i\left(\dfrac{\theta j}{k} + \dfrac{2nj}{k}\pi\right), n \in \mathbb{Z}$

and so

$$z^{\frac{j}{k}} = e^{\frac{j}{k}\ln r} e^{i\left(\frac{\theta j}{k} + \frac{2nj}{k}\pi\right)} = r^{\frac{j}{k}} e^{i\left(\frac{\theta j}{k} + \frac{2nj}{k}\pi\right)} \ , n \in \mathbb{Z}$$

gives k different values $n = 0, 1, 2, 3, \cdots, k-1$ and repeats this values for other n's.

case iv. if $c \notin \mathbb{Q}$ (that is c is irrational real or complex number)

Then there are infinitely many different values of

$$z^c = e^{c \log z} = e^{c(\ln r + i(\theta + 2\pi n))}.$$

Example. Find the values of $2^{\frac{1}{9} + \frac{i}{50}}$.

Solution. $2^{\frac{1}{9}+\frac{i}{50}} = e^{\left(\frac{1}{9}+\frac{i}{50}\right)\log 2}$

$$\log 2 = \ln 2 + i(0 + 2n\pi) = \ln 2 + i2n\pi$$

$$\left(\frac{1}{9} + \frac{i}{50}\right)\log 2 = \frac{1}{9}\ln 2 + i\frac{2n\pi}{9} + i\frac{\ln 2}{50} - \frac{n\pi}{25} \Rightarrow$$

$$2^{\frac{1}{9}+\frac{i}{50}} = 2^{\frac{1}{9}} e^{-\frac{n\pi}{25}} e^{i\left(\frac{\ln 2}{50} + \frac{2n\pi}{9}\right)}, \quad n \in \mathbb{Z}$$

principal value is $2^{\frac{1}{9}} e^{i\frac{\ln 2}{50}} = 2^{\frac{1}{9}}\left(\cos\frac{\ln 2}{50} + i \sin\frac{\ln 2}{50}\right)$

Properties For $z \neq 0$

i) $z^{-c} = \dfrac{1}{z^c}$

ii) $z^c z^d = z^{c+d}$

iii) $\dfrac{z^c}{z^d} = z^{c-d}$

iv) $(z^c)^n = z^{cn}$ for $n \in \mathbb{Z}$

Proof: Exercise!

Remark. (iv) may not hold if we remove the condition "$n \in \mathbb{Z}$".

Example. Show that $(i^2)^i \neq i^{2i}$.

Solution $(i^2)^i = (-1)^i = e^{i\log(-1)} = e^{i(\ln|-1| + i(\pi + 2n\pi))} = e^{-(2n+1)\pi}, \ n \in \mathbb{Z}$

and $(i)^{2i} = e^{2i(\ln|-i| + i(\frac{\pi}{2} + 2n\pi))} = e^{-\pi - 4n\pi} = e^{-(4n+1)\pi}, \ n \in \mathbb{Z}$

Clearly, $e^{-3\pi} \in (i^2)^i$, but $e^{-3\pi} \notin (i)^{2i}$

Example. Find $f'(z)$ if f is the principal branch of z^c.

Solution. $f(z) = e^{c\,\mathrm{Log}z}$

$$\Rightarrow \quad f'(z) = e^{c\,\mathrm{Log}z}\,\frac{c}{z} = c\,e^{c\,\mathrm{Log}z}\,e^{-\mathrm{Log}z} = c\,\underbrace{e^{(c-1)\,\mathrm{Log}z}}_{\text{principal branch of } z^{c-1}}$$

If we reserve z^c for (single-valued) principal branch of z^c for all c, we may write "$\frac{d}{dz} z^c = c z^{c-1}$".

For any $b \neq 0$, we can define the exponential function with base b, as

$$b^z = e^{z \log b}$$

If we specify a branch of the logarithm then b^z will be single-valued and we can use the rules of differentiation to show that the resulting branch of b^z is an analytic function and

$$\frac{d}{dz} b^z = b^z \log_\alpha(b)$$

where $\log_\alpha(z)$ is any branch of the logarithm whose branch cut does not include the point b.

Example Show that if $z \neq 0$, then z^0 has a unique value.

Solution $z^0 = e^{0\log z} = e^{0(\ln r + i(\theta + 2n\pi))} = e^0 = 1.$

Example: Construct an example that shows that the principal value of $(z_1 z_2)^{\frac{1}{3}}$ need not equal to the product of the principal values of $z_1^{\frac{1}{3}}$ and $z_2^{\frac{1}{3}}$

Solution Try $z_1 = -\sqrt{3} + i$, $z_2 = -1 + i\sqrt{3}$.

Trigonometric and hyperbolic functions.

We define $\sin z$ and $\cos z$ by

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \quad \text{and} \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

Theorem. $\sin z$ and $\cos z$ are entire functions, with

$\frac{d}{dz} \sin z = \cos z$ and $\frac{d}{dz}\cos z = -\sin z$.

Proof. We'll prove the theorem for $\sin z$.

Ratio test shows that the radius of convergence for $\sin z$ is infinity, and

$$\frac{d}{dz}\sin z = \frac{d}{dz}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{d}{dz}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n (2n+1) z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!} = \cos z. \quad \blacksquare$$

We define $\tan z$, $\cot z$, $\sec z$ and $\csc z$ as follows:

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}.$$

Remark All complex trigonometric functions agree with their real counterparts.

Properties For all $z \in \mathbb{C}$

i) $\sin(-z) = -\sin z$

ii) $\cos(-z) = \cos z$

iii) $\sin^2 z + \cos^2 z = 1$

Proof i), ii) exercise!

iii) Let $f(z) = \sin^2 z + \cos^2 z$, then $f'(z) = 2\sin z\cos z + 2\cos z(-\sin z) = 0 \Rightarrow$ $f(z)$ is constant $\Rightarrow f(z) = f(0) = \sin^2 0 + \cos^2 0 = 1$ for all $z \in \mathbb{C}$. $\blacksquare$

Properties

i) $\frac{d}{dz}\tan z = \sec^2 z$

ii) $\frac{d}{dz}\cot z = -\csc^2 z$

iii) $\frac{d}{dz}\sec z = \sec z \tan z$

iv) $\frac{d}{dz}\csc z = -\csc z \cot z$

Proof. Exercise!

Example. Write $\sin z$ and $\cos z$ in the form $u(x,y)+iv(x,y)$.

Solution One can easily show that

$$e^{iz} = \cos z + i\sin z \quad \text{for all } z \in \mathbb{C}. \quad \text{(Exercise!)}$$

Hence $e^{-iz} = \cos(-z) + i\sin(-z) = \cos z - i\sin z$

Thus, $\dfrac{e^{iz} - e^{-iz}}{2i} = \sin z$

and so
$$\sin(x+iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}$$
$$= \frac{(\cos x + i\sin x)e^{-y} - (\cos(-x) + i\sin(-x))e^{y}}{2i}$$
$$= \frac{(-i\cos x + \sin x)e^{-y} + (i\cos x + \sin x)e^{y}}{2}$$
$$= \sin x\left(\frac{e^{y}+e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y}-e^{-y}}{2}\right)$$
$$= \underbrace{\sin x \cosh y}_{u(x,y)} + i\,\underbrace{\cos x \sinh y}_{v(x,y)}.$$

Similarly, $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ and

$$\cos(x+iy) = \frac{e^{ix-y} + e^{-ix+y}}{2} = \frac{(\cos x + i\sin x)e^{-y} + (\cos x - i\sin x)e^{y}}{2}$$
$$= \cos x\left(\frac{e^{y}+e^{-y}}{2}\right) - i\sin x\left(\frac{e^{y}-e^{-y}}{2}\right)$$
$$= \underbrace{\cos x \cosh y}_{u(x,y)} + i\,\underbrace{(-\sin x \sinh y)}_{v(x,y)}$$

<u>Properties</u> For all $z \in \mathbb{C}$

i) $\sin(z+2\pi) = \sin z$

ii) $\cos(z+2\pi) = \cos z$

iii) $\sin(z+\pi) = -\sin z$

iv) $\cos(z+\pi) = -\cos z$

v) $\tan(z+\pi) = \tan z$

vi) $\cot(z+\pi) = \cot z$

<u>Proof</u> i) $\sin((x+iy)+2\pi) = \sin((x+2\pi)+iy)$

$$= \sin(x+2\pi)\cosh y + i\cos(x+2\pi)\sinh y$$

$$= \sin x\cosh y + i\cos x\sinh y = \sin(x+iy).$$

Proof of ii)-vi) are left as exercises.

<u>Properties</u> For all $z, z_1, z_2 \in \mathbb{C}$

i) $\sin(z_1+z_2) = \sin z_1\cos z_2 + \cos z_1\sin z_2$

ii) $\cos(z_1+z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2$

iii) $\sin 2z = 2\sin z\cos z$

iv) $\cos 2z = \cos^2 z - \sin^2 z$

v) $\sin\left(\frac{\pi}{2}+z\right) = \sin\left(\frac{\pi}{2}-z\right) = \cos z$

<u>Proof</u> i) $\sin(z_1+z_2) = \dfrac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$

$$\sin z_1\cos z_2 = \left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) = \frac{e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}-e^{-i(z_1+z_2)}}{4i}$$

$$\cos z_1\sin z_2 = \left(\frac{e^{iz_1}+e^{-iz_1}}{2}\right)\left(\frac{e^{iz_2}-e^{-iz_2}}{2i}\right) = \frac{e^{i(z_1+z_2)}-e^{i(z_1-z_2)}+e^{-i(z_1-z_2)}-e^{-i(z_1+z_2)}}{4i}$$

Therefore $\sin z_1\cos z_2 + \cos z_1\sin z_2 = \dfrac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i} = \sin(z_1+z_2).$

Proof of (ii) - (v) are left as exercises

A solution to the equation $f(z)=0$ is called a <u>zero</u> of the given function f.

<u>Example.</u> Show that $\sin z = 0$ if and only if $z = n\pi$ for some $n \in \mathbb{Z}$, $\cos z = 0$ if and only if $z = (n+\frac{1}{2})\pi$ for some $n \in \mathbb{Z}$.

<u>Solution.</u> $\sin z = \sin x \cosh y + i \cos x \sinh y = 0$ if and only if $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$. Since $\cosh y = \frac{e^y + e^{-y}}{2}$ is never 0, $\sin x = 0 \Rightarrow x = n\pi \Rightarrow \cos x \sinh y = (-1)^n \sinh y = 0 \Rightarrow \sinh y = 0$ $\Rightarrow y = 0$, so the only zeros for $\sin z$ are those given by $z = n\pi$ for any integer n. The result for $\cos z$ is left as an exercise.

<u>Example.</u> Show that $|\sin z|^2 = \sin^2 x + \sinh^2 y$ and $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

<u>Solution</u> $\sin z = \sin x \cosh y + i \cos x \sinh y \Rightarrow$

$$|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \cosh^2 y - \sin^2 x \sinh^2 y + \sin^2 x \sinh^2 y + \cos^2 x \sinh^2 y$$

$$= \sin^2 x \underbrace{(\cosh^2 y - \sinh^2 y)}_{1} + \sinh^2 y \underbrace{(\sin^2 x + \cos^2 x)}_{1}$$

$$\Rightarrow |\sin z|^2 = \sin^2 x + \sinh^2 y.$$

Proof for $|\cos z|^2 = \cos^2 x + \sinh^2 y$ is similar. ∎

<u>Remark</u> $\sin z$ is an unbounded function because $\sinh y \to \infty$ as $y \to \infty$ while the real function $\sin x$ is bounded (in modulus) by 1.

<u>Example</u> Solve the equation $\cos z = \cosh 2$.

<u>Solution.</u> $\cos z = \cos x \cosh y - i \sin x \sinh y = \cosh 2 \iff \cos x \cosh y = \cosh 2$ and $\sin x \sinh y = 0$. $\sin x \sinh y = 0 \iff \sin x = 0$ or $\sinh y = 0 \iff x = n\pi$ or $y = 0$

i) if $y = 0$, $\cos x \cosh y = \cos x = \cosh 2$, but this is impossible because $\cosh 2 > 1$

ii) if $x = n\pi$, $\cos x \cosh y = (-1)^n \cosh y = \cosh 2 \Rightarrow n$ is even and $y = \pm 2$. Thus the solution set is $z = 2n\pi \pm 2i = 2(n\pi \pm i), n \in \mathbb{Z}$.

We define complex hyperbolic functions as follows:

$$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right) \qquad \sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right),$$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z} \quad \text{and} \quad \operatorname{csch} z = \frac{1}{\sinh z}.$$

Properties.

i) $\frac{d}{dz} \cosh z = \sinh z$

ii) $\frac{d}{dz} \sinh z = \cosh z$

iii) $\frac{d}{dz} \tanh z = \operatorname{sech}^2 z$

iv) $\frac{d}{dz} \coth z = -\operatorname{csch}^2 z$

v) $\frac{d}{dz} \operatorname{sech} z = -\operatorname{sech} z \tanh z$

vi) $\frac{d}{dz} \operatorname{csch} z = -\operatorname{csch} z \coth z.$

vii) $\cosh z = \cosh x \cos y + i \sinh x \sin y$

viii) $\sinh z = \sinh x \cos y + i \cosh x \sin y$

ix) $\cosh(iz) = \cos z, \quad \sinh(iz) = i \sin z, \quad \sin(iz) = i \sinh z, \quad \cos(iz) = \cosh z.$

x) $\cosh^2 z - \sinh^2 z = 1$

xi) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$

$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$

xii) $\cosh(z + 2\pi i) = \cosh z$

$\sinh(z + 2\pi i) = \sinh z$

xiii) $\cosh(-z) = \cosh z$

$\sinh(-z) = -\sinh z$

Inverse Trigonometric and Hyperbolic Functions

Because trigonometric and hyperbolic functions are all periodic, they are many-to-one; hence their inverses are necessarily multivalued. The formulas for the inverse trigonometric functions are

$$\arcsin z = -i\log\left(iz + (1-z^2)^{\frac{1}{2}}\right) \qquad (*)$$

$$\arccos z = -i\log\left(z + i(1-z^2)^{\frac{1}{2}}\right)$$

and $$\arctan z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

we will justify $(*)$ and leave the others as exercises:

If $w = \arcsin z$ then $\sin w = z$, that is

$$z = \frac{e^{iw}-e^{-iw}}{2i} \iff 2iz = e^{iw}-e^{-iw} \iff \left(e^{iw}\right)^2 - 2iz\,e^{iw} - 1 = 0$$

we can solve the quadratic equation for e^{iw} and obtain

$$e^{iw} = \frac{2iz + (-4z^2+4)^{\frac{1}{2}}}{2} = iz + (1-z^2)^{\frac{1}{2}} \Rightarrow$$

$$w = -i\log\left(iz + (1-z^2)^{\frac{1}{2}}\right)$$

To construct a specific branch of $\arcsin z$, we must first select a branch of the square root and then select a branch of the logarithm.

Derivatives of inverse trigonometric functions for any selected branch is given by

$$\frac{d}{dz}\arcsin z = \frac{1}{(1-z^2)^{\frac{1}{2}}} \qquad (**)$$

$$\frac{d}{dz}\arccos z = \frac{-1}{(1-z^2)^{\frac{1}{2}}}$$

$$\frac{d}{dz}\arctan z = \frac{1}{1+z^2}$$

we will justify (**) and leave the others as exercises

$$w = \arcsin z \Leftrightarrow z = \sin w \Leftrightarrow \frac{d}{dz} z = \frac{d}{dz} \sin w \Leftrightarrow 1 = \cos w \frac{dw}{dz}$$

$$\Rightarrow \frac{dw}{dz} = \frac{1}{\cos w} = \frac{1}{(1-\sin^2 w)^{\frac{1}{2}}} = \frac{1}{(1-z^2)^{\frac{1}{2}}}$$

<u>Example.</u> Evaluate $\arcsin\sqrt{2}$.

<u>Solution.</u>

$$\arcsin\sqrt{2} = -i\log\left[i\sqrt{2} + (1-(\sqrt{2})^2)^{\frac{1}{2}}\right]$$

$$= -i\log\left[i\sqrt{2} + (-1)^{\frac{1}{2}}\right] = -i\log\left[i\sqrt{2} \pm i\right]$$

$$= -i\log(i(\sqrt{2}\pm 1))$$

$$= -i\left(\ln(\sqrt{2}\pm 1) + i\left(\frac{\pi}{2} + 2n\pi\right)\right)$$

$$= \frac{\pi}{2} + 2n\pi - i\ln(\sqrt{2}\pm 1), \quad n \in \mathbb{Z}$$

Since $\sqrt{2}-1 = \frac{(\sqrt{2}-1)(\sqrt{2}+1)}{(\sqrt{2}+1)} = \frac{1}{\sqrt{2}+1}$, $\ln(\sqrt{2}-1) = \ln\frac{1}{\sqrt{2}+1} = -\ln(\sqrt{2}+1)$,

we may $\arcsin\sqrt{2} = \frac{\pi}{2} + 2n\pi \pm i\ln(\sqrt{2}+1), \quad n \in \mathbb{Z}$

<u>Example.</u> Evaluate $f(\sqrt{2})$ and $f'(\sqrt{2})$, if f is the branch of $\arcsin z$ for which i is selected as the value of the square root $[1-(\sqrt{2})^2]^{\frac{1}{2}}$ and the principal value of the logarithm is used

<u>Solution</u>

$$f(\sqrt{2}) = -i\,\mathrm{Log}(i\sqrt{2}+i) = -i\,\mathrm{Log}(i(1+\sqrt{2}))$$

$$= -i\left(\ln(1+\sqrt{2}) + i\frac{\pi}{2}\right) = \frac{\pi}{2} - i\ln(1+\sqrt{2})$$

$$f'(\sqrt{2}) = \frac{1}{(1-(\sqrt{2})^2)^{\frac{1}{2}}} = \frac{1}{(-1)^{\frac{1}{2}}} = \frac{1}{i} = -i.$$

The inverse hyperbolic functions are

$$\operatorname{arcsinh} z = \log\left(z + (z^2+1)^{\frac{1}{2}}\right)$$

$$\operatorname{arccosh} z = \log\left(z + (z^2-1)^{\frac{1}{2}}\right)$$

$$\operatorname{arctan} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and their derivatives (for suitably selected branch) are

$$\frac{d}{dz}\operatorname{arcsinh} z = \frac{1}{(z^2+1)^{\frac{1}{2}}}$$

$$\frac{d}{dz}\operatorname{arccosh} z = \frac{1}{(z^2-1)^{\frac{1}{2}}}$$

$$\frac{d}{dz}\operatorname{arctanh} z = \frac{1}{1-z^2}$$

verification of all formulas above are left as exercises

Example. Evaluate arctanh i

Solution

$$\operatorname{arctanh} i = \frac{1}{2}\log\left(\frac{1+i}{1-i}\right)$$

$$= \frac{1}{2}\log\left(\frac{(1+i)^2}{2}\right)$$

$$= \frac{1}{2}\log\left(\frac{1+2i-1}{2}\right) = \frac{1}{2}\log(i)$$

$$= \frac{1}{2}\left(\ln|i| + i\left(\frac{\pi}{2} + 2n\pi\right)\right) \quad, n \in \mathbb{Z}$$

$$= i\left(\frac{\pi}{4} + n\pi\right), \quad n \in \mathbb{Z}$$

Complex Integration

Complex integrals

In order to introduce integrals of $f(z)$, we need to first consider derivatives of complex-valued functions ω of a real variable t.

We write $\omega(t) = u(t) + iv(t)$ where the functions u and v are real-valued functions of t. The derivative $\omega'(t)$ (or $\frac{d}{dt}\omega(t)$) of ω at a point t is defined as

$$\omega'(t) = u'(t) + iv'(t)$$

provided that u' and v' exists at t.

<u>Example</u>. Find $\omega'(t)$, if

a) $\omega(t) = 4t + 5it^2$

b) $\omega(t) = e^{(1+2i)t}$

<u>Solution</u>. a) $\omega'(t) = \frac{d}{dt}(4t) + i\frac{d}{dt}(5t^2) = 4 + 10it$.

b)
$$\omega(t) = e^t e^{i2t} = e^t(\cos(2t) + i\sin(2t)) = e^t\cos(2t) + ie^t\sin(2t)$$

$$\omega'(t) = e^t\cos(2t) - e^t\sin(2t)\,2 + i(e^t\sin(2t) + e^t\cos(2t)\,2)$$

$$= e^t\cos(2t)(1+2i) - e^t\sin(2t)(2-i)$$

$$= e^t\cos(2t)(1+2i) + i^2e^t\sin(2t)(2-i)$$

$$= e^t\cos(2t)(1+2i) + ie^t\sin(2t)(1+2i)$$

$$= (1+2i)e^t(\cos(2t) + i\sin(2t)) = (1+2i)e^te^{i2t} = (1+2i)e^{t(1+2i)}.$$

<u>Example</u> show that $(c\,\omega(t))' = c\,\omega'(t)$ for any constant $c \in \mathbb{C}$.

<u>Solution</u> Let $c = a+ib$ and $\omega(t) = u(t) + iv(t)$. Then

$$c\omega(t) = (au(t) - bv(t)) + i(bu(t) + av(t)) \quad \text{and}$$

$$(c\omega(t))' = (au'(t) - bv'(t)) + i(bu'(t) + av'(t))$$

$$= (a+bi)(u'(t) + iv'(t)) = c\omega'(t). \blacksquare$$

when $w(t)$ is a complex-valued function of a real variable t and is written

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued, the definite integral of $w(t)$ over an interval $a \leq x \leq b$ is defined as

$$\int_a^b w(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

when the individual integrals on the right exists.

Thus,

$$\mathrm{Re}\int_a^b w(t)\,dt = \int_a^b \mathrm{Re}\, w(t)\,dt \quad \text{and} \quad \mathrm{Im}\int_a^b w(t)\,dt = \int_a^b \mathrm{Im}\, w(t)\,dt.$$

<u>Properties</u> For complex-valued f and g,

(i) $\int_a^b (f(t)+g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

(ii) $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$

(iii) $\int_a^b c f(t)\,dt = c\int_a^b f(t)\,dt$ for any complex constant $c \in \mathbb{C}$.

(iv) $\int_a^b f(t)\,dt = -\int_b^a f(t)\,dt$

(v) $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

<u>Proof:</u> Exercise!

Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way.

Example. Evaluate

a) $\int_0^1 (1+it)^2\,dt$

b) $\int_0^1 (t-i)^3\,dt$

c) $\int_0^{\pi/2} e^{t+it}\,dt$

Solution

a)
$$\int_0^1 (1+it)^2 dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$
$$= \left(t - \frac{t^3}{3}\right)\Big|_0^1 + i\,(t^2)\Big|_0^1 = 1 - \frac{1}{3} + i = \frac{2}{3} + i.$$

b)
$$\int_0^1 (t-i)^3 dt = \int_0^1 (t^3 - 3t^2 i - 3t + i)\,dt$$
$$= \int_0^1 (t^3 - 3t)\,dt + i\int_0^1 (1-3t^2)\,dt$$
$$= \left(\frac{t^4}{4} - 3\frac{t^2}{2}\right)\Big|_0^1 + i\,(t - t^3)\Big|_0^1 = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}.$$

c)
$$e^{t+it} = e^t e^{it} = e^t(\cos t + i\sin t) = e^t\cos t + i e^t \sin t$$
$$\int_0^{\frac{\pi}{2}} e^{t+it} dt = \int_0^{\frac{\pi}{2}} e^t \cos t\,dt + i\int_0^{\frac{\pi}{2}} e^t \sin t\,dt.$$
$$\int_0^{\frac{\pi}{2}} e^t \cos t\,dt = \int_0^{\frac{\pi}{2}} e^t\,d(\sin t) = \left(e^t \sin t\right)\Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \sin t\,d(e^t)$$
$$= \left(e^t \sin t\right)\Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} e^t \sin t\,dt$$
$$= \left(e^t \sin t\right)\Big|_0^{\pi/2} + \int_0^{\frac{\pi}{2}} e^t\,d(\cos t)$$
$$= \left(e^t \sin t + e^t \cos t\right)\Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} \cos t\,d(e^t)$$
$$= \left(e^t(\sin t + \cos t)\right)\Big|_0^{\pi/2} - \int_0^{\frac{\pi}{2}} e^t \cos t\,dt \implies$$
$$2\int_0^{\frac{\pi}{2}} e^t \cos t\,dt = \left(e^t(\sin t + \cos t)\right)\Big|_0^{\pi/2} \implies \int_0^{\frac{\pi}{2}} e^t \cos t\,dt = \left(\frac{e^t(\sin t + \cos t)}{2}\right)\Bigg|_0^{\frac{\pi}{2}} \implies$$
$$\int_0^{\pi/2} e^t \cos t\,dt = \frac{e^{\frac{\pi}{2}}}{2} - \frac{1}{2} = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$$

One can similarly show that $\int_0^{\frac{\pi}{2}} e^t \sin t\,dt = \frac{1}{2}\left(e^{\pi/2} + 1\right)$.

Hence
$$\int_0^{\pi/2} e^{t+it} dt = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right) + \frac{i}{2}\left(e^{\frac{\pi}{2}} + 1\right).$$

The fundamental theorem of Calculus, involving antiderivatives can be extended so as to apply to integrals of type $\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$:

Suppose that the functions

$$w(t) = u(t) + iv(t) \quad \text{and} \quad W(t) = U(t) + iV(t)$$

are continuous on the interval $a \leq t \leq b$. If $W'(t) = w(t)$ when $a \leq t \leq b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$. Hence

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

$$= U(t)\Big|_a^b + iV(t)\Big|_a^b = (U(b) - U(a)) + i(V(b) - V(a))$$

$$= (U(b) + iV(b)) - (U(a) + iV(a)) = W(b) - W(a) = W(t)\Big|_a^b .$$

<u>Example.</u> Evaluate

(a) $\displaystyle\int_0^{\frac{\pi}{4}} e^{it}dt$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} e^{(t+it)} dt$

<u>Solution</u> a) $\frac{d}{dt}\left(\frac{e^{it}}{i}\right) = \frac{1}{i}\frac{d}{dt}(\cos t + i\sin t) = \frac{1}{i}(-\sin t + i\cos t)$

$$= \cos t + i\sin t = e^{it} \Rightarrow$$

$$\int_0^{\frac{\pi}{4}} e^{it}dt = \frac{1}{i}e^{it}\Big|_0^{\frac{\pi}{4}} = \frac{1}{i}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) - \frac{1}{i} = \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} + i$$

$$= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

b) One can easily show that $e^{t+it} = e^{t(1+i)} = \frac{d}{dt}\frac{e^{t(1+i)}}{1+i}$

Hence $\displaystyle\int_0^{\frac{\pi}{2}} e^{(t+it)}dt = \frac{e^{t(1+i)}}{1+i}\Big|_0^{\frac{\pi}{2}} = \frac{1}{1+i}\left(e^{\frac{\pi}{2}+i\frac{\pi}{2}} - 1\right) = \frac{1}{1+i}\left(e^{\frac{\pi}{2}}i - 1\right)$

$$= \frac{1}{2}(1-i)(ie^{\frac{\pi}{2}} - 1) = \frac{1}{2}(e^{\frac{\pi}{2}} - 1) + \frac{i}{2}(e^{\frac{\pi}{2}} + 1).$$

Contours and contour integrals

Let $x = x(t)$ and $y = y(t)$ be two continuous functions on $[a,b]$. Consider the set

$$\mathcal{L} = \{ z : z = z(t) = x(t) + iy(t), \quad a \le t \le b \}$$

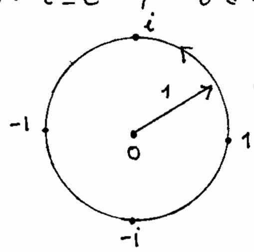
Let us order this set according to the natural order on $[a,b]$ — that is, $z_2 = z(t_2)$ follows $z_1 = z(t_1)$ if $t_1 \le t_2$.

The set $\mathcal{L}$ ordered in such a way is called a curve (= an arc). The point $z(a)$ is the beginning and $z(b)$ is the end of $\mathcal{L}$.

Example. $x(t) = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

$$z(t) = \cos t + i \sin t = e^{it}$$

$$\mathcal{L} = \{ z : z = e^{it}, \quad 0 \le t \le 2\pi \}$$



counterclockwise

-1 follows i

$-1 = z(\pi) \qquad i = z(\frac{\pi}{2}) \qquad \frac{\pi}{2} < \pi$

Example. $\mathcal{L}_1 = \{ z : z = e^{-it}, \quad 0 \le t \le 2\pi \}$

Set is the same but the order is different

i follows $-1 \qquad -1 = z(\pi) \qquad i = z(\frac{3\pi}{2}) \qquad \pi < \frac{3\pi}{2}$

$\mathcal{L} \ne \mathcal{L}_1$ but $\mathcal{L} = -\mathcal{L}_1$ (sets are the same but the orders are different!)

A curve is called simple if $z(t_1) \ne z(t_2)$ whenever $t_1 \ne t_2$. ($\mathcal{L}$ does not cross itself)

A curve is called simple closed if $z(t_1) \ne z(t_2)$ whenever $t_1 \ne t_2$ except $z(a) = z(b)$.

Example. $\mathcal{L}$ and $\mathcal{L}_1$ above are simple closed curves.

Example. $L_2 = \{z : z = e^{2it},\ 0 \le t \le 2\pi\}$ is not simple because
$z(\frac{\pi}{2}) = -1 = z(\frac{3\pi}{2})$ and $0 < \frac{\pi}{2} < \frac{3\pi}{2} < 2\pi$.

The curve $L = \{z : z = z(t),\ a \le t \le b\}$ is called smooth if $z(t)$ is continuously differentiable on $[a,b]$ and moreover, $z'(t) \ne 0$ for all $t \in [a,b]$.

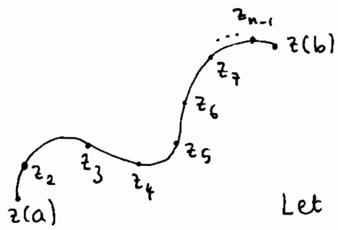
L is called piecewise smooth if it consists of finitely many of smooth curves, that is
$L = \{z : z = z(t),\ a \le t \le b\}$, $a = a_0 < a_1 < a_2 < \cdots < a_n = b\}$
such that $L_k = \{z : z = z(t),\ a_k \le t \le a_{k+1}\}$ is smooth $(k = 0, 1, \ldots, n-1)$

The curve is called a contour if it is simple and piecewise smooth.

A closed contour is called a Jordan contour.

Jordan's theorem. Let L be a Jordan curve, then we have $\mathbb{C} = G_1 \cup L \cup G_2$ $\quad G_1 \cap G_2 = \emptyset$ where G_1 and G_2 are regions, moreover G_1 is bounded, G_2 is unbounded and $\partial G_1 = \partial G_2 = L$. ■

Let L be a contour (possibly closed) $L = \{z = z(t),\ a \le t \le b\}$.



A set $\{z_1 = z(t_1),\ z_2 = z(t_2), \ldots, z_n = z(t_n)\}$ is called a division of L if $a = t_1 < t_2 < \cdots < t_n = b$.

Let f be a function defined on L.

Take points ζ_k between z_k and z_{k+1} $(k = 1, 2, \ldots, n-1)$
$$\zeta_k = z(\tau_k), \quad \tau_k \in [t_k, t_{k+1}]$$

Form the sum
$$\sum_{k=1}^{n-1} f(\zeta_k)(z_{k+1} - z_k)$$
(It is called an integral sum connected with the division).

Assume, there exists a complex number I such that $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\left| I - \sum_{k=1}^{n-1} f(\zeta_k)(z_{k+1} - z_k) \right| < \varepsilon \quad \text{whenever} \quad \max_{1 \le k \le n-1} |z_{k+1} - z_k| < \delta .$$

Then we say that $f(z)$ is integrable along $\mathcal{L}$ and denote

$$I = \int_{\mathcal{L}} f(z)\,dz$$

$$\left(\int_{\mathcal{L}} f(t)\,dt = \lim_{n\to\infty} \sum_{k=1}^{n-1} f(\zeta_k)(z_{k+1} - z_k) \quad \text{as} \quad \max_{1 \le k \le n-1} |z_{k+1} - z_k| \to 0 \right)$$

<u>Theorem</u>. If $f(z)$ is continuous on $\mathcal{L}$ where $\mathcal{L}$ is a contour then $f(z)$ is integrable along $\mathcal{L}$, and if $f(z) = u(x,y) + iv(x,y)$, then

$$\int_{\mathcal{L}} f(z)\,dz = \int_{\mathcal{L}} (u\,dx - v\,dy) + i \int_{\mathcal{L}} (v\,dx + u\,dy)$$

<u>Proof</u>. Let $f(z) = u(x,y) + iv(x,y)$, $z_k = x_k + iy_k$, $\zeta_k = \xi_k + i\eta_k$, then

$$\sum_{k=1}^{n-1} f(\zeta_k)(z_{k+1} - z_k) = \sum_{k=1}^{n-1} \big(u(\xi_k, \eta_k) + iv(\xi_k, \eta_k)\big)\big((x_{k+1} - x_k) + i(y_{k+1} - y_k)\big)$$

$$= \sum_{k=1}^{n-1} u(\xi_k, \eta_k)(x_{k+1} - x_k) - v(\xi_k, \eta_k)(y_{k+1} - y_k)$$

$$+ i \sum_{k=1}^{n-1} v(\xi_k, \eta_k)(x_{k+1} - x_k) + u(\xi_k, \eta_k)(y_{k+1} - y_k)$$

$$\to \int_{\mathcal{L}} (u\,dx - v\,dy) + i \int_{\mathcal{L}} (v\,dx + u\,dy) \quad \blacksquare$$

<u>Theorem</u> If $f(z)$ is continuous on $\mathcal{L}$, where $\mathcal{L}$ is contour defined by $\mathcal{L} = \{z : z = z(t),\ a \le t \le b\}$ then

$$\int_{\mathcal{L}} f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt . \qquad (*)$$

<u>Proof</u> It is well know from Calculus that

$$\int_{\mathcal{L}} P(x,y)\,dx + Q(x,y)\,dy = \int_a^b \big(P(x(t),y(t))\,x'(t) + Q(x(t),y(t))\,y'(t)\big)\,dt$$

we may use this result with that of the previous theorem to obtain $(*)$. $\blacksquare$

Remarks. 1) Above theorem makes the problem of evaluating complex-valued functions along contours easy, as it reduces the task to evaluation of complex-valued functions over real intervals, a procedure we have studied earlier.

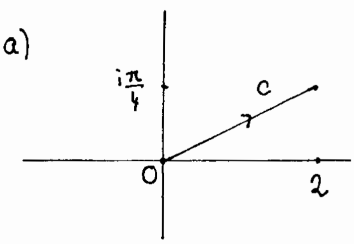
2) As theorem shows $\int_{\mathcal{L}} f(z)dz$ does not depend on the parametrization of the contour $\mathcal{L}$.

Example. Evaluate

a) $\int_C e^z dz$ where C is the line segment joining 0 to $2+i\frac{\pi}{4}$.

b) $\int_C \frac{1}{z-2} dz$ where C is the upper semicircle with radius 1 centered at $x=2$ oriented in a positive direction.

Solution.

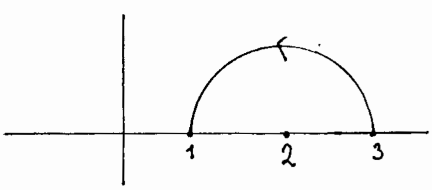
a) 

a parametrization for C is

$$z(t) = (1-t)0 + t\left(2+i\frac{\pi}{4}\right)$$
$$= \left(2+i\frac{\pi}{4}\right)t, \qquad 0 \le t \le 1, \text{ and } z'(t) = 2+i\frac{\pi}{4}.$$

Therefore,
$$\int_C e^z dz = \int_0^1 e^{(2+i\frac{\pi}{4})t}\left(2+i\frac{\pi}{4}\right)dt$$

Since
$$\frac{d}{dt}\left(e^{(2+i\frac{\pi}{4})t}\right) = \left(2+i\frac{\pi}{4}\right)e^{(2+i\frac{\pi}{4})t}$$
we have
$$\int_C e^z dz = e^{(2+i\frac{\pi}{4})t}\Big|_{t=0}^{t=1} = e^{2+i\frac{\pi}{4}} - e^0 = e^2 e^{i\frac{\pi}{4}} - 1 = e^2\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) - 1$$
$$= \left(\frac{e^2}{\sqrt{2}} - 1\right) + i\frac{e^2}{\sqrt{2}}.$$

b) 

a parametrization for C is

$$z(t) = 2 + e^{it}, \quad 0 \le t \le \pi \text{ and } z'(t) = ie^{it}$$

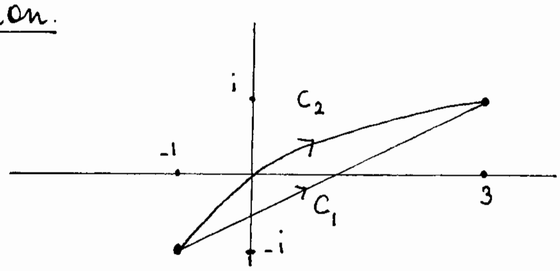
Thus
$$\int_C \frac{1}{z-2}dz = \int_0^\pi \frac{1}{e^{it}} ie^{it} dt = i\int_0^\pi dt = i\pi.$$

Example. Show that

$$\int_{C_1} z\,dz = \int_{C_2} z\,dz$$

where C_1 is the line segment from $-1-i$ to $3+i$ and C_2 is the portion of the parabola $x = y^2 + 2y$ joining $-1-i$ to $3+i$.

Solution.



$$x = y^2 + 2y = y^2 + 2y + 1 - 1$$
$$= (y+1)^2 - 1$$

C_1: $\quad z(t) = (1-t)(-1-i) + t(3+i) = -1 - i + t + it + 3t + it = (-1+4t) + i(2t-1)\,, \quad 0 \le t \le 1$

$$z'(t) = 4 + 2i \quad \text{and} \quad \int_{C_1} z\,dz = \int_0^1 \big((-1+4t) + i(2t-1)\big)(4+2i)\,dt \quad \Rightarrow$$

$$\int_{C_1} z\,dz = (4+2i)\left(\int_0^1 (-1+4t)\,dt + i\int_0^1 (2t-1)\,dt\right) = (4+2i)\left((-t+2t)\Big|_0^1 + i(t^2-t)\Big|_0^1\right)$$

$$= (4+2i)(1) = 4 + 2i.$$

C_2: $\quad z(t) = t^2 + 2t + it\,, \quad -1 \le t \le 1 \quad \Rightarrow \quad z'(t) = 2t + 2 + i$ and

$$\int_{C_2} z\,dz = \int_{-1}^{1} (t^2 + 2t + it)(2t + 2 + i)\,dt$$

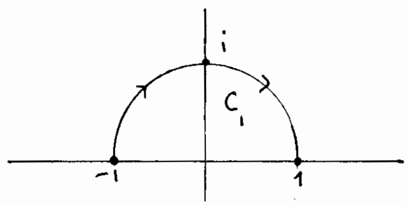
$$= \int_{-1}^{1} (2t^3 + 2t^2 + it^2 + 4t^2 + 4t + 2it + 2it^2 + 2it - t)\,dt$$

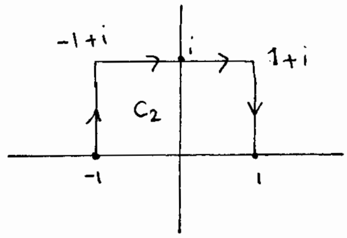
$$= \int_{-1}^{1} (2t^3 + 6t^2 + 3t)\,dt + i\int_{-1}^{1} (3t^2 + 4t)\,dt$$

$$= \left(\frac{t^4}{2} + 2t^3 + 3\frac{t^2}{2}\right)\Big|_{-1}^{1} + i\,(t^3 + 2t^2)\Big|_{-1}^{1}$$

$$= 4 + i2 = 4 + 2i.$$

Example. Show that $\int_{C_1} \bar{z}\,dz \neq \int_{C_2} \bar{z}\,dz$ where C_1 is the semicircular path from -1 to 1 and C_2 is the polygonal path from -1 to 1, respectively show in the figure:





For C_1: $\quad z(t) = e^{-it}, \quad \pi \le t \le 2\pi \quad \Rightarrow \quad z'(t) = -i e^{-it}$

$$\int_{C_1} \bar{z}\, dt = \int_{\pi}^{2\pi} \overline{e^{-it}} \, (-i e^{-it})\, dt = -i \int_{\pi}^{2\pi} e^{it} e^{-it}\, dt = -i \int_{\pi}^{2\pi} dt = -\pi i$$

For C_2:

$$z(t) = \begin{cases} -1+it & 0 \le t \le 1 \quad (z'(t) = i) \\ (2t-3)+i & 1 \le t \le 2 \quad (z'(t) = 2) \\ 1+(-t+3)i & 2 \le t \le 3 \quad (z'(t) = -i) \end{cases}$$

$$\int_{C_2} \bar{z}\, dt = \int_0^1 (-1-it)\, i\, dt + \int_1^2 ((2t-3)-i)\, 2\, dt + \int_2^3 (1+(t-3)i)(-i)\, dt$$

$$= \left(-it + \frac{t^2}{2}\right)\Big|_0^1 + 2\left(\left(t^2 - 3t - it\right)\Big|_1^2\right) - i\left(t + \frac{t^2}{2} i - 3ti\right)\Big|_2^3$$

$$= -i + \frac{1}{2} + 2(4 - 6 - 2i - 1 + 3 + i) - i\left(3 + \frac{9}{2} i - 9i - 2 - 2i + 6i\right)$$

$$= -i + \frac{1}{2} - 2i - i - \frac{1}{2} = -4i.$$

Note that the value of the contour integral along C_1 isn't the same as the value of the contour integral along C_2, although both integrals have the same initial and terminal points.

Properties

i) $$\int_{-C} f(z)\, dz = -\int_C f(z)\, dz$$

ii) $$\int_C (f(z) + g(z))\, dz = \int_C f(z)\, dz + \int_C g(z)\, dz$$

iii) $$\int_C c f(z)\, dz = c \int_C f(z)\, dz$$ for any constant $c \in \mathbb{C}$.

(iv) If two contours C_1 and C_2 are placed end to end so that the terminal point of C_1 coincides with the initial point of C_2, then the contour $C = C_1 + C_2$ is a continuation of C_1, and

$$\int_{C_1+C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

<u>Theorem</u> If $f(t) = u(t) + iv(t)$ is a continuous function of the real parameter t, then

$$\left| \int_a^b f(t)\,dt \right| \leq \int_a^b |f(t)|\,dt$$

<u>Proof</u>. wlog, $\int_a^b f(t)dt \neq 0$. (if so assertion is trivial)

Then $\int_a^b f(t)dt = r_0 e^{i\theta_0}$ for some $r_0 > 0$ and $\theta_0 \in \mathbb{R}$.

So $$r_0 = \underbrace{e^{-i\theta_0}}_{\text{constant}} \int_a^b f(t)dt = \int_a^b e^{-i\theta_0} f(t)dt$$

Then $$r_0 = Re(r_0) = Re\left(\int_a^b e^{-i\theta} f(t)\,dt\right) = \int_a^b Re(e^{-i\theta} f(t))\,dt \,.$$

Since $$Re(e^{-i\theta_0} f(t)) \leq |e^{-i\theta_0} f(t)| = |e^{-i\theta_0}||f(t)| = |f(t)|$$

we have $$\int_a^b Re(e^{-i\theta} f(t))\,dt \leq \int_a^b |f(t)|\,dt$$

since $r_0 = \left|\int_a^b f(t)\,dt\right|$, the result follows. ■

<u>Theorem.</u> If $f(z) = u(x,y) + iv(x,y)$ is continuous on the contour C, then

$$\left|\int_C f(z)\,dz\right| \leq ML,$$

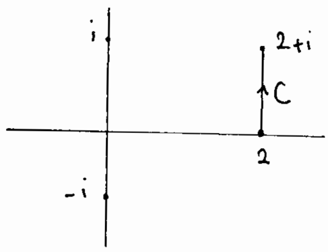
where L is the length of the contour C and M is an upper bound for the modulus $|f(z)|$ on C; that is $|f(z)| \leq M$ for all $z \in C$.

Proof: Omitted. $\blacksquare$

Example Show that $\left| \int_C \frac{1}{z^2+1}\, dz \right| \leq \frac{1}{2\sqrt{5}}$,

where C is the straight-line segment from 2 to $2+i$.

Solution



$|z^2+1| = |(z+i)(z-i)| = |z+i||z-i|$

simple geometry shows that

$|z-i| \geq 2$ and $|z+i| \geq \sqrt{5}$ for all $z \in C$.

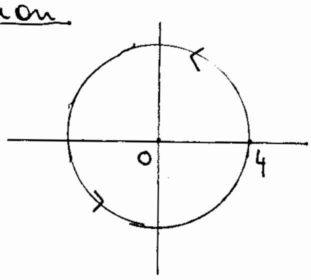
so $\left|\frac{1}{z^2+1}\right| \leq \frac{1}{2\sqrt{5}}$

Clearly, $L =$ length of $C = 1$ and so, by the theorem, we have

$$\left| \int_C \frac{1}{1+z^2}\, dz \right| \leq \frac{1}{2\sqrt{5}} \cdot 1 = \frac{1}{2\sqrt{5}}.$$

Example Evaluate $\int_C z\, dz$ where C is the circle of radius 4 centered at the origin, oriented clockwise.

Solution



$C: z(t) = 4e^{it}$, $0 \leq t \leq 2\pi$ $\quad z'(t) = 4ie^{it} \Rightarrow$

$$\int_C z\, dz = \int_0^{2\pi} 4e^{it}\, 4ie^{it}\, dt = 16i \int_0^{2\pi} e^{2it}\, dt$$

$$= 16i \left. \frac{e^{2it}}{2i} \right|_0^{2\pi}$$

$$= 8\left(e^{4\pi i} - 1\right) = 0.$$

The Cauchy-Goursat Theorem

Recall that each simple closed contour C divides the plane into two domains. One domain is bounded and is called the <u>interior</u> of C, the other domain is unbounded and called the <u>exterior</u> of C.

Recall that a <u>domain</u> $\mathcal{D}$ is a connected open set. In particular, if z_1 and z_2 are any pair of points in $\mathcal{D}$, then they can be joined by a curve that lies entirely in $\mathcal{D}$. A domain $\mathcal{D}$ is said to be a <u>simply connected domain</u>, if the interior of any simple closed contour C contained in $\mathcal{D}$ is contained in $\mathcal{D}$.

A domain that is not simply connected is said to be a <u>multiply connected domain</u>.

Let the simple closed contour C have the parametrization $C \quad z(t) = x(t) + iy(t)$ for $a \leq t \leq b$. Recall that if C is parametrized so that the interior of C is kept on the left as $z(t)$ moves around C, then we say that C is <u>oriented positively</u> (counterclockwise) otherwise, C is <u>oriented negatively</u> (clockwise).

Recall the Green's theorem from Calculus:

<u>Theorem (Green's theorem)</u> Let C be a simple closed contour with positive orientation and let R be the domain that forms the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x and Q_y at all points of C and R, then

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(Q_x(x,y) - P_y(x,y)\right)dx\,dy .$$

Theorem (Cauchy-Goursat theorem) Let f be analytic in a simply connected domain $\mathcal{D}$. If C is a simple closed contour that lies in $\mathcal{D}$, then

$$\int_C f(z)\,dz = 0$$

Cauchy's proof (with additional assumption that f' is continuous)

Recall that, if $f(z) = u(x,y) + iv(x,y)$,

$$\int_C f(z)\,dz = \int_C u(x,y)\,dx - v(x,y)\,dy + i\int_C v(x,y)\,dx + u(x,y)\,dy,$$

Applying Green's theorem, we obtain

$$\int_C f(z)\,dz = \iint_R \left(-v_x(x,y) - u_y(x,y)\right)dx\,dy + i\iint_R \left(u_x(x,y) - v_y(x,y)\right)dx\,dy$$

By Cauchy-Riemann equations ($u_x = v_y$; $v_x = -u_y$)
$-v_x(x,y) - u_y(x,y) = 0$ and $u_x(x,y) - v_y(x,y) = 0$ and so

$$\int_C f(z)\,dz = 0. \blacksquare$$

Goursat proved the Cauchy theorem by assuming only the analyticity of f.

~~Goursat's proof~~ Reading exercise. $\blacksquare$

Example. Let C be any simple closed contour, then

$$\int_C e^z\,dz = 0, \quad \int_C \cos z\,dz = 0 \text{ and } \int_C z^n\,dz = 0 \text{ for } n \in \mathbb{N}\cup\{0\},$$

because e^z, $\cos z$ and z^n are entire functions.

Example. Let C be a simple closed contour which is not circumscribing the origin, then

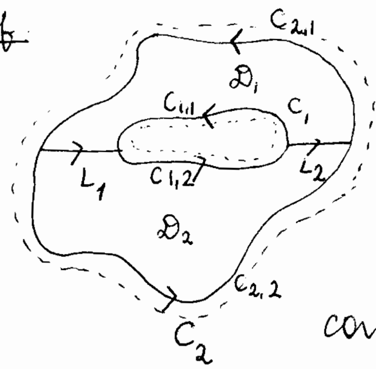
$$\int_C \frac{1}{z^n}\,dz = 0.$$

Theorem (Deformation of contour)

Let C_1 and C_2 be two simple closed positively oriented contours such that C_1 lies interior to C_2. If f is analytic in a domain $\mathcal{D}$ that contains both C_1 and C_2 and the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

Proof.



We construct two disjoint contours or cuts, L_1 and L_2 that joins C_1 to C_2. The contour C_1 is cut into two contour $C_{1,1}$ and $C_{1,2}$, and the contour C_2 is cut into $C_{2,1}$ and $C_{2,2}$.

We now form two new contours

$$K_1 = -C_{1,1} + L_2 + C_{2,1} + L_1 \quad \text{and} \quad K_2 = C_{2,2} - L_2 - C_{1,2} - L_1$$

The function f will be analytic on a simple connected domain $\mathcal{D}_1$ that contains K_1, and f will be analytic on the simple connected domain $\mathcal{D}_2$ that contains K_2. We apply the Cauchy-Goursat theorem to the contours K_1 and K_2, giving

$$\int_{K_1} f(z)dz = 0 \quad \text{and} \quad \int_{K_2} f(z)dz = 0.$$

Adding contours gives

$$\begin{aligned} K_1 + K_2 &= -C_{1,1} + L_2 + C_{2,1} + L_1 + C_{2,2} - L_2 - C_{1,2} - L_1 \\ &= (C_{2,1} + C_{2,2}) - (C_{1,1} + C_{1,2}) = C_2 - C_1. \end{aligned}$$

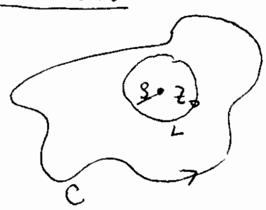
Thus

$$\int_{C_2} f(z)dz - \int_{C_1} f(z)dz = \int_{K_1} f(z)dz + \int_{K_2} f(z)dz = 0. \quad \blacksquare$$

Example Let z_0 be a fixed complex number. If C is a simple closed contour with positive orientation such that z_0 lies interior to C, then

$$\int_C \frac{dz}{z-z_0} = 2\pi i \quad \text{and} \quad \int_C \frac{dz}{(z-z_0)^n} = 0 \quad \text{if} \quad n \in \mathbb{Z}, n \neq 1.$$

Solution



Let $\rho > 0$ be so that $\{z : |z - z_0| \leq \rho\}$ is completely contained in the interior of C.

Let $L = \{z : |z - z_0| = \rho\}$ be oriented positively.

Then

$$\int_C \frac{dz}{z-z_0} = \int_L \frac{dz}{z-z_0}$$

L has the parametrization

$$L \quad z(t) = z_0 + \rho e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \quad \text{and so}$$

$$\int_C \frac{dz}{z-z_0} = \int_L \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{1}{\rho e^{i\theta}} i\rho e^{i\theta} d\theta = i \int_0^{2\pi} d\theta = 2\pi i$$

Similarly,

$$\int_C \frac{dz}{(z-z_0)^n} = \int_L \frac{dz}{(z-z_0)^n} = \int_0^{2\pi} \frac{1}{\rho^n e^{in\theta}} i\rho e^{i\theta} d\theta$$

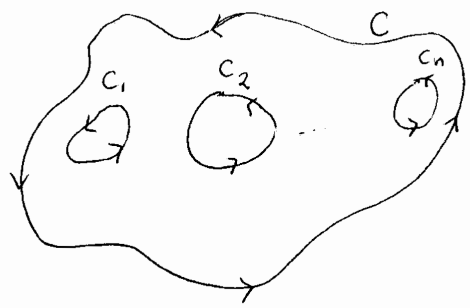
$$= i\rho^{1-n} \int_0^{2\pi} e^{i(1-n)\theta} d\theta = i\rho^{1-n} \left. \frac{e^{i(1-n)\theta}}{i(1-n)} \right|_0^{2\pi}$$

$$= \frac{\rho^{1-n}}{1-n} - \frac{\rho^{1-n}}{1-n} = 0.$$

Theorem. (Extended Cauchy-Goursat theorem)

Let $C, C_1, C_2, \ldots, C_n$ be simple, closed, positively oriented contours such that C_k's are interior to C for $k = 1, 2, \ldots, n$, and the interior of C_k has no points in common with the interior of C_j if $k \neq j$. Let f be analytic on a domain $\mathcal{D}$ that contains all the contours and the region between C and $C_1 + C_2 + \cdots + C_n$. Then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz.$$

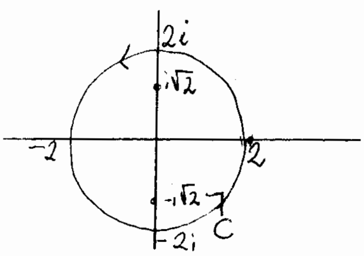


Proof. Exercise ■

Example. Find $\int_C \frac{2z}{z^2+2}\,dz$, where C is the circle $\{z : |z| = 2\}$ with positive orientation.

Proof (By partial fraction decomposition)

$$\frac{2z}{z^2+2} = \frac{1}{z+i\sqrt{2}} + \frac{1}{z-i\sqrt{2}}$$

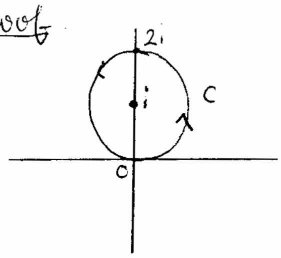


$$\int_C \frac{2z}{z^2+2}\,dz = \int_C \frac{dz}{z+i\sqrt{2}} + \int_C \frac{dz}{z-i\sqrt{2}}$$

$$= 2\pi i + 2\pi i \quad \text{by the previous example.}$$

Example Find $\int_C \frac{2z}{z^2+2}\,dz$ if $C = \{z : |z-i| = 1\}$ having positive orientation.

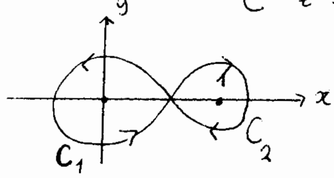
Proof.



$$\int_C \frac{2z}{z^2+2}\,dz = \underbrace{\int_C \frac{dz}{z+i\sqrt{2}}}_{0} + \underbrace{\int_C \frac{dz}{z-i\sqrt{2}}}_{2\pi i} = 2\pi i$$

by Cauchy-Goursat theorem because $\frac{1}{z+i\sqrt{2}}$ is analytic inside and on C.

Example Find $\int_C \frac{z-2}{z^2-z}\,dz$, where C is the contour shown below



Solution

Clearly, $\quad \dfrac{z-2}{z^2-z} = \dfrac{2}{z} - \dfrac{1}{z-1}$.

Let C_1 be the left (simple, closed) part of C and C_2 be the right part of C. Then

$$\int_C \frac{z-2}{z^2-z}\,dz = \int_{C_1} \frac{z-2}{z^2-z}\,dz + \int_{C_2} \frac{z-2}{z^2-z}\,dz.$$

and

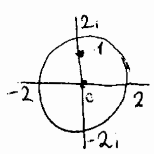
$$\int_{C_1} \frac{z-2}{z^2-z}\,dz = 2\int_{C_1} \frac{dz}{z} - \underbrace{\int_{C_1} \frac{dz}{z-1}}_{0} = 2(2\pi i) = 4\pi i$$

$$\int_{C_2} \frac{z-2}{z^2-z}\,dz = -\int_{-C_2} \frac{z-2}{z^2-z}\,dz = -\Big(\underbrace{\int_{-C_2} \frac{1}{z}\,dz}_{0} - \underbrace{\int_{-C_2} \frac{dz}{z-1}}_{2\pi i}\Big) = 2\pi i$$

Therefore, $\quad \displaystyle\int_C \frac{z-2}{z^2-z}\,dz = 6\pi i$

Example. Find $\displaystyle\int_C \frac{2z-1}{z^2-z}\,dz$, where $C = \{z : |z| = 2\}$ oriented positively.

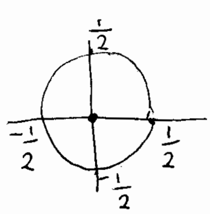
Solution. Clearly, $\quad \dfrac{2z-1}{z^2-z} = \dfrac{1}{z-1} + \dfrac{1}{z}$, and



$$\int_C \frac{1}{z-1}\,dz + \int_C \frac{1}{z}\,dz = 2\pi i + 2\pi i = 4\pi i$$

Example Find $\displaystyle\int_C \frac{2z-1}{z^2-z}\,dz$, where $C = \{z : |z| = \tfrac{1}{2}\}$ oriented positively.

Solution



$$\int_C \frac{2z-1}{z^2-z}\,dz = \underbrace{\int_C \frac{dz}{z-1}}_{0} + \underbrace{\int_C \frac{dz}{z}}_{2\pi i} = 2\pi i$$

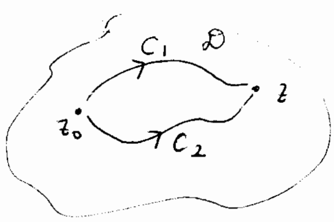
The fundamental theorems of integration

Theorem. Let f be analytic in the simply connected domain $\mathcal{D}$. If z_0 is a fixed value in $\mathcal{D}$ and if C is any contour in $\mathcal{D}$ with initial point z_0 and terminal point z, then the function

$$F(z) = \int_C f(\xi)\,d\xi = \int_{z_0}^{z} f(\xi)\,d\xi$$

is well-defined and analytic in $\mathcal{D}$, with its derivative given by $F'(z) = f(z)$.

Proof Step 1. (The integral is independent of the path of integration and thus the function F is well-defined and the notation $\int_{z_0}^{z} f(\xi)d\xi$ is suitable)



Let C_1 and C_2 be two contours in $\mathcal{D}$, both with initial point z_0 and terminal point z. Then $C_2 - C_1$ is a simple closed contour, and the Cauchy–Goursat theorem implies that

$$\int_{C_2 - C_1} f(\xi)d\xi = 0 = \int_{C_2} f(\xi)\,d\xi - \int_{C_1} f(\xi)\,d\xi \Rightarrow$$

$$\int_{C_1} f(\xi)d\xi = \int_{C_2} f(\xi)d\xi .$$

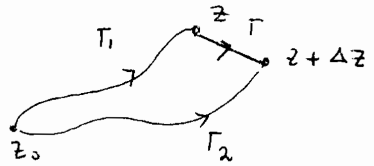
Step 2. ($F'(z) = f(z)$) Let z be held fixed, and let $|\Delta z|$ be chosen small enough so that the point $z + \Delta z$ also lies in the domain $\mathcal{D}$. Let $K = f(z)$, then

$$\int_{z}^{z+\Delta z} f(z)d\xi = \int_{z}^{z+\Delta z} K\,d\xi = K\Delta z = f(z)\Delta z$$

Therefore,
$$F(z+\Delta z) - F(z) = \int_{z_0}^{z+\Delta z} f(\xi)d\xi - \int_{z_0}^{z} f(\xi)d\xi$$

$$= \int_{\Gamma_2} f(\xi)d\xi - \int_{\Gamma_1} f(\xi)d\xi = \int_{\Gamma} f(\xi)d\xi$$

where Γ is the straight-line segment joining z to $z+\Delta z$, and Γ_1 and Γ_2 join z_0 to z and z_0 to $z+\Delta z$, respectively.



Since f is continuous at z, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \varepsilon \quad \text{whenever} \quad |\zeta - z| < \delta$$

if $\Delta z < \delta$, we have

$$\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right| = \frac{1}{|\Delta z|} \left| \int_\Gamma f(\zeta)d\zeta - \int_\Gamma f(z)d\zeta \right|$$

$$= \frac{1}{|\Delta z|} \left| \int_\Gamma f(\zeta) - f(z)\, d\zeta \right|$$

$$\leqslant \frac{1}{|\Delta z|} \varepsilon . |\Delta z| = \varepsilon$$

Thus, $\left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) \right|$ tends to 0 as $\Delta z \to 0$, so $F'(z) = f(z)$ ▨

<u>Theorem</u> Let f be analytic in a simply connected domain $\mathcal{D}$. If z_0 and z_1 are any two points in $\mathcal{D}$ joined by a contour C, then

$$\int_C f(z)dz = \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0) \qquad (*)$$

where F is an antiderivative of f in $\mathcal{D}$.

<u>Proof</u>. If we choose $F(z) = \int_{z_0}^{z} f(\zeta)d\zeta$ and set $z = z_1$, we clearly have $(*)$

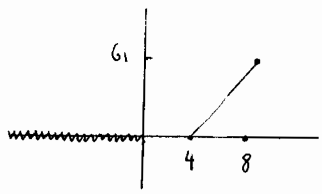
If G is any other antiderivative of f in $\mathcal{D}$, then $G'(z) = F'(z)$ for all $z \in \mathcal{D}$. Thus, the function $H(z) = G(z) - F(z)$ is analytic in $\mathcal{D}$, and $H'(z) = G'(z) - F'(z) = 0$, for all $z \in \mathcal{D}$. Thus $H(z) = K$ for some constant $K \in \mathbb{C}$. Therefore,

$$G(z) = F(z) + K, \quad \text{so} \quad G(z_1) - G(z_0) = F(z_1) - F(z_0). \text{▨}$$

Remark The contour integral of an analytic function is independent of path.

Example. Find $\int_C \frac{dz}{2z^{\frac{1}{2}}}$, where $z^{\frac{1}{2}}$ is the principal branch of the square root function and C is the line segment joining 4 to $8+6i$.

Solution



Clearly, $z^{\frac{1}{2}}$ is analytic on $\mathbb{C}\setminus(-\infty,0]$ and $\frac{d}{dz} z^{\frac{1}{2}} = \frac{1}{2}\frac{1}{z^{\frac{1}{2}}}$ is also analytic in $\mathbb{C}\setminus(-\infty,0]$, and

$$\int_C \frac{dz}{2z^{\frac{1}{2}}} = (8+6i)^{\frac{1}{2}} - 4^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}(8+6i)} - e^{\frac{1}{2}\mathrm{Log}\,4}$$

$$= e^{\frac{1}{2}(\ln 10 + i\theta)} - e^{\frac{1}{2}(\ln 4 + i0)} \quad \text{where } \theta = \arcsin\frac{3}{5}$$

$$= \sqrt{10}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right) - 2$$

$$= \sqrt{10}\left(\frac{3}{\sqrt{10}} + i\frac{1}{\sqrt{10}}\right) - 2 = 3+i-2 = 1+i$$

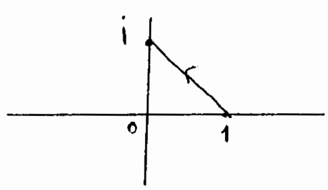
$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\frac{4}{5} = 2\cos^2\frac{\theta}{2} - 1 \Leftrightarrow \frac{9}{10} = \cos^2\frac{\theta}{2}$

$\Rightarrow \cos\frac{\theta}{2} = \frac{3}{\sqrt{10}}$ and $\sin\frac{\theta}{2} = \frac{1}{\sqrt{10}}$

Example Find $\int_C \cos z\, dz$, where C is the line segment between 1 and i.

Solution.

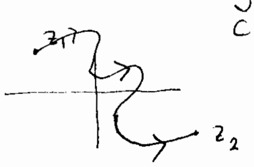


Clearly, $\frac{d}{d}\sin z = \cos z$ and so

$$\int_C \cos z\, dz = \sin i - \sin 1 = \frac{e^{-1} - e^{1}}{2i} - \sin 1$$

$$= i\left(\frac{e^{1} - e^{-1}}{2}\right) - \sin 1 = -\sin 1 + i\sinh 1$$

Example. Find $\int_C \frac{dz}{z}$, where C is the contour shown below

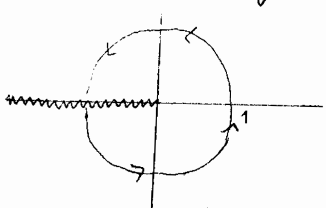


Solution $\operatorname{Log} z$ is analytic in $\mathbb{C}\setminus(-\infty,0]$ and $\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}$ there, so

$$\int_C \frac{dz}{z} = \operatorname{Log} z_2 - \operatorname{Log} z_1.$$

Example Find $\int_C \frac{dz}{z}$ where $C = \{z : |z| = 1\}$.

Solution $\operatorname{Log} z$ is analytic and has the derivative $\frac{1}{z}$ in $\mathbb{C}\setminus(-\infty,0]$, we may suppose that -1 is omitted from C.



Therefore, if we let z_2 approach -1 on C through the upper half-plane and z_1 approach -1 on C through the lower half-plane,

$$\int_C \frac{dz}{z} = \lim_{\substack{z_2\to -1,\ \operatorname{Im} z_2 > 0 \\ z_1 \to -1\ \operatorname{Im} z_1 < 0}} \int_{z_1}^{z_2} \frac{dz}{z}$$

$$= \lim_{z_2\to -1,\ \operatorname{Im} z_2>0} \operatorname{Log} z_2 - \lim_{z_1\to -1,\ \operatorname{Im} z_1<0} \operatorname{Log} z_1$$

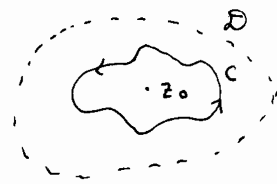
$$= i\pi - (-i\pi) = 2\pi i.$$

Exercise Evaluate $\int_C \operatorname{Log} z\, dz$, where C is the line segment from 1 to $1+i$

Integral representations for analytic functions

Theorem (Cauchy's integral formula)

Let f be analytic in the simply connected domain $\mathcal{D}$ and let C be a simple closed positively oriented contour that lies in $\mathcal{D}$. If z_0 is a point that lies interior to C, then



$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

Proof: Let $\varepsilon > 0$ be arbitrary.

f is analytic in $\mathcal{D} \Rightarrow f$ is continuous at z_0

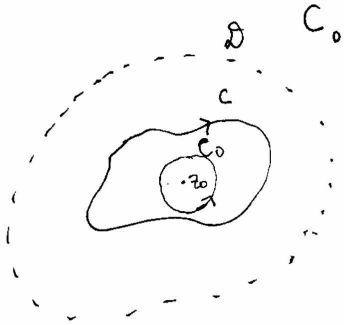
$\Rightarrow \exists \delta_1 > 0$ such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever} \quad |z - z_0| \leq \delta_1 .$$

z_0 is interior to $C \Rightarrow \exists \delta_2 > 0$ such that $\{z : |z| \leq \delta_2\}$ is interior to C. Choose $\delta = \min\{\delta_1, \delta_2\}$ and let

$C_0 = \{z : |z - z_0| = \delta\}$. Clearly, C_0 is interior to C

and
$$\oint_{C_0} \frac{dz}{z - z_0} = 2\pi i \quad \text{and hence} \quad \frac{1}{2\pi i} \oint_{C_0} \frac{f(z_0)}{z - z_0}\, dz = f(z_0).$$



On the other hand, since $\dfrac{f(z)}{z - z_0}$ is analytic on a region containing C, C_0 and the region between them

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z - z_0}\, dz .$$

Therefore
$$\left| \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz - f(z_0) \right| = \left| \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z - z_0}\, dz - \frac{1}{2\pi i} \oint_{C_0} \frac{f(z_0)}{z - z_0}\, dz \right|$$

$$= \left| \frac{1}{2\pi i} \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| .$$

If $z \in C_0$ then $|z - z_0| < \delta_1$ and so $|f(z) - f(z_0)| < \varepsilon$

Hence if $z \in C_0$

$$\left| \frac{f(z)-f(z_0)}{z-z_0} \right| < \frac{\varepsilon}{\delta} \quad \text{and so}$$

$$\left| \frac{1}{2\pi i} \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0}\, dz \right| \leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{\delta} \cdot (\text{length of } C_0) = \frac{1}{2\pi} \cdot \frac{\varepsilon}{\delta} \cdot 2\pi\delta = \varepsilon .$$

Thus $\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz - f(z_0) \right| < \varepsilon$ for any $\varepsilon > 0$. This implies

that $$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz = f(z_0). \quad \blacksquare$$

<u>Examples</u> Find

1. $$\oint_C \frac{\sin z}{4z+\pi}\, dz$$

2. $$\oint_C \frac{e^{i\pi z}}{2z^2-5z+2}\, dz$$

where $C = \{z : |z| < 1\}$.

<u>Solutions</u>

1. Let $f(z) = \sin z$ and $z_0 = -\frac{\pi}{4}$. Clearly f is analytic in a region containing C (in fact $\sin z$ is an entire function) and z_0 is a point interior to C. $\left(\left| -\frac{\pi}{4} \right| = \frac{\pi}{4} \approx 0.79 < 1 \right)$

Then by Cauchy's integral formula

$$-\frac{1}{\sqrt{2}} = \sin\left(-\frac{\pi}{4}\right) = f\left(-\frac{\pi}{4}\right) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz = \frac{1}{2\pi i} \oint_C \frac{\sin z}{z-(-\frac{\pi}{4})}\, dz = \frac{4}{2\pi i} \oint_C \frac{\sin z}{4z+\pi}\, dz$$

$$\Rightarrow \oint_C \frac{\sin z}{4z+\pi}\, dz = -\frac{2\pi i}{4} \cdot \frac{1}{\sqrt{2}} = -\frac{\sqrt{2}\pi i}{4}.$$

2. Note that $2z^2 - 5z + 2 = 2(z-2)\left(z - \frac{1}{2}\right)$. Let $f(z) = \frac{e^{i\pi z}}{2(z-2)}$ and $z_0 = \frac{1}{2}$. Clearly, f is analytic in a region

containing C and z_0 is interior to C, then by Cauchy's integral formula

$$f\left(\tfrac{1}{2}\right) = \frac{e^{i\pi\frac{1}{2}}}{2\left(\frac{1}{2}-2\right)} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_C \frac{\frac{e^{i\pi z}}{2(z-2)}}{\left(z-\frac{1}{2}\right)}\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{e^{i\pi z}}{2(z-2)\left(z-\frac{1}{2}\right)}\,dz = \frac{1}{2\pi i}\oint_C \frac{e^{i\pi z}}{2z^2-5z+2}\,dz$$

and so

$$\oint_C \frac{e^{i\pi z}}{2z^2-5z+2}\,dz = \frac{2\pi i \cdot e^{i\frac{\pi}{2}}}{2\left(\frac{1}{2}-2\right)} = -\frac{2}{3}\pi i \cdot i = \frac{2\pi}{3}.$$

__Theorem__ Let G be a region bounded by a simple closed contour $\mathcal{L}$. Let f be a function analytic on $G \cup \mathcal{L}$. Then, for $\forall z \in G$, $\forall k \in \mathbb{N}$, there exists $f^{(k)}(z)$, and moreover

$$f^{(k)}(z) = \frac{k!}{2\pi i}\oint_{\mathcal{L}} \frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta$$

__Remark.__ Above theorem is known as "Cauchy's integral formula for derivatives" and implies that "If f is analytic in a region G, then it is infinitely many times differentiable in G."

To prove the theorem we need a lemma.

__Lemma.__ Let $\mathcal{L}$ be a simple contour, let $\mathcal{D}$ be a closed disk $\{z : |z-z_0| \le R\}$. Let $F(\zeta, z)$ be a function defined for $\forall \zeta \in L$, $\forall z \in \mathcal{D}$. Assume

i) $F(\zeta, z)$ is continuous on $\mathcal{L} \times \mathcal{D}$

ii) for any fixed $\zeta \in \mathcal{L}$, $F(\zeta, z)$ is analytic as a function of z in $\mathcal{D}$

iii) the derivative $F_z(\zeta, z)$ is continuous on $\mathcal{L} \times \mathcal{D}$.

Then the function

$$f(z) = \int_{\mathcal{L}} F(\zeta, z)\, d\zeta \quad \text{is analytic in interior of } \mathcal{D}$$

and moreover

$$f'(z) = \int_{\mathcal{L}} F_z(\zeta, z)\, d\zeta \quad \text{for all } z \in \mathrm{int}(\mathcal{D}).$$

Proof. Take $z \in \mathrm{int}(\mathcal{D})$. Let h be so small that $z+h \in \mathrm{int}(\mathcal{D})$

Consider

$$\frac{f(z+h) - f(z)}{h} - \int_{\mathcal{L}} F_z(\zeta, z)\, d\zeta$$

$$\int_{\mathcal{L}} \left(\frac{F(\zeta, z+h) - F(\zeta, z)}{h} - F_z(\zeta, z) \right) d\zeta = \int_{\mathcal{L}} \left(\frac{1}{h} \int_z^{z+h} F_z(\zeta, u)\, du - F_z(\zeta, z) \right) d\zeta$$

$$= \int_{\mathcal{L}} \left(\frac{1}{h} \int_z^{z+h} \left(F_z(\zeta, u) - F_z(\zeta, z) \right) du \right) d\zeta$$

Since $F_z(\zeta, u)$ is continuous on $L \times \mathcal{D}$, then for any $\varepsilon > 0$ $\exists \delta > 0$ such that

$$|F_z(\zeta, u) - F_z(\zeta, z)| < \varepsilon \quad \text{whenever} \quad |u - z| < \delta, \ \zeta \in L$$

If $|h| < \delta$ then

$$\left| \frac{1}{h} \int_z^{z+h} \left(F_z(\zeta, u) - F_z(\zeta, z) \right) du \right| \leq \frac{1}{|h|} \cdot |h| \cdot \varepsilon = \varepsilon$$

and so

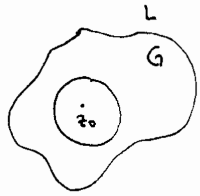
$$\left| \frac{f(z+h) - f(z)}{h} - \int_{\mathcal{L}} F_z(\zeta, z)\, d\zeta \right| \leq (\text{length of } L)\, \varepsilon$$

This implies that $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = f'(z) = \int_{\mathcal{L}} F_z(\zeta, z)\, d\zeta$. ∎

Proof of the theorem. (By induction)

$k = 1$. Set $F(\zeta, z) = \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z}$, $\zeta \in \mathcal{L}$, $z \in \mathcal{D}$.

$$F_z(\zeta, z) = \frac{1}{2\pi i} \frac{f(\zeta)}{(\zeta - z)^2}$$



$\mathcal{D}_0$ = a closed neighborhood of z_0

By Cauchy's formula

$$f(z) = \int_{\mathcal{L}} F(\zeta, z)\, d\zeta \,.$$

The conditions of the lemma are satisfied. Thus, by the lemma

$$f'(z) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2} \,.$$

Assume the assertion of the theorem is valid for some $k \in \mathbb{N}$. By the assumption

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{L} \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}} \,.$$

Set $\tilde{F}(\zeta, z) = \dfrac{k!}{2\pi i} \dfrac{f(\zeta)}{(\zeta - z)^{k+1}}$, $\zeta \in \mathcal{L}$, $z \in \mathcal{D}$

all conditions of the lemma are satisfied for $\tilde{F}(\zeta, z)$ and $\tilde{F}_z(\zeta, z) = \dfrac{(k+1)!}{2\pi i} \dfrac{f(\zeta)}{(\zeta - z)^{k+2}}$ to obtain the result for $k+1$ we apply the lemma. ☒

We now state two important corollaries of the theorem.

<u>Corollary</u> If f is analytic in the domain $\mathcal{D}$, then, for integers $n \geq 0$, all derivatives $f^{(n)}(z)$ exist for $z \in \mathcal{D}$ (and therefore analytic in $\mathcal{D}$)

<u>Proof</u> Exercise!

Corollary. If u is a harmonic function at each point (x,y) in the domain $\mathcal{D}$, then all partial derivatives u_x, u_y, u_{xx}, u_{xy} and u_{yy} exist and are harmonic functions.

Proof. Exercise!

Example. Evaluate

1. $\displaystyle\oint_C \frac{e^{z^2}}{(z-i)^4}\,dz$ where $C=\{z:|z|=2\}$

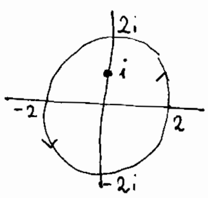
2. $\displaystyle\oint_C \frac{e^{z}+\cos z}{z}\,dz$ where $C=\{z:|z|=1\}$

3. $\displaystyle\oint_C \frac{1}{(z+1)(z-1)^2}\,dz$ where $C=\{z:|z-1|=1\}$

4. $\displaystyle\oint_C \frac{\sinh(z^2)}{z^3}\,dz$ where $C=\{z:|z|=1\}$

Solutions.

1.



Let $f(z)=e^{z^2}$, clearly f is entire, and let $z_0=i$. Applying the theorem (Cauchy integral formula for derivatives), we obtain

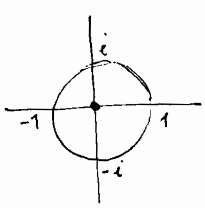
$$f^{(3)}(i)=\frac{3!}{2\pi i}\int_C \frac{e^{z^2}}{(z-i)^4}\,dz \Rightarrow \int_C \frac{e^{z^2}}{(z-i)^4}\,dz=\frac{2\pi i}{3!}\,f^{(3)}(i)$$

$f'(z)=e^{z^2}.2z$, $f''(z)=e^{z^2}(2z)^2+e^{z^2}.2$, $f^{(3)}(z)=e^{z^2}(2z)^3+e^{z^2}2(2z).2+e^{z^2}(2z).2$

$\Rightarrow f^{(3)}(i)=e^{-1}(-8i)+e^{-1}2(2i)2+e^{-1}(2i)2=\dfrac{4i}{e} \Rightarrow$

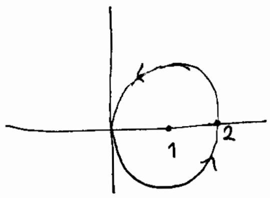
$$\int_C \frac{e^{z^2}}{(z-i)^4}\,dz=\frac{2\pi i}{6}\cdot\frac{4i}{e}=-\frac{4\pi}{3e}.$$

2.



Let $f(z)=e^{z}+\cos z$, and $z_0=0$. Then

$$f(0)=\frac{1}{2\pi i}\int_C \frac{e^{z}+\cos z}{z}\,dz \Rightarrow \int_C \frac{e^{z}+\cos z}{z}\,dz=f(0)2\pi i$$
$$=2.2\pi i=4\pi i$$

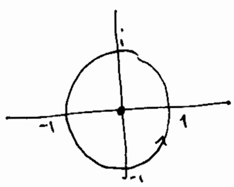
3. 

Let $f(z) = \frac{1}{z+1}$ and $z_0 = 1$

Clearly f is analytic in a domain containing C. Then

$$f'(1) = \frac{1}{2\pi i} \oint_C \frac{\frac{1}{(z+1)}}{(z-1)^2}\, dz = \frac{1}{2\pi i} \oint_C \frac{1}{(z+1)(z-1)^2}\, dz \quad \text{and}$$

$$f'(z) = \frac{-1}{(z+1)^2} \Rightarrow f'(1) = \frac{-1}{4} \Rightarrow \oint_C \frac{1}{(z+1)(z-1)^2}\, dz = -\frac{2\pi i}{4} = -\frac{\pi i}{2}.$$

4. 

Let $f(z) = \sinh(z^2)$ and $z_0 = 0 \Rightarrow$

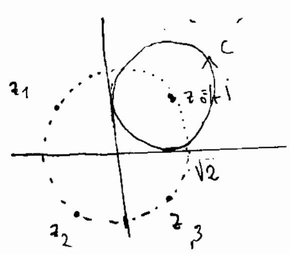
$$f''(0) = \frac{2}{2\pi i} \oint_C \frac{\sinh(z^2)}{z^3}\, dz \quad \text{and}$$

$$f'(z) = \cosh(z^2)\, 2z, \quad f''(z) = \sinh(z^2)(2z)^2 + \cosh z \cdot 2 \Rightarrow f''(0) = 2$$

$$\Rightarrow \oint_C \frac{\sinh(z^2)}{z^3}\, dz = 2\pi i.$$

<u>Example.</u> Find $\oint_C \frac{1}{z^4+4}\, dz$ where $C = \{z : |z-1-i| < 1\}$

<u>solution</u> $z^4 + 4 = 0 \Leftrightarrow z^4 = -4 \Leftrightarrow z^4 = 4e^{i\pi} \Leftrightarrow z = 4^{\frac{1}{4}} e^{\frac{i(\pi + 2k\pi)}{4}} = \sqrt{2}\, e^{i\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)}$ $k = 0,1,2,3$



$$z^4 + 4 = (z - z_0)\underbrace{(z - z_1)(z - z_2)(z - z_3)}_{g(z)}$$

Let $f(z) = \frac{1}{g(z)}$, (which is analytic in a domain containing C) and $z_0 = 1+i$. By Cauchy's integral formula

$$f(1+i) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-1-i}\, dz = \frac{1}{2\pi i} \oint_C \frac{1}{z^4+4}\, dz \Rightarrow$$

$$\oint_C \frac{1}{z^4+4}\, dz = \frac{2\pi i}{g(1+i)} = \frac{2\pi i}{(1+i-(-1+i))(1+i-(-1-i))(1+i-(1-i))} = \frac{\cancel{2}\pi \cancel{i}}{\cancel{2}(2+2i)2\cancel{i}} = \frac{\pi}{4+4i}\,\frac{4-4i}{4-4i} = \frac{4\pi - 4\pi i}{32} = \frac{\pi}{8} - \frac{\pi i}{8}.$$

The Theorems of Morera and Liouville, and Extensions

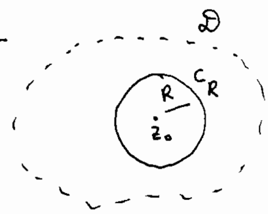
Theorem (Morera) Let f be a continuous function in a simply connected domain $\mathcal{D}$. If $\oint_C f(z)\,dz = 0$ for every closed contour C in $\mathcal{D}$, then f is analytic in $\mathcal{D}$.

Proof. Exercise!
Hint: Define $F(z) = \int_{z_0}^{z} f(\xi)\,d\xi$ for some point z_0 in $\mathcal{D}$ and take the integral along some contour that begins at z_0 and ends at z lying entirely in $\mathcal{D}$. Show that F is well defined and analytic in $\mathcal{D}$ (by using the continuity of f) and $F'(z) = f(z)$. Then, apply corollary of the Cauchy's integral formula for derivatives to F. ■

Theorem. (Gauss' mean value theorem) If f is analytic in a simply connected domain $\mathcal{D}$ that contains the circle $C_R(z_0) = \{z : |z - z_0| = R\}$, then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta .$$

Proof.



C_R: $z(\theta) = z_0 + Re^{i\theta}$, $0 \leq \theta \leq 2\pi$

By Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_R} \frac{f(z)}{z - z_0}\,dz = \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{\cancel{Re^{i\theta}}} \cancel{R i e^{i\theta}}\,d\theta$$

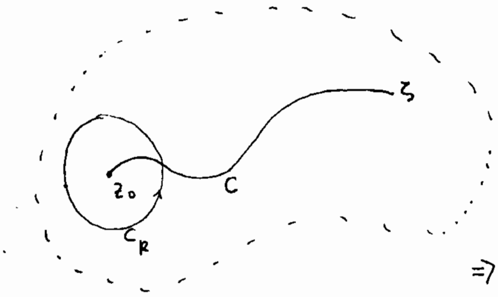
$$= \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta})\,d\theta . \quad ■$$

Theorem (Maximum modulus principle) Let f be analytic and nonconstant in the domain $\mathcal{D}$. Then $|f(z)|$ does not attain a maximum value at any point z_0 in $\mathcal{D}$.

Proof. Assume the contrary and suppose that there exists a point $z_0 \in \mathcal{D}$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \mathcal{D}$.

Let $C_R(z_0) = \{z : |z - z_0| = R\}$ be any circle contained in $\mathcal{D}$. Let $0 < r \leq R$.
By the previous theorem, we have

$$|f(z_0)| = \left|\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+re^{i\theta})|\,d\theta$$



$$\le \frac{1}{2\pi}\int_0^{2\pi}|f(z_0)|\,d\theta = |f(z_0)| \Rightarrow$$

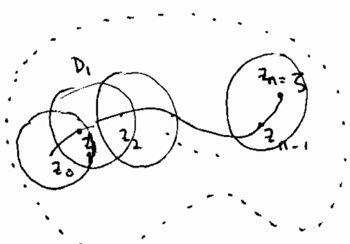
$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi}|f(z_0+re^{i\theta})|\,d\theta$$

$$\Rightarrow \frac{1}{2\pi}\int_0^{2\pi}\underbrace{\{|f(z_0)| - |f(z_0+re^{i\theta})|\}}_{\text{continuous and nonnegative}}\,d\theta = 0$$

$\Rightarrow |f(z_0)| - |f(z_0+re^{i\theta})| = 0 \quad \Rightarrow \quad |f(z_0)| = |f(z_0+re^{i\theta})|$ for any r, $0 \le r \le R$ and any θ, $0 \le \theta \le 2\pi$.

This implies that modulus of $f(z)$ is constant in $\{z : |z-z_0| \le R\}$, then by a previous theorem $f(z)$ is constant in $\{z : |z-z_0| \le R\}$, that is $f(z) = f(z_0)$ for all $z \in \{z : |z-z_0| \le R\}$.

Now, let ζ be an arbitrary point in $\mathcal{D}$ and C be a contour that joins z_0 to ζ and $0 < d < \text{dist}(\mathcal{D}, C)$. Choose points $z_1, z_2, \ldots, z_{n-1}$ on C (say $z_n = \zeta$) with $|z_{k+1} - z_k| < d$. So the disks



$D_k = \{z : |z - z_k| \le d\}$ are contained in $\mathcal{D}$ and cover C.

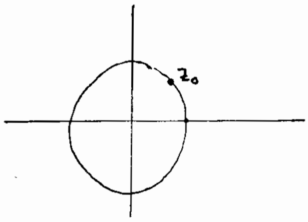
By the previous arguments $f(z_0) = f(z_1)$, that is $|f(z)|$ also reaches its maximum value at z_1. A similar argument shows that $f(z) = f(z_1) = f(z_0)$ for all z in $\mathcal{D}_1$. Repeating this argument, we arrive at $f(\zeta) = f(z_0)$. Therefore, f is constant in $\mathcal{D}$, which is a contradiction. ▩

<u>Theorem.</u> (Maximum modulus principle) Let f be analytic and nonconstant in the bounded domain $\mathcal{D}$. If f is continuous on the closed region $R = \mathcal{D} \cup \partial\mathcal{D}$, then $|f(z)|$ assumes its maximum value, and does so only at point(s) z_0 on $\partial\mathcal{D}$.

<u>Proof.</u> Exercise! ▩

Example. Let $f(z) = az+b$. Prove that $\max\limits_{|z|\leq 1} |f(z)| = |a|+|b|$.

solution



If $|z|\leq 1$ then

$$|f(z)| = |az+b| \leq |az|+|b| = |a||z|+|b| \leq |a|+|b|$$

$$\Rightarrow \max_{|z|\leq 1} |f(z)| \leq |a|+|b|$$

Let $a = |a|e^{i\alpha}$ and $b = |b|e^{i\beta}$. If $z_0 = e^{i(\beta-\alpha)}$, then

$$|f(z_0)| = \left|\, |a| e^{i\alpha} e^{i(\beta-\alpha)} + |b|e^{i\beta}\right| = \underbrace{|e^{i\beta}|}_{1} \left|\, |a| \underbrace{e^{i(\alpha-\beta)} e^{i(\beta-\alpha)}}_{1} + |b|\right| = \big||a|+|b|\big| = |a|+|b|.$$

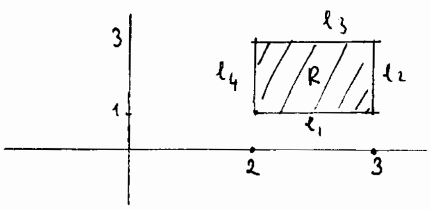
Clearly $|z_0| = 1$ and so $\max\limits_{|z|\leq 1} |f(z)| = |a|+|b|$. ▨

Example. Let $f(z) = z^2$ and $R = \{ z = x+iy : 2\leq x\leq 3 \text{ and } 1\leq y\leq 3\}$ Find

(a) $\max\limits_{z\in R} |f(z)|$, (b) $\min\limits_{z\in R} |f(z)|$, (c) $\max\limits_{z\in R} \mathrm{Re}(f(z))$, (d) $\min\limits_{z\in R} \mathrm{Im}(f(z))$.

Solution.

(a),(b) $|f(z)| = |z^2| = |z|^2 = |x+iy|^2 = x^2+y^2 = F(x,y)$



Critical points: $F_x = 2x = 0$ $F_y = 2y = 0 \Rightarrow (0,0) \notin R$

boundary points:

on l_1: $2\leq x\leq 3$, $y=1$ $F(x) = x^2+1 \Rightarrow F'(x) = 2x = 0$
$\Rightarrow x=0$, but $0\notin [2,3]$.

on l_2: $x=3$, $1\leq y\leq 3$ $F(y) = 9+y^2 \Rightarrow F'(y) = 2y = 0 \Rightarrow y=0$, but $0\notin [1,3]$

on l_3: $2\leq x\leq 3$, $y=3$ $F(x) = x^2+9 \Rightarrow F'(x) = 2x = 0 \Rightarrow x=0$, but $0\notin [2,3]$.

on l_4: $1\leq y\leq 3$, $x=2$ $F(y) = 4+y^2 \Rightarrow F'(y) = 2y = 0 \Rightarrow y=0$, but $0\notin [1,3]$.

Corners: (end points of l_1, l_2, l_3 and l_4): $(2,1)$, $(3,1)$, $(2,3)$, $(3,3)$

$F(2,1) = 4+1 = 5$, $F(3,1) = 9+1 = 10$, $F(2,3) = 4+9 = 13$, $F(3,3) = 9+9 = 18$. $\Rightarrow$

$\max\limits_{z\in R} |f(z)| = 18$ and $\min\limits_{z\in R} |f(z)| = 5$.

(c) $\mathrm{Re}\, f(z) = \mathrm{Re}\,(x+iy)^2 = \mathrm{Re}(x^2-y^2+2ixy) = x^2-y^2 = G(x,y)$.

Critical points: $G_x = 2x = 0$, $G_y = -2y = 0 \Rightarrow (x,y) = (0,0) \notin R$.

l_1: $G(x) = x^2-1 \Rightarrow G'(x) = 2x = 0 \Rightarrow x=0 \notin (2,3)$

l_2: $G(y) = 9-y^2 \Rightarrow G'(y) = -2y = 0 \Rightarrow y=0 \notin (1,3)$

l_3: $G(x) = x^2-9 \Rightarrow G'(x) = 2x = 0 \Rightarrow x=0 \notin (2,3)$

l_4: $G(y) = 4-y^2 \Rightarrow G'(y) = -2y = 0 \Rightarrow y=0 \notin (1,3)$

Corners: $G(2,1) = 3$, $G(2,3) = -5$, $G(3,1) = 8$, $G(3,3) = 0 \Rightarrow \max\limits_{z\in R} \mathrm{Re}\, f(z) = 8$.

(d) Exercise!

Theorem (*Cauchy's inequalities*) Let f be analytic in the simply connected domain $\mathcal{D}$ that contains the circle $C_R(z_0) = \{z : |z - z_0| = R\}$. If $|f(z)| \leq M$ holds for all points $z \in C_R(z_0)$, then

$$|f^{(n)}(z_0)| \leq \frac{n!\,M}{R^n} \quad \text{for } n = 0, 1, 2, \ldots$$

Proof. By the Cauchy integral formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_R(z_0)} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Note that $C_R(z_0)$ can be parametrized by $C_R(z_0) : z(\theta) = z_0 + Re^{i\theta}$, $0 \leq \theta \leq 2\pi$ and so

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + Re^{i\theta})}{R^{n+1} e^{i(n+1)\theta}} \cdot Rie^{i\theta}\, d\theta = \frac{n!}{2\pi R^n} \int_0^{2\pi} f(z_0 + Re^{i\theta})\, e^{-in\theta}\, d\theta$$

and hence

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(z_0 + Re^{i\theta}) e^{-in\theta}|\, d\theta$$

$$= \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(z_0 + Re^{i\theta})|\, d\theta \leq \frac{Mn!}{2\pi R^n} \int_0^{2\pi} d\theta = \frac{Mn!}{R^n}. \quad \blacksquare$$

We use the above theorem to prove the following important theorem.

Theorem (*Liouville's theorem*) If f is an entire function and is bounded for all values of z in the complex plane, then f is constant.

Proof. Suppose that $\exists M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be arbitrary, and $C_R(z_0) = \{z : |z - z_0| = R\}$ where $R > 0$ is arbitrary. By Cauchy's inequality with $n = 1$, we have

$$|f'(z_0)| \leq \frac{M}{R}.$$

Since R can be arbitrarily large, we must have $f'(z_0) = 0$.

Since z_0 is arbitrary, we have $f'(z) = 0 \quad \forall z \in \mathbb{C}$.

since $f'(z)=0 \quad \forall z \in \mathbb{C}$, f should be constant. ■

Example. Show that $f(z) = \sin z$ is unbounded.

Solution If it were, it would be constant, but it is not. ■

Finally, we can use Liouville's theorem to prove the "Fundamental theorem of algebra".

Theorem. (The fundamental theorem of algebra) If P is a polynomial of degree $n \geq 1$, then P has at least one zero.

Proof Assume contrary. Then $P(z) \neq 0 \quad \forall z \in \mathbb{C}$. Set $f(z) = \frac{1}{P(z)}$.

Clearly, f is an entire function.

If $\quad P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, we have

$$|f(z)| = \frac{1}{|P(z)|} = \frac{1}{|z|^n \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \right|}.$$

Clearly, $a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \to a_n$ as $|z| \to \infty$ and hence $|f(z)| \to 0$ as $|z| \to \infty$. In particular, $\exists R > 0$ such that $|f(z)| \leq 1$ for all $|z| \geq R$.

On the other hand $|f(z)| = \sqrt{(u(x,y))^2 + (v(x,y))^2}$ is continuous function and hence bounded on the compact set $\{z : |z| \leq R\}$ by some positive number say M.

Therefore $|f(z)|$ is bounded by $\max\{M, 1\}$ on $\mathbb{C}$.

By Liouville's theorem f is constant, so that the degree of f is zero, a contradiction. ■

Corollary Let P be a polynomial of degree $n \geq 1$. Then P can be expressed as the product of linear factors. That is,

$$P(z) = A(z - z_1)(z - z_2) \cdots (z - z_n), \qquad (*)$$

where $z_1, z_2, \ldots, z_n$ are the zeros of P, counted according to multiplicity, and A is constant.

Proof. The theorem ensures that $P(z)$ has a zero z_1. Then, $P(z) = (z-z_1)\,Q_1(z)$ (see exercise below)

where $Q_1(z)$ is a polynomial of degree $n-1$. The same argument, applied to $Q_1(z)$ gives us that there is a number z_2 such that

$$P(z) = (z-z_1)(z-z_2)Q_2(z),$$

where $Q_2(z)$ is a polynomial of degree $n-2$. Continuing in this way, we obtain (*) ∎

Exercise Let z_0 be a zero of the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \quad , \ (a_n \neq 0) \text{ of degree } n \ (n \geq 1)$$

show in the following way that

$$P(z) = (z-z_0)\,Q(z),$$

where $Q(z)$ is a polynomial of degree $n-1$.

a) verify that

$$z^k - z_0^k = (z-z_0)(z^{k-1} + z^{k-2} z_0 + \cdots + z z_0^{k-2} + z_0^{k-1}) \quad (k=2,3,\ldots)$$

b) Use the factorization in part (a) to show that

$$P(z) - P(z_0) = (z-z_0)\,Q(z)$$

where $Q(z)$ is a polynomial of degree $n-1$, and deduce the desired result from this.

Example. Let f be an entire function with the property that $|f(z)| \geq 1$ for all z. Show that f is constant.

Solution If $|f(z)| \geq 1$, f is never zero and so $g(z) = \frac{1}{f(z)}$ is an entire function and moreover $|g(z)| = \frac{1}{|f(z)|} \leq 1 \Rightarrow g(z) = C$ by Liouville's theorem $\Rightarrow$ $f(z) = \frac{1}{g(z)} = \frac{1}{C} \Rightarrow f$ is also constant.

Example. Let f be an entire function such that $|f(z)| \leq M|z|$ for all z.

a) Show that, for $n \geq 2$, $f^{(n)}(z) = 0$ for all z.

b) Use part (a) to show that $f(z) = az + b$.

Solution.

a) By Cauchy's integral formula

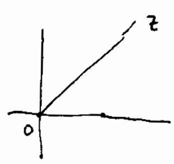
$$|f^{(n)}(z)| \leq \frac{n!}{2\pi R^n} \int_0^{2\pi} |f(z_0 + Re^{i\theta})| \, d\theta$$

$$\leq \frac{n!}{R^n} M(|z_0| + R) \text{ for all } R > 0.$$

If $n \geq 2$, this implies that $f^{(n)}(z) = 0$.

b) $f''(z) = 0 \Rightarrow f'(z) = C$. By the fundamental theorems of integration, $\int_0^z f'(\xi)\, d\xi = f(z) - f(0)$ and so

$$\int_0^z C \, d\xi = f(z) - f(0)$$



let $\xi = tz$, $0 \leq t \leq 1$

then $\int_0^z C \, d\xi = \int_0^1 Cz \, dt = Cz \int_0^1 dt = Cz$

$\Rightarrow f(z) = Cz + f(0)$

By redenoting $C = a$ and $f(0) = b$, we obtain

$f(z) = az + b$. ■

Taylor and Laurent Series

Uniform Convergence

For a function f defined on a set T, the sequence of functions $\{S_n\}$ converges to f at the point $z_0 \in T$, provided

$$\lim_{n\to\infty} S_n(z_0) = f(z_0).$$

That is, for the particular point z_0, for each $\varepsilon > 0$, there exists a positive integer N_{ε, z_0} such that

$$|S_n(z_0) - f(z_0)| < \varepsilon \quad \text{whenever} \quad n \geq N_{\varepsilon, z_0}.$$

If this is true for all $z_0 \in T$, we say that $\{S_n\}$ <u>converges pointwise</u> to f on T.

The sequence $\{S_n\}$ <u>converges uniformly</u> to f on T if for every $\varepsilon > 0$, $\exists$ a positive integer N_ε such that

$$|S_n(z) - f(z)| < \varepsilon \quad \text{for all } z \in T \text{ for all } n \geq N_\varepsilon.$$

If $S_n(z)$ is the partial sum of the series $\sum_{k=0}^{\infty} c_k (z-\alpha)^k$, we say that series converges uniformly to f on T.

<u>Example.</u> Show that the sequence $f_n(z) = e^z + \frac{1}{n}$ converges uniformly to $f(z) = e^z$ on $\mathbb{C}$.

<u>Proof.</u> Let $\varepsilon > 0$ be given. Choose $N_\varepsilon = \left[\!\left[\frac{1}{\varepsilon}\right]\!\right] + 1$. Clearly $N_\varepsilon > \frac{1}{\varepsilon}$ and $|f_n(z) - f(z)| = \frac{1}{n} \leq \frac{1}{N_\varepsilon} < \varepsilon$ whenever $n \geq N_\varepsilon$.

Example. Show that the series $\sum_{k=0}^{\infty} z^k$ converges to $\frac{1}{1-z}$ on $\mathcal{D}_1(0) = \{z : |z| < 1\}$ pointwise but not uniformly.

Solution. We know that $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ for all $z \in \mathcal{D}_1(0)$.

Clearly,
$$|S_n(z) - f(z)| = \left|1 + z + \cdots + z^n - \frac{1}{1-z}\right|$$
$$= \left|\frac{1-z^{n+1}}{1-z} - \frac{1}{1-z}\right| = \frac{|z|^{n+1}}{|1-z|}.$$

Clearly, for any $1 > \varepsilon > 0$ and any $n \in \mathbb{N}$, we can find a point $z_0 = \varepsilon^{\frac{1}{n+1}} \in D_1(0)$ such that
$$|S_n(z_0) - f(z_0)| = \frac{\varepsilon}{1 - \varepsilon^{\frac{1}{n+1}}} > \varepsilon.$$
This shows that $S_n \not\to f$ uniformly on $\mathcal{D}_1(0)$.

Theorem (Weierstrass M-test) Suppose that the infinite series $\sum_{k=0}^{\infty} u_k(z)$ has the property that for each k, $|u_k(z)| \leq M_k$ for all $z \in T$. If $\sum_{k=0}^{\infty} M_k$ converges, then $\sum_{k=0}^{\infty} u_k(z)$ converges uniformly on T.

Proof. Exercise!

Example. Prove that $\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ converges uniformly on $\{z : |z| \geq 2\}$.

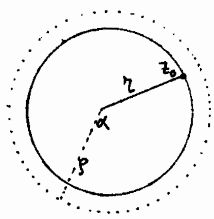
Solution If $|z| \geq 2$, then $|z^2 - 1| \geq |z|^2 - 1 \geq 3$ and so
$$\left|\frac{1}{(z^2-1)^k}\right| \leq \frac{1}{3^k}.$$
Since $\left|\frac{1}{3}\right| < 1$, $\sum_{k=0}^{\infty} \frac{1}{3^k}$ converges.

If we apply the Weierstrass M-test with $u_k(z) = \frac{1}{(z^2-1)^k}$ and $M_k = \frac{1}{3^k}$, we conclude that $\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ converges uniformly on $\{z : |z| \geq 2\}$.

Theorem Suppose that the power series $\sum_{k=0}^{\infty} c_k(z-\alpha)^k$ has radius of convergence $\rho > 0$. Then for each r, $0 < r < \rho$, the series converges uniformly on the closed disk $\bar{D}_r(\alpha) = \{z : |z-\alpha| \leq r\}$.

Proof. We have proved before that $\sum_{k=0}^{\infty} c_k(z-\alpha)^k$ converges absolutely for $z \in D_\rho(\alpha) = \{z : |z-\alpha| < \rho\}$. Then if z_0 is a point such that $|z_0 - \alpha| = r$, then

$$\sum_{k=0}^{\infty} |c_k(z-\alpha)^k| = \sum_{k=0}^{\infty} |c_k| r^k \quad \text{converges.}$$



Moreover for all $z \in \bar{D}_r(\alpha)$

$$|c_k(z-\alpha)^k| = |c_k||z-\alpha|^k \leq |c_k| r^k \quad \text{and so}$$

$\sum_{k=0}^{\infty} c_k(z-\alpha)^k$ converges uniformly by Weierstrass M-test (with $M_k = |c_k| r^k$). ∎

Theorem. Suppose $\{S_k\}$ is a sequence of continuous functions defined on a set T containing the contour C. If $\{S_k\}$ converges uniformly to f on the set T, then

(i) f is continuous on T, and

(ii) $$\lim_{k\to\infty} \int_C S_k(z)\,dz = \int_C \lim_{k\to\infty} S_k(z)\,dz = \int_C f(z)\,dz.$$

Proof. (i) Exercise!

(ii) Let $\varepsilon > 0$ be given. Since $\{S_k\}$ converges uniformly to f on T $\exists N_\varepsilon > 0$ such that

$$|S_k(z) - f(z)| < \frac{\varepsilon}{L} \quad \text{for all } z \in T \quad \text{where } L$$

is the length of C. Therefore

$$\max_{z\in C} |S_k(z) - f(z)| < \frac{\varepsilon}{L} \quad \text{and}$$

$$\left| \int_C S_k(z)\,dz - \int_C f(z)\,dz \right| = \left| \int_C (S_k(z) - f(z))\,dz \right|$$

$$\leq \frac{\varepsilon}{L} \cdot L = \varepsilon \qquad \text{for all } k > N_\varepsilon$$

This shows that

$$\lim_{k \to \infty} \int_C S_k(z)\,dz = \int_C f(z)\,dz = \int_C \left(\lim_{k \to \infty} S_k(z)\right) dz. \quad \blacksquare$$

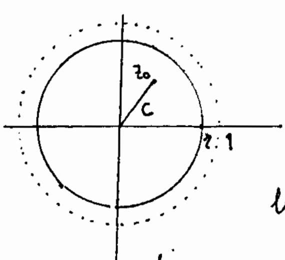
<u>Corollary</u>. If $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ converges uniformly to $f(z)$ on T and C is a contour contained in T, then

$$\sum_{n=0}^{\infty} \int_C c_n (z-\alpha)^n\,dz = \int_C \sum_{n=0}^{\infty} c_n (z-\alpha)^n\,dz = \int_C f(z)\,dz.$$

<u>Example</u>. Show that $\sum_{n=1}^{\infty} \frac{1}{n} z^n = -\operatorname{Log}(1-z)$ for all $z \in D_1(0) = \{z : |z| < 1\}$.

<u>Solution</u>

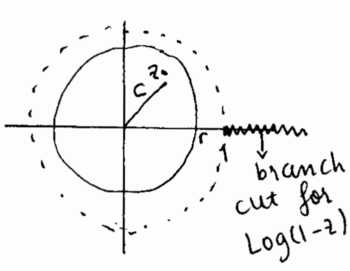
Let $z_0 \in D_1(0)$ be arbitrary. Choose r so that $|z_0| < r < 1$



Since $\sum_{n=0}^{\infty} z^n$ converges uniformly to $\frac{1}{1-z}$ in $D_r(0)$

$\sum_{n=0}^{\infty} \int_C z^n\,dz = \int_C \frac{1}{1-z}\,dz$ for any contour lying entirely in $D_r(0)$. Let C be the line segment joining 0 to z_0. Then

$\int_C z^n\,dz = \int_0^{z_0} z^n\,dz = \left.\frac{z^{n+1}}{n+1}\right|_0^{z_0} = \frac{z_0^{\,n+1}}{n+1}$. Clearly $-\operatorname{Log}(1-z)$ is analytic in $\mathbb{C} \setminus [1, \infty)$ and $\frac{d}{dz}\left(-\operatorname{Log}(1-z)\right) = \frac{1}{1-z}$. Thus



$$\int_C \frac{1}{1-z}\,dz = \int_0^{z_0} \frac{1}{1-z}\,dz = \left. -\operatorname{Log}(1-z) \right|_0^{z_0}$$

$$= -\operatorname{Log}(1-z_0) + \underbrace{\operatorname{Log} 1}_{\ln|1| + i\operatorname{Arg}(1) = 0} = -\operatorname{Log}(1-z_0).$$

and hence $-\operatorname{Log}(1-z_0) = \sum_{n=0}^{\infty} \frac{z_0^{\,n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z_0^{\,n}}{n}$ for all $z_0 \in D_1(0)$.

Taylor Series Representations

If f is analytic at $z=\alpha$, then the series

$$f(\alpha)+f'(\alpha)(z-\alpha)+\frac{f^{(2)}(\alpha)(z-\alpha)^2}{2!}+\cdots+\frac{f^{(k)}(\alpha)(z-\alpha)^k}{k!}+\cdots=\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k$$

is called the Taylor series for f centered at $z=\alpha$. When the center is $\alpha=0$, the series is called the Maclaurin series for f.

Lemma. If $z, z_0, \alpha \in \mathbb{C}$ with $z\neq z_0$ and $z\neq\alpha$, then

$$\frac{1}{z-z_0}=\frac{1}{z-\alpha}+\frac{z_0-\alpha}{(z-\alpha)^2}+\frac{(z_0-\alpha)^2}{(z-\alpha)^3}+\cdots+\frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}}+\frac{1}{z-z_0}\,\frac{(z_0-\alpha)^{n+1}}{(z-\alpha)^{n+1}}.$$

Proof. Remind that $\sum_{k=0}^{n} z^k=\frac{1-z^{n+1}}{1-z}=\frac{1}{1-z}-\frac{z^{n+1}}{1-z}$. So

$$\frac{1}{1-z}=1+z+\cdots+z^n+\frac{z^{n+1}}{1-z}.\quad(*)$$

Clearly, $\frac{1}{z-z_0}=\frac{1}{z-\alpha+\alpha-z_0}=\frac{1}{z-\alpha}\left(\frac{1}{1-\frac{z_0-\alpha}{z-\alpha}}\right)$

putting $\frac{z_0-\alpha}{z-\alpha}$ instead of z in $(*)$, we obtain

$$\frac{1}{z-z_0}=\frac{1}{z-\alpha}+\frac{z_0-\alpha}{(z-\alpha)^2}+\quad+\frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}}+\frac{1}{z-z_0}\,\frac{(z_0-\alpha)^{n+1}}{(z-\alpha)^{n+1}}.\ \blacksquare$$

Theorem. Suppose f is analytic in a domain G and that $D_R(\alpha)$ is any disk contained in G. Then the Taylor series for f converges to $f(z)$ for all z in $D_R(\alpha)$; that is

$$f(z)=\sum_{k=0}^{\infty}\frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k \quad \text{for all } z\in D_R(\alpha)$$

Furthermore, for any r, $0<r<R$, the convergence is uniform on the closed subdisk $\overline{D}_r(\alpha)=\{z:|z-\alpha|\leq r\}$.

Proof. Let $z_0 \in D_R(\alpha)$ and let $|z_0-\alpha| = r$. Choose ρ so that $r < \rho < R$. Let $C_\rho(\alpha) = \{z : |z-\alpha| = \rho\}$. Then by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \frac{f(z)}{z-z_0}\, dz$$

By lemma,

$$f(z_0) = \frac{1}{2\pi i} \int_{C_\rho(\alpha)} \left[\frac{1}{z-\alpha} + \frac{z_0-\alpha}{(z-\alpha)^2} + \cdots + \frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}} + \frac{1}{(z-z_0)} \frac{(z_0-\alpha)^{n+1}}{(z-\alpha)^{n+1}} \right] f(z)\, dz$$

$$= \frac{1}{2\pi i} \int_{C_\rho(\alpha)} \frac{f(z)}{z-\alpha}\, dz + \frac{(z_0-\alpha)}{2\pi i} \int_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^2}\, dz + \cdots + \frac{(z_0-\alpha)^n}{2\pi i} \int_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

$$+ E_n(z_0)$$

where
$$E_n(z_0) = \frac{1}{2\pi i} \int_{C_\rho(\alpha)} \frac{(z_0-\alpha)^{n+1} f(z)}{(z-z_0)(z-\alpha)^{n+1}}\, dz$$

Remind that
$$\frac{k!}{2\pi i} \int_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{k+1}}\, dz = f^{(k)}(\alpha), \quad k = 0, 1, 2, \ldots$$

So,

$$f(z_0) = f(\alpha) + f'(\alpha)(z_0-\alpha) + \cdots + \frac{f^{(n)}(\alpha)}{n!}(z_0-\alpha)^n + E_n(z_0).$$

If we show $E_n(z_0) \to 0$ as $n \to \infty$, the proof is done. Let $M = \max_{z \in C} |f(z)|$. Note that for $z \in C_\rho(\alpha)$,

$$|z - z_0| = |(z-\alpha) - (z_0-\alpha)| \geq |z-\alpha| - |z_0-\alpha| = \rho - r$$

So,
$$\left| \frac{(z_0-\alpha)^{n+1} f(z)}{(z-z_0)(z-\alpha)^{n+1}} \right| \leq \frac{r^{n+1} M}{(\rho-r)\rho^{n+1}}$$
and (by ML-inequality)

$$|E_n(z)| \leq \frac{1}{2\pi} \cdot \frac{r^{n+1} M}{(\rho-r)\rho^{n+1}} \cdot 2\pi\rho = \underbrace{\frac{M\rho}{\rho-r}}_{\text{constant}} \left(\frac{r}{\rho}\right)^{n+1} \to 0 \text{ as } n \to \infty$$

because $0 < \frac{r}{\rho} < 1$. ∎

A <u>singular point</u> of a function is a point at which the function fails to be analytic. A <u>nonremovable singular point</u> of a function has the property that it is impossible to redefine the value of the function at that point so as to make it analytic there.

<u>Corollary.</u> Suppose that f is analytic in the domain G that contains the point α. Let z_0 be a nonremovable singular point of minimum distance to the point α. If $|z_0-\alpha| = R$, then

(i) the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (z-\alpha)^k$ converges to $f(z)$ on all of $D_R(\alpha)$, and

(ii) if $|z_1-\alpha| = S > R$, the Taylor series $\sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (z_1-\alpha)^k$ does not converge to $f(z_1)$

<u>Proof.</u> (i) Obvious.

(ii) If $|z_0-\alpha| = R$, then $z_0 \in D_S(\alpha) = \{z : |z-\alpha| < S\}$ whenever $S > R$. If for some z_1, with $|z_1-\alpha| = S > R$, the Taylor series converged to $f(z_1)$, the the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!} (z-\alpha)^k$ would be at least equal to S. We could then make f differentiable at z_0 by redefining $f(z_0)$ to equal to the value of the series at z_0, thus contradicting the fact that z_0 is a nonremovable singular point. ∎

Example. Show that $\dfrac{1}{(1-z)^2} = \sum_{k=0}^{\infty} (k+1) z^k$ for all $z \in \mathcal{D}_1(0)$.

Solution Let $f(z) = \dfrac{1}{(1-z)^2}$. Clearly, f is analytic in

$D_1(0) = \{z : |z|<1\}$ and so $f(z) = \sum_{k=0}^{\infty} \dfrac{f^{(k)}(0)}{k!} z^k$, $z \in \mathcal{D}_1(0)$.

$f'(z) = -2(1-z)^{-3}(-1)$, $f''(z) = -2(-3)(1-z)^{-4}(-1)(-1), \dots, f^{(k)}(z) = (k+1)!(1-z)^{-(k+2)}$

$\Rightarrow \dfrac{f^{(k)}(0)}{k!} = \dfrac{(k+1)!}{k!} = k+1 \Rightarrow f(z) = \sum_{k=0}^{\infty} (k+1) z^k$. ▨

Example. Show that, for $z \in \mathcal{D}_1(0) = \{z : |z|<1\}$,

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty} z^{2n}, \quad \text{and} \quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Solution. We have shown before that

$$\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n \quad \text{for} \quad |w|<1$$

Let $w = z^2$. Clearly $|w|<1 \Leftrightarrow |z^2| = |z|^2 < 1 \Leftrightarrow |z|<1$

and so for $z \in \mathcal{D}_1(0)$ $\dfrac{1}{1-z^2} = \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} z^{2n}$.

Similarly, let $w = -z^2$. Clearly $|w|<1 \Leftrightarrow |z|<1$ and

for $z \in \mathcal{D}_1(0)$ $\dfrac{1}{1-(-z^2)} = \dfrac{1}{1+z^2} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}$.

Theorem (Uniqueness of power series) Suppose that in some disk $\mathcal{D}_r(\alpha)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n = \sum_{n=0}^{\infty} b_n (z-\alpha)^n.$$

Then $a_n = b_n$, for $n = 0, 1, 2, \dots$

Proof By a theorem we have seen in Complex Analysis I (in "power series" chapter) $a_n = \dfrac{f^{(n)}(\alpha)}{n!} = b_n$, for $n = 0, 1, 2, \ldots$ ∎

Example Find the Maclaurin series of $f(z) = \sin^3 z$.

Solution method 1 find $f^{(k)}(z)$, evaluate and $z=0$ and form $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$. (Exercise!)

Method 2. Use the trigonometric identity
$$\sin^3 z = \frac{3}{4}\sin z - \frac{1}{4}\sin 3z$$
and the power series expansion
$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}, \quad z \in \mathbb{C}.$$

$$\sin^3 z = \frac{3}{4}\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} - \frac{1}{4}\sum_{k=0}^{\infty} \frac{(-1)^k (3z)^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} \frac{3(-1)^k(1-3^{2k})}{4(2k+1)!} z^{2k+1} = \sum_{k=0}^{\infty} (-1)^k \frac{3(1-9^k)}{4(2k+1)!} z^{2k+1}.$$

By the uniqueness of power series, this last expression is the Maclaurin series for $\sin^3 z$.

Theorem. Let f and g have the power series representations $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n (z-\alpha)^n$, for $z \in D_{r_1}(\alpha)$,

$g(z) = \displaystyle\sum_{n=0}^{\infty} b_n (z-\alpha)^n$, for $z \in D_{r_2}(\alpha)$.

If $r = \min\{r_1, r_2\}$ and β is any complex constant, then

$$\beta f(z) = \sum_{n=0}^{\infty} \beta a_n (z-\alpha)^n, \quad \text{for } z \in D_{r_1}(\alpha),$$

$$f(z)+g(z)=\sum_{n=0}^{\infty}(a_n+b_n)(z-\alpha)^n, \quad \text{for} \quad z\in \mathcal{D}_r(\alpha), \text{ and}$$

$$f(z)g(z)=\sum_{n=0}^{\infty}c_n(z-\alpha)^n, \quad \text{for} \quad z\in \mathcal{D}_r(\alpha),$$

where $\quad c_n=\sum_{k=0}^{n}a_k b_{n-k}$

<u>Proof.</u> Exercise!

<u>Example.</u> Show $\frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n$ for $z\in\mathcal{D}_1(0)$ by using the Cauchy product of the series!

<u>Solution.</u> let $f(z)=g(z)=\frac{1}{1-z}=\sum_{n=0}^{\infty}z^n$ (that is, $a_n=b_n=1$)

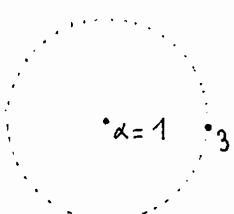
Then $\quad f(z)g(z)=\frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}c_n z^n \quad$ where $\quad c_n=\sum_{k=0}^{n}a_k b_{n-k}$.

So $\quad c_n=(a_0b_n+a_1b_{n-1}+\cdots+a_nb_0)=\underbrace{1+1+\cdots+1}_{(n+1)\text{ terms}}=n+1$

So $\quad \frac{1}{(1-z)^2}=\sum_{n=0}^{\infty}(n+1)z^n.$

<u>Example.</u> Find the Taylor series centered at $\alpha=1$ and state where it converges for $f(z)=\frac{1-z}{z-3}$.

<u>Solution.</u>



f fails to be analytic at $z=3$, so the series converges in $\{z:|z-1|<2\}$, and

$$\frac{1-z}{z-3}=\frac{1-z}{z-1-2}=\frac{1}{2}\,\frac{z-1}{1-\frac{z-1}{2}}=\frac{z-1}{2}\,\frac{1}{1-\left(\frac{z-1}{2}\right)_w}$$

$$=\frac{z-1}{2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n=\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^{n+1}$$

Example. Find $f^{(3)}(0)$ for $f(z) = \sum_{n=1}^{\infty} \frac{(1+i)^n}{n} z^n$

Solution $\frac{f^{(n)}(0)}{n!} = \frac{(1+i)^n}{n} \Rightarrow f^{(3)}(0) = 3! \cdot \frac{(1+i)^3}{3} \Rightarrow$

$f^{(3)}(0) = 2(1+3i-3-i) = -4+4i.$

Laurent Series Representations

Let c_n be a complex number for $n \in \mathbb{Z}$. The doubly infinite series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$, called a Laurent series, is defined by

$$\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n = \sum_{n=-\infty}^{-1} c_n (z-\alpha)^n + \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

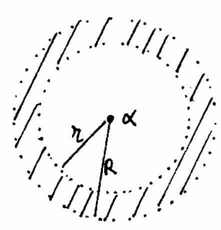
provided the series on the right-hand side of this equation converge.

Given $0 \le r < R$, we define the annulus centered at α with radii r and R by

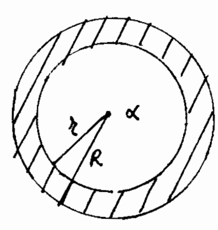
$$A(\alpha, r, R) = \{z : r < |z-\alpha| < R\}.$$

The closed annulus centered at α with radii r and R is denoted by

$$\bar{A}(\alpha, r, R) = \{z : r \le |z-\alpha| \le R\}$$



$A(\alpha, r, R)$



$\bar{A}(\alpha, r, R)$

__Theorem__. Suppose that the Laurent Series $\sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$ converges on the annulus $A(\alpha, r, R)$. Then the series converges uniformly on any closed subannulus $\bar{A}(\alpha, s, t)$ where $r < s < t < R$.

__Proof__. Exercise!

__Theorem__. (Laurent's theorem) Suppose that $0 \leq r < R$, and that f is analytic in the annulus $A = A(\alpha, r, R)$. If ρ is any number such that $r < \rho < R$, then for all $z_0 \in A$, f has the Laurent series representation

$$f(z_0) = \sum_{n=-\infty}^{\infty} c_n (z_0 - \alpha)^n \qquad (*)$$

where

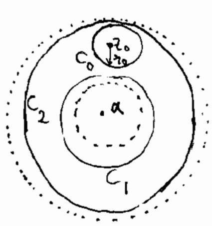
$$c_n = \frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz \quad , \; n \in \mathbb{Z}$$

and $C_\rho(\alpha) = \{z : |z - \alpha| = \rho\}$.

Moreover, the convergence in $(*)$ is uniform on any closed subannulus $\bar{A}(\alpha, s, t)$ with $r < s < t < R$.

__Proof__. Let $z_0 \in A$ be arbitrary. Choose r_0 so that $\{z : |z - z_0| \leq r\} \subset A$. Let $C_0 = \{z : |z - z_0| = r\}$. Then by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{C_0} \frac{f(z)}{z - z_0}\, dz .$$



Choose r_1 and r_2 so that C_0 lies in the region between $C_1 = \{z : |z - \alpha| = r_1$ and $C_2 = \{z : |z - \alpha| = r_2\}$.

Then by the Cauchy-Goursat (extended) theorem, we have

$$\frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz + \frac{1}{2\pi i}\int_{C_0}\frac{f(z)}{z-z_0}\,dz$$

Therefore,

$$f(z_0) = \frac{1}{2\pi i}\oint_{C_2}\frac{f(z)}{z-z_0}\,dz - \frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{z-z_0}\,dz.$$

If $z \in C_2$, then

$$\frac{1}{z-z_0} = \frac{1}{(z-\alpha)-(z_0-\alpha)} = \frac{1}{z-\alpha}\;\frac{1}{1-\frac{z_0-\alpha}{z-\alpha}} = \sum_{n=0}^{\infty}\frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}}$$

and the series converges uniformly on C_2,

Similarly,

If $z \in C_1$, then

$$\frac{1}{z-z_0} = -\sum_{n=0}^{\infty}\frac{(z-\alpha)^n}{(z_0-\alpha)^{n+1}}$$

and the series converges uniformly on C_1.

Therefore

$$f(z_0) = \frac{1}{2\pi i}\int_{C_2}\sum_{n=0}^{\infty}\frac{(z_0-\alpha)^n}{(z-\alpha)^{n+1}}f(z)\,dz + \frac{1}{2\pi i}\int_{C_1}\sum_{n=0}^{\infty}\frac{(z-\alpha)^n}{(z_0-\alpha)^{n+1}}f(z)\,dz$$

Because of uniform convergence,

$$f(z_0) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-\alpha)^{n+1}}\,dz\right)(z_0-\alpha)^n + \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_1}(z-\alpha)^n f(z)\,dz\right)\frac{1}{(z_0-\alpha)^{n+1}}$$

by reindexing, we get

$$f(z_0) = \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-\alpha)^{n+1}}\,dz\right)(z_0-\alpha)^n + \sum_{n=1}^{\infty}\left(\frac{1}{2\pi i}\int_{C}\frac{f(z)}{(z-\alpha)^{-n+1}}\,dz\right)(z_0-\alpha)^{-n}.$$

Applying the extended Cauchy-Goursat theorem again, we conclude that the integrals taken over C_2 and C_1 give the same result if they are taken over the contour $C_\rho(\alpha)$, where ρ is any number such that $r < \rho < R$. This yields

$$f(z_0) = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}} dz \right] (z_0-\alpha)^n$$

$$+ \sum_{n=1}^{\infty} \left[\frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{-n+1}} dz \right] (z_0-\alpha)^{-n}.$$

$$= \sum_{n=0}^{\infty} c_n (z_0-\alpha)^n + \sum_{n=1}^{\infty} c_{-n} (z_0-\alpha)^{-n} = \sum_{n=-\infty}^{\infty} c_n (z_0-\alpha)^n.$$

Because $z_0 \in A$ is arbitrary, the proof is done. ∎

<u>Remark</u> If f is analytic inside $D_R(\alpha)$, the Cauchy-Goursat theorem shows that coefficient for the negative power of $(z_0-\alpha)$ equal to zero and in this case, the Laurent series reduces to the Taylor series.

<u>Theorem</u> Suppose that f is analytic in the annulus $A(\alpha, r, R)$ and has the Laurent series $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$, for all $z \in A(\alpha, r, R)$.

i) If $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-\alpha)^n$ for all $z \in A(\alpha, r, R)$, then $b_n = c_n$ for all n

ii) For all $z \in A(\alpha, r, R)$, the derivatives of $f(z)$ may be obtained by termwise differentiation of its Laurent series.

<u>Proof</u> ii) Exercise!

(i) The series $\sum_{n=-\infty}^{\infty} b_n (z-\alpha)^n$ converges uniformly on $C_\rho^+(\alpha)$ for $0 \leq r < \rho < R$. Then

$$c_n = \frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \frac{f(z)}{(z-\alpha)^{n+1}}\, dz$$

$$= \frac{1}{2\pi i} \oint_{C_\rho(\alpha)} \left((z-\alpha)^{-n-1} \sum_{m=-\infty}^{\infty} b_m (z-\alpha)^m \right) dz$$

$$= \sum_{n=-\infty}^{\infty} \frac{b_m}{2\pi i} \left(\oint_{C_\rho(\alpha)} (z-\alpha)^{-n+m-1}\, dz \right)$$

Remind that $\oint_{C_\rho(\alpha)} (z-\alpha)^{-n+m-1}\, dz = \begin{cases} 0 & \text{if } -n+m-1 \neq -1 \quad (m \neq n) \\ 2\pi i & \text{if } -n+m-1 = -1 \quad (m=n) \end{cases}$

So

$c_n = b_n$. ∎

<u>Example</u>. Find the Laurent Series expansion for $f(z) = \frac{\cos z - 1}{z^4}$ that involves powers of z.

<u>Solution</u> $f(z)$ is analytic in the (infinite) annulus $A(0,0,\infty)$  . Remind that $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$

So $\cos z - 1 = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$ and so

$$\frac{\cos z - 1}{z^4} = -\frac{1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \cdots$$

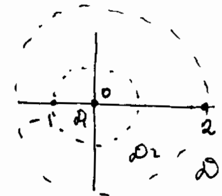
By the previous theorem this is the Laurent series expansion of $\frac{\cos z - 1}{z^4}$ in $|z|>0$.

Example. Find three different Laurent series representations for the function $f(z) = \dfrac{3}{2+z-z^2}$ involving powers of z.

Solution

$$f(z) = \frac{3}{(1+z)(2-z)} = \frac{A}{1+z} + \frac{B}{2-z} \Rightarrow A(2-z) + B(1+z) = 3 \Rightarrow \begin{matrix} B = 1 \\ A = 1 \end{matrix}$$

$$\Rightarrow \quad f(z) = \frac{1}{1+z} + \frac{1}{2-z}$$



clearly,
$$\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-1)^n z^n, \quad |z|<1$$

$$\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z\left(1-\left(-\frac{1}{z}\right)\right)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}, \quad |z|>1$$

$$\frac{1}{2-z} = \frac{1}{2(1-\frac{z}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{2^n} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad |z|<2$$

$$\frac{1}{2-z} = \frac{1}{-z(1-\frac{2}{z})} = -\frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} = -\sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}}, \quad |z|>2$$

Thus,
$$f(z) = \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2^{n+1}} \right) z^n \quad \text{in} \quad |z|<1$$

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} = \sum_{n=-\infty}^{-1} (-1)^{n+1} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}, \quad 1<|z|<2$$

and
$$f(z) = \sum_{n=0}^{\infty} \left((-1)^n - 2^n \right) \frac{1}{z^{n+1}} = \sum_{n=-\infty}^{-1} \left((-1)^{n+1} - 2^{-n-1} \right) z^n, \quad |z|>2.$$

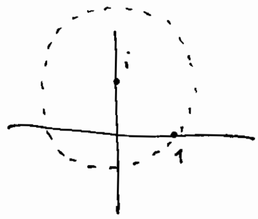
Example. Find the Laurent series expansion for $e^{-\frac{1}{z^2}}$ centered at $\alpha = 0$.

Solution $\quad e^z = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{n!}$ for all $z \in \mathbb{C}$. So

$$e^{-\frac{1}{z^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{2n}} \quad \text{for all } z \in \mathbb{C}\setminus\{0\}, \text{ (that is for } |z|>0)$$

Example. Find the Laurent series expansion for $f(z) = \frac{1}{1-z}$, which is valid in $|z-i| > \sqrt{2}$.

Solution.



$$f(z) = \frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = -\frac{1}{z-i}\,\frac{1}{1-\frac{1-i}{z-i}}. \quad \text{let } w = \frac{1-i}{z-i}.$$

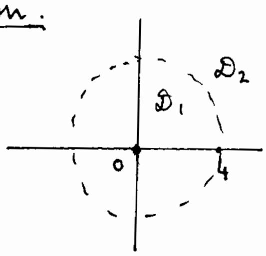
if $|z-i| > \sqrt{2}$ then $|w| = \frac{\sqrt{2}}{|z-i|} < 1$ and

$$f(z) = -\frac{1}{z-i}\sum_{n=0}^{\infty}\frac{(1-i)^n}{(z-i)^n} = -\sum_{n=0}^{\infty}\frac{(1-i)^n}{(z-i)^{n+1}} = -\sum_{n=1}^{\infty}\frac{(1-i)^{n-1}}{(z-i)^n} = -\sum_{n=-\infty}^{-1}(1-i)^{-n-1}(z-i)^n,$$

$|z-i| > \sqrt{2}$.

Example. Find two Laurent series for $z^{-1}(4-z)^{-2}$ involving powers of z and state where they are valid.

Solution.



$$\frac{1}{z(4-z)^2} = \frac{A}{z} + \frac{B}{4-z} + \frac{C}{(4-z)^2} \Rightarrow$$

$$A(4-z)^2 + Bz(4-z) + Cz = 1$$

$$\Rightarrow C = \frac{1}{4}, \quad A = \frac{1}{16}, \quad B = \frac{1}{6}$$

$$f(z) = \frac{1}{16}\frac{1}{z} + \frac{1}{16}\frac{1}{4-z} + \frac{1}{4}\frac{1}{(4-z)^2}$$

$\frac{1}{z}$ is already a power of z (and valid for $|z| > 0$)

$$\frac{1}{4-z} = \frac{1}{4}\frac{1}{1-\frac{z}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{4^n} = \sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}} \quad \text{for } |z| < 4$$

$$\frac{1}{4-z} = -\frac{1}{z}\frac{1}{1-\frac{4}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{4^n}{z^n} = -\sum_{n=0}^{\infty}\frac{4^n}{z^{n+1}} = -\sum_{n=1}^{\infty}\frac{4^{n-1}}{z^n} \quad \text{for } |z| > 4$$

$$\frac{1}{(4-z)^2} = \frac{1}{16}\frac{1}{(1-\frac{z}{4})^2} = \frac{1}{16}\sum_{n=0}^{\infty}(n+1)\frac{z^n}{4^n} = \sum_{n=0}^{\infty}(n+1)\frac{z^n}{4^{n+2}} \quad \text{for } |z| < 4$$

$$\frac{1}{(4-z)^2} = \frac{1}{z^2}\frac{1}{(1-\frac{4}{z})^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}(n+1)\frac{4^n}{z^n} = \sum_{n=0}^{\infty}(n+1)\frac{4^n}{z^{n+2}} = \sum_{n=2}^{\infty}(n-1)\frac{4^{n-2}}{z^n} \quad \text{for } |z| > 4$$

$$\Rightarrow f(z) = \frac{1}{16z} + \frac{1}{16}\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}} + \frac{1}{4}\sum_{n=0}^{\infty}(n+1)\frac{z^n}{4^{n+2}} \quad \text{for } 0 < |z| < 4, \text{ and}$$

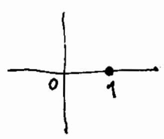
$$f(z) = \frac{1}{16}\frac{1}{z} - \frac{1}{16}\sum_{n=1}^{\infty}\frac{4^{n-1}}{z^n} + \frac{1}{4}\sum_{n=2}^{\infty}(n-1)\frac{4^{n-2}}{z^n} \quad \text{for } 4 < |z|.$$

Singularities, zeros, and poles

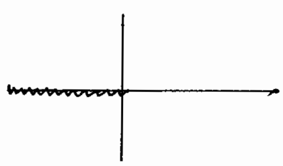
The point α is called a <u>singular point</u>, or <u>singularity</u> of f if f is not analytic at the point α, but every neighborhood $D_R(\alpha)$ of α contains at least one point at which f is analytic.

The point α is called an <u>isolated singularity</u> of f if f is not analytic at α but there exists a real number $R>0$ such that f is analytic everywhere in the punctured disk $\{z : 0<|z-\alpha|<R\} = A(\alpha, 0, R)$

<u>Example</u> $f(z) = \frac{1}{1-z}$ has an isolated singularity at $z=1$.



let $g(z) = \text{Log}\, z$. Clearly, the origin and each point on the negative real axis are singularities of g, and none of them are isolated.



<u>Classification of (isolated) singularities.</u>

If f has an isolated singularity at α, then it has a Laurent series in $A(\alpha, 0, R)$ for some α, say $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-\alpha)^n$

(i) If $c_n = 0$ for $n = -1, -2, -3, \dots$, then f has a <u>removable singularity</u> at α. (i.e. $f(z) = c_0 + c_1 z + c_2 z^2 + \dots$)

(ii) If k is a positive integer such that $c_{-k} \neq 0$, but $c_n = 0$ for $n < -k$, then f has a <u>pole of order k</u> at α.
(i.e. $f(z) = \frac{c_{-k}}{z^k} + \frac{c_{-k+1}}{z^{k-1}} + \dots + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \dots$)

(iii) If $c_n \neq 0$ for infinitely many negative integers n, then f has an <u>essential singularity</u> at α.

<u>Example</u> classify the isolated singularities of

1. $\dfrac{\sin z}{z}$

2. $\dfrac{\cos z - 1}{z^2}$.

3. $\dfrac{\sin z}{z^3}$

4. $\dfrac{e^z}{z}$

5. $z^2 \sin \dfrac{1}{z}$.

<u>Solution.</u> Remind that $\sin z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$

$\cos z = \sum_{n=0}^{\infty} \dfrac{(-1)^n z^{2n}}{(2n)!} = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \cdots$

and $e^z = \sum_{n=0}^{\infty} \dfrac{z^n}{n!} = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!} + \cdots$.

Then

1. $\dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + \cdots + (-1)^n \dfrac{z^{2n}}{(2n+1)!} + \cdots \Rightarrow \dfrac{\sin z}{z}$ has a removable singularity at 0.

2. $\dfrac{\cos z - 1}{z^2} = -\dfrac{1}{2!} + \dfrac{z^2}{4!} - \dfrac{z^4}{6!} + \cdots + (-1)^n \dfrac{z^{2(n-2)}}{(2n)!} + \cdots \Rightarrow \dfrac{\cos z - 1}{z^2}$ has a removable singularity at 0

3. $\dfrac{\sin z}{z^3} = \dfrac{1}{z^2} - \dfrac{1}{3!} + \dfrac{z^2}{5!} + \cdots \Rightarrow \dfrac{\sin z}{z^3}$ has a pole of order 2 (a double pole) at 0

4. $\dfrac{e^z}{z} = \dfrac{1}{z} + 1 + \dfrac{z}{2!} + \cdots \Rightarrow \dfrac{e^z}{z}$ has a pole of order 1 (a simple pole) at 0

5. $z^2 \sin \dfrac{1}{z} = z - \dfrac{1}{3!}\dfrac{1}{z} + \dfrac{1}{5!}\dfrac{1}{z^3} - \dfrac{1}{7!}\dfrac{1}{z^5} + \cdots \Rightarrow z^2 \sin \dfrac{1}{z}$ has an essential singularity at 0.

A function f analytic in $\mathcal{D}_R(\alpha) = \{z : |z-\alpha| < R\}$ has a <u>zero of order k</u> at the point α if and only if

$$f^{(n)}(\alpha) = 0, \quad \text{for} \quad n = 0, 1, \dots, k-1 \quad \text{but} \quad f^{(k)}(\alpha) \neq 0.$$

<u>Example.</u> Show that $f(z) = z \sin z^2$ has a zero of order 3 at $z=0$.

<u>Solution.</u> $f(0) = 0$, $f'(z) = \sin z^2 + z \cos z^2 \cdot 2z \Rightarrow f'(0) = 0$,

$f''(z) = \cos z^2 \cdot 2z + 4z \cos z^2 + 2z^2 (-\sin z^2) 2z \Rightarrow f''(0) = 0$

$$f'''(z) = 6\cos z^2 + 6z(-\sin z^2)2z - 12z^2 \sin z^2 - 4z^3 \cos z^2 (2z)$$
$$= 6\cos z^2 - 24 z^2 \sin z^2 - 8z^4 \cos z^2 \Rightarrow f'''(0) = 6 \neq 0$$

$\Rightarrow f$ has a zero of order 3 at $z=0$.

<u>Theorem</u> A function f analytic in $\mathcal{D}_R(\alpha)$ has a zero of order k at the point α if and only if its Taylor series given by $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ has $c_0 = c_1 = \dots = c_{k-1} = 0$, but $c_k \neq 0$

<u>Proof.</u> Exercise!

<u>Example.</u> $f(z) = z \sin z^2 = z\left(z^2 - \frac{(z^2)^3}{3!} + \frac{(z^2)^5}{5!} - \dots\right) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} + \dots$

$\Rightarrow c_0 = c_1 = c_2 = 0$ in Taylor series expansion of $f(z) \Rightarrow$ $f(z)$ has a zero of order 3 at 0.

<u>Theorem</u> Suppose that the function f is analytic in $\mathcal{D}_R(\alpha)$. Then f has a zero of order k at the point α if and only if f can be expressed in the form

$$f(z) = (z-\alpha)^k g(z),$$

where g is analytic at the point α and $g(\alpha) \neq 0$.

Proof. Suppose f has a zero of order k at the point α, then

$$f(z) = \sum_{n=k}^{\infty} c_n (z-\alpha)^n = (z-\alpha)^k \sum_{n=0}^{\infty} c_{n+k} (z-\alpha)^n \quad \text{where} \quad c_k \neq 0.$$

Let $g(z) = \sum_{n=0}^{\infty} c_{n+k}(z-\alpha)^n = c_k + \sum_{n=1}^{\infty} c_{n+k}(z-\alpha)^n$.

Clearly g is analytic in $\mathcal{D}_R(\alpha)$ and $g(\alpha) = c_k \neq 0$.

Conversely, suppose that $f(z) = (z-\alpha)^k g(z)$ where g is analytic at α, and $g(\alpha) \neq 0$. Then $g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n$

so $g(\alpha) = b_0 \neq 0$. Then $f(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^{n+k} = \sum_{n=k} b_{n-k} (z-\alpha)^n$

$\Rightarrow$ $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$ where $c_0 = c_1 = \cdots = c_{k-1} = 0$ and $c_n = b_{n-k}$ for $n \geq k$ since $c_k = b_0 \neq 0$, f has a zero of order k at α. ∎

Corollary. If f and g are analytic at $z = \alpha$ and have zeros of orders m and n, respectively, at $z = \alpha$, then their product $h(z) = f(z)g(z)$ has a zero of order $m+n$ at $z = \alpha$.

Example. $f(z) = z^3 \sin z$. z^3 has a zero of order 3 at $z=0$ and $\sin z$ has a simple zero at $z=0 \Rightarrow f$ has a zero of order 4 at $z=0$.

Theorem. A function f analytic in the punctured disk $\mathcal{D}_R^*(\alpha) = A(\alpha, 0, R) = \{z : 0 < |z-\alpha| < R\}$ has a pole of order k at the point α if and only if it can be expressed in the form

$$f(z) = \frac{h(z)}{(z-\alpha)^k},$$

where the function h is analytic at the point α, and $h(\alpha) \neq 0$.

Proof. Exercise!

Corollary. If f is analytic and has a zero of order k at the point α, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at α.

Corollary. If f has a pole of order k at the point α, then $g(z) = \frac{1}{f(z)}$ has a removable singularity at α. If we define $g(\alpha) = 0$, then $g(z)$ has a zero of order k at α.

Corollary. If f and g have poles of orders m and n, respectively, at the point α, then their product $h(z) = f(z)g(z)$ has a pole of order $m+n$ at α.

Corollary. Let f and g be analytic with zeros of orders m and n, respectively, at α. Then their quotient $h(z) = \frac{f(z)}{g(z)}$ has the following behavior.

i) If $m > n$, then h has a removable singularity at α. If we define $h(\alpha) = 0$, then h has a zero of order $m-n$ at α

ii) If $m < n$, then h has a pole of order $n-m$ at α.

iii) If $m = n$, then h has a removable singularity at α and can be defined so that h is analytic at α by $h(\alpha) = \lim_{z \to \alpha} h(z)$.

Example. Locate the zeros and poles of $f(z)$ and determine their order, if

1. $f(z) = \frac{\tan z}{z}$
2. $f(z) = \frac{1}{5z^4 + 26z^2 + 5}$
3. $f(z) = \frac{\pi \cot(\pi z)}{z^2}$.

Solution 1. $\frac{\tan z}{z} = \frac{\sin z}{z \cos z}$. $\sin z$ has simple zeros at $n\pi$, $n \in \mathbb{Z}$, z has simple zero at 0, $\cos z$ has simple zeros at $\left(\frac{2n+1}{2}\right)\pi$. So $\frac{\tan z}{z}$ has simple zeros at $n\pi$, $n \in \mathbb{Z} \setminus \{0\}$ (At $z=0$, the singularity is removable) and simple poles at $\frac{2n+1}{2}\pi$, $n \in \mathbb{Z}$.

2. $f(z) = \frac{1}{5z^4 + 26z^2 + 5} = \frac{1}{(5z^2+1)(z^2+5)} = \frac{1}{5(z - \frac{i}{\sqrt{5}})(z + \frac{i}{\sqrt{5}})(z - i\sqrt{5})(z + \frac{i}{\sqrt{5}})} \Rightarrow$

f has simple poles at $z = \pm \frac{i}{\sqrt{5}}$ and $z = \pm i\sqrt{5}$.

3. $f(z) = \frac{\pi \cot(\pi z)}{z^2} = \frac{\pi \cos(\pi z)}{z^2 \sin(\pi z)}$.

$\cos \pi z$ has simple zeros at $\frac{2n+1}{2}$, $n \in \mathbb{Z}$, and

$\sin \pi z$ has simple zeros at n, $n \in \mathbb{Z}$, and z^2 has a double pole at 0, so f has a pole of order 3 (triple pole) at 0, simple poles at n, $n \in \mathbb{Z} \setminus \{0\}$ and simple zeros at $\frac{2n+1}{2}$, $n \in \mathbb{Z}$.

Applications of Taylor and Laurent Series

Theorem. Suppose that f is analytic in a domain $\mathcal{D}$ containing α and that $f(\alpha)=0$. If f is not identically zero in $\mathcal{D}$, then there exists a punctured disk $\mathcal{D}_R^*(\alpha)$ in which f has no zeros.

(That is, zeros of analytic functions are isolated.)

Proof. By Taylor's theorem, there exists some disk $\mathcal{D}_R(\alpha)$ about α such that

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}(z-\alpha)^n \quad \text{for all } z\in\mathcal{D}_R(\alpha).$$

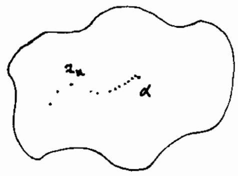
If all the Taylor coefficients $\frac{f^{(n)}(\alpha)}{n!}$ of f were zero, then f would be identically zero on $\mathcal{D}_R(\alpha)$.

Let z be an arbitrary point in $\mathcal{D}$ find points $a_1, a_2, \dots, a_n \in \mathcal{D}$ and $r_1, r_2, \dots, r_n \in \mathbb{R}^+$ such that $a_1\in\mathcal{D}_R(\alpha)$, $a_2\in D_{r_1}(a_1), \dots, a_n\in\mathcal{D}_{r_{n-1}}(a_{n-1})$, $z\in \mathcal{D}_{r_n}(a_n)$ and $D_{r_j}(a_j)\subset\mathcal{D}$ for all j. Thus, f has Taylor series expansion in all of $D_{r_j}(a_j)$ with centers a_j.

If f is identically zero in $\mathcal{D}_R(\alpha)$, then all Taylor coefficients found at a_1 will be zero and so f will be identically zero in $\mathcal{D}_{r_1}(a_1)$. Using this argument repeatedly, we arrive that f is identically zero in $\mathcal{D}_{r_n}(a_n)$ and so $f(z)=0$. Since z is arbitrary, this means that f is identically zero in $\mathcal{D}$, a contradiction. Therefore $\frac{f^{(n)}(\alpha)}{n!}$ are nonzero for some n. Let k be the smallest integer such that $\frac{f^{(k)}(\alpha)}{k!}\neq 0$. Then f has a zero of order k at α and can be written in the form $f(z)=(z-\alpha)^k g(z)$,

where g is analytic at α and $g(\alpha)\neq 0$. Since g is a continuous function, there exists a disk $D_r(\alpha)$ throughout which g is nonzero. Therefore $f(z) \neq 0$ in the punctured disk $D_r^*(\alpha)$. ∎

<u>Corollary</u>. Suppose that f is analytic in the domain $\mathcal{D}$ and that $\alpha \in \mathcal{D}$. If there exists a sequence of points $\{z_n\}$ in $\mathcal{D}$ such that $z_n \to \alpha$, and $f(z_n)=0$, then $f(z)=0$ for all $z \in \mathcal{D}$.



<u>Proof</u>. since f is analytic, it is continuous and so convergence preserving. Then $f(\alpha) = \lim_{n\to\infty} f(z_n) = 0$. But this shows that zeros of f are not isolated. Hence it should be identically zero. ∎

<u>Corollary</u>. Suppose that f and g are analytic in the domain $\mathcal{D}$, where $\alpha \in \mathcal{D}$. If there exists a sequence $\{z_n\}$ in $\mathcal{D}$ such that $z_n \to \alpha$, and $f(z_n) = g(z_n)$ for all n, then $f(z) = g(z)$ for all $z \in \mathcal{D}$.

<u>Proof</u>. Apply the previous corollary for $f(z) - g(z)$. ∎

<u>Corollary</u> (L'Hôpital's rule) Suppose that f and g are analytic at α. If $f(\alpha)=0$ and $g(\alpha)=0$, but $g'(\alpha) \neq 0$, then

$$\lim_{z\to\alpha} \frac{f(z)}{g(z)} = \frac{f'(\alpha)}{g'(\alpha)}.$$

Proof. Because $g'(\alpha) \neq 0$, g is not identically zero, and by the previous theorem there is a punctured disk $\mathcal{D}_r^*(\alpha)$ in which $g(z) \neq 0$. Thus the quotient $\dfrac{f(z)}{g(z)} = \dfrac{f(z) - f(\alpha)}{g(z) - g(\alpha)}$

is defined for all $z \in \mathcal{D}_r^*(\alpha)$, and we can write

$$\lim_{z\to\alpha} \frac{f(z)}{g(z)} = \lim_{z\to\alpha} \frac{f(z)-f(\alpha)}{g(z)-g(\alpha)} = \lim_{z\to\alpha} \frac{\frac{f(z)-f(\alpha)}{z-\alpha}}{\frac{g(z)-g(\alpha)}{z-\alpha}} = \frac{f'(\alpha)}{g'(\alpha)}. \blacksquare$$

<u>Theorem</u> (Division of power series) Suppose that f and g are analytic at α with the power series representations $f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ and $g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n$ for all $z \in \mathcal{D}_R(\alpha)$. If $g(\alpha) \neq 0$, then the quotient $\frac{f}{g}$ has the power series representation $\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$, where the coefficients satisfy the equation

$$a_n = b_0 c_n + \cdots + b_{n-1} c_1 + b_n c_0 .$$

In other words, we can obtain the series for the quotient $\frac{f(z)}{g(z)}$ by the familiar process of dividing the series for $f(z)$ by the series for $g(z)$, using the standard long division algorithm.

<u>Example.</u> Find the first few terms of the Maclaurin series for the function $f(z) = \sec z$ if $|z| < \frac{\pi}{2}$, and compute $f^{(4)}(0)$

<u>Solution</u> $\sec z = \frac{1}{\cos z}$, then we apply the previous theorem with $f(z) = 1$ and $g(z) = \cos z$. Clearly $a_0 = 1$, $a_1 = a_2 = \cdots = 0$ and $b_0 = 1$, $b_1 = 0$, $b_2 = -\frac{1}{2}$, $b_3 = 0, \ldots$, $b_{2n} = \frac{(-1)^n}{(2n)!}$, $b_{2n+1} = 0, \ldots$

$a_0 = b_0 c_0 \Rightarrow c_0 = 1$; $a_1 = b_0 c_1 + b_1 c_0 \Rightarrow 0 = c_1$; $a_2 = b_0 c_2 + b_1 c_1 + b_2 c_0 \Rightarrow$ $0 = c_2 - \frac{1}{2} \Rightarrow c_2 = \frac{1}{2}$; $a_3 = b_0 c_3 + \underset{0}{b_1} c_2 + b_2 \underset{0}{c_1} + \underset{0}{b_3} c_0 \Rightarrow 0 = c_3$;

$a_4 = b_0 c_4 + \underset{0}{b_1} c_3 + b_2 c_2 + b_3 \underset{0}{c_1} + b_4 c_0 \Rightarrow 0 = c_4 - \frac{1}{4} + \frac{1}{24} \Rightarrow c_4 = \frac{5}{24}$, then

$\sec z = 1 + \frac{1}{2} z^2 + \frac{5}{24} z^4 + \cdots$. $\frac{5}{24} = \frac{f^{(4)}(0)}{4!} \Rightarrow f^{(4)}(0) = 5.$ $\blacksquare$

Now, we will consider behaviors of complex functions at points near the different types of isolated singularities.

Theorem (Riemann) Suppose that f is analytic in $\mathcal{D}_r^*(\alpha)$. If f is bounded in $\mathcal{D}_r^*(\alpha)$, then either f is analytic at α or f has a removable singularity at α.

Proof. Let g is defined by

$$g(z) = \begin{cases} (z-\alpha)^2 f(z) & \text{when } z \neq \alpha, \\ 0 & \text{when } z = \alpha. \end{cases}$$

Clearly, g is analytic in $\mathcal{D}_r^*(\alpha)$. On the other hand

$$g'(\alpha) = \lim_{z\to\alpha} \frac{g(z)-g(\alpha)}{z-\alpha} = \lim_{z\to\alpha} (z-\alpha) f(z) = 0$$ because f is bounded. Thus g is analytic at α and so analytic in $\mathcal{D}_r(\alpha)$. Since $g(\alpha) = g'(\alpha) = 0$, f has the following Taylor series expansion in $\mathcal{D}_r(\alpha)$

$$g(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(\alpha)}{n!} (z-\alpha)^n, \quad \text{for all } z \in \mathcal{D}_r(\alpha).$$

Therefore,

$$f(z) = \frac{1}{(z-\alpha)^2} g(z) = \sum_{n=2}^{\infty} \frac{g^{(n)}(\alpha)}{n!} (z-\alpha)^{n-2} = \sum_{n=0}^{\infty} \frac{g^{(n+2)}(\alpha)}{(n+2)!} (z-\alpha)^n$$

and so f is analytic or has a removable singularity at $z = \alpha$. ∎

Corollary If f is analytic in $\mathcal{D}_r^*(\alpha)$, then f can be defined to be analytic at α if and only if $\lim_{z\to\alpha} f(z)$ exists and is finite.

Proof. Exercise!

Theorem Suppose that f is analytic in $D_r^*(\alpha)$. The function f has a pole of order k at α if and only if $\lim_{z\to\alpha} |f(z)| = \infty$.

Proof. Suppose, first, that f has a pole of order k at α. Then $f(z) = \dfrac{h(z)}{(z-\alpha)^k}$, where h is analytic at α, and $h(\alpha)\neq 0$.

Because $\lim_{z\to\alpha} |h(z)| = |h(\alpha)| \neq 0$ and $\lim_{z\to\alpha} |(z-\alpha)| = 0$, we conclude

that $\lim_{z\to\alpha} |f(z)| = \infty$.

Conversely, suppose that $\lim_{z\to\alpha} |f(z)| = \infty$. Then there exists $r>0$ such that $|f(z)|>1$ in $D_r(\alpha)$. Then $g(z) = \dfrac{1}{f(z)}$ is

analytic and bounded in $D_r^*(\alpha)$. By the previous theorem we may define g at α so that g is analytic in $D_r(\alpha)$. Clearly $|g(\alpha)| = \lim_{z\to\alpha} \dfrac{1}{|f(z)|} = 0 \Rightarrow g(\alpha)=0$. That is, α is a zero of g. If we show that g has a zero of finite order k at α, this implies that f has a pole of order k at α, and the proof is done. Now suppose g has a zero of infinite order at α then $g^{(n)}(\alpha)=0$, for all n, and hence $g(z) = \sum_{n=0}^{\infty} \dfrac{g^{(n)}(\alpha)}{n!}(z-\alpha)^n = 0$ for all $z \in D_r(\alpha)$. But this is impossible, because $f(z) = \dfrac{1}{g(z)}$ is analytic in $D_r^*(\alpha)$. ∎

Theorem The function f has an essential singularity at α if and only if $\lim_{z\to\alpha} |f(z)|$ does not exist.

Proof. By the previous corollary and the theorem, this is the only remaining possibility. ■

Example. Show that g defined by

$$g(z) = \begin{cases} e^{-\frac{1}{z^2}} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$$

is not continuous at 0.

Solution. $e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots \Rightarrow$

$g(z) = 1 - \frac{1}{z^2} + \frac{1}{2z^4} + \cdots \Rightarrow g$ has an essential

singularity at $z=0$ and by the previous theorem, $\lim_{z \to 0} g(z)$ does not exist and hence can't be continuous at $z=0$.

Residue Theory

The Residue Theorem

Let f have a nonremovable isolated singularity at the point z_0. Then f has the Laurent series representation for all z in some punctured disk $D_r^*(z_0)$ given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n .$$

The coefficient a_{-1} is called <u>the residue of f at z_0</u>.

We use the notation $\operatorname*{Res}_{z=z_0} f(z) = a_{-1}$.

<u>Example.</u> Find $\operatorname*{Res}_{z=0} f(z)$ if

1. $f(z) = e^{\frac{2}{z}}$
2. $f(z) = \dfrac{3}{2z+z^2-z^3}$.

<u>Solution.</u> 1. $e^{\frac{2}{z}} = 1 + \frac{2}{z} + \frac{1}{2}\frac{4}{z^2} + \frac{1}{6}\cdot\frac{8}{z^3} + \dots + \frac{1}{n!}\cdot\frac{2^n}{z^n} + \dots$.

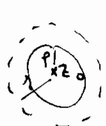
$$\Rightarrow \quad \operatorname*{Res}_{z=0} e^{\frac{2}{z}} = 2 .$$

2. $$\frac{3}{2z+z^2-z^3} = \frac{3}{z(1+z)(2-z)} = \frac{A}{z} + \frac{B}{1+z} + \frac{C}{2-z} \Rightarrow A = \frac{3}{2},\ B=-1,\ C=\frac{1}{2}$$

and $$f(z) = \frac{3}{2}\cdot\frac{1}{z} - \frac{1}{1+z} + \frac{1}{4}\,\frac{1}{1-\frac{z}{2}} = \frac{3}{2}\,\frac{1}{z} - \sum_{n=0}^{\infty}(-1)^n z^n + \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^n}{2^n}$$

$$\Rightarrow \quad \operatorname*{Res}_{z=0} \frac{3}{2z+z^2-z^3} = \frac{3}{2} .$$

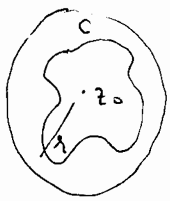
<u>Remark.</u> Remind that if f is analytic in $D_r^*(z_0)$ the Laurent series coefficients a_n are computed by



$$a_n = \frac{1}{2\pi i}\oint_{C_\rho(z_0)} \frac{f(z)}{(z-z_0)^{n+1}}\,dz \quad \text{for some } 0<\rho<r .$$

Then $$\operatorname*{Res}_{z=z_0} f(z) = a_{-1} = \frac{1}{2\pi i}\oint_{C_\rho(z_0)} f(z)\,dz .$$

instead of $C_\rho(z_0)$, we can take any simple closed contour lying in $\mathcal{D}_1^*(z_0)$ and containing z_0 as an interior point. (Use deformation of contour theorem)



Therefore, if z_0 is the only singularity of f that lies inside C, then

$$\oint_C f(z)dz = 2\pi i \operatorname*{Res}_{z=z_0} f(z)$$

<u>Example.</u> Evaluate $\displaystyle\oint_{C_1(0)} e^{\frac{2}{z}} dz$

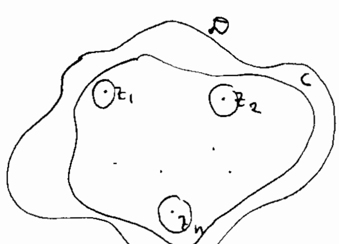
<u>Solution</u>

$$\oint_{C_1(0)} e^{\frac{2}{z}} dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i \cdot 2 = 4\pi i .$$

<u>Theorem</u> (Cauchy's Residue theorem) Let $\mathcal{D}$ be a simply connected domain and let C be a simple closed positively oriented contour that lies in $\mathcal{D}$. If f is analytic inside and on C except at the points $z_1, z_2, \ldots, z_n$ that lie inside C, then

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) .$$

<u>Proof.</u> Use the extended Cauchy-Goursat theorem and the ideas discussed in the previous remark. ∎



<u>Question.</u> Is there any other method than finding Laurent series expansion to calculate the residues.

<u>Answer.</u> Yes

Theorem (Residues at poles) If f has a pole of order k at z_0, then $\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(k-1)!} \lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}} (z-z_0)^k f(z)$.

Proof Suppose f has a pole of order k at z_0. Then f can be written in the form

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

then $$f(z)(z-z_0)^k = a_{-k} + a_{-k+1}(z-z_0) + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots$$

If we differentiate $k-1$ times, we get

$$\frac{d^{k-1}}{dz^{k-1}}\left(f(z)(z-z_0)^k\right) = (k-1)!\, a_{-1} + k!\, a_0(z-z_0) + ((k+1)k\cdots 3)a_1(z-z_0)^2 + \cdots$$

and when we let $z \to z_0$, we get

$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left(f(z)(z-z_0)^k\right) = (k-1)!\, a_{-1}. \blacksquare$$

Example. Find $\operatorname*{Res}_{z=0} \frac{\pi\cot(\pi z)}{z^2}$.

solution $\frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\cos(\pi z)}{z^2\sin(\pi z)}$ has a pole of order 3 at 0.

Then by the previous theorem,

$$\operatorname*{Res}_{z=0} \frac{\pi\cot(\pi z)}{z^2} = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\frac{\pi\cot(\pi z)}{z^2}\cdot z^3\right)$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left(\pi z\cot(\pi z)\right)$$

$$= \frac{1}{2}\lim_{z\to 0}\left(\frac{d}{dz}\left(\pi\cot(\pi z) - \pi^2 z\csc^2(\pi z)\right)\right)$$

$$= \frac{1}{2}\lim_{z\to 0}\left(-\pi^2\csc^2(\pi z) - \pi^2\csc^2(\pi z) + 2\pi^3 z\csc^2(\pi z)\cot(\pi z)\right)$$

$$= \lim_{z\to 0}\left(\pi^2\csc^2(\pi z)\left[\pi z\cot(\pi z) - 1\right]\right)$$

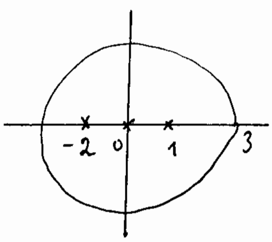
$$= \lim_{z\to 0} \pi^2 \frac{\pi z\cos(\pi z) - \sin(\pi z)}{\sin^3(\pi z)}$$

$$\overset{L.R}{=} \lim_{z\to 0} \pi^2 \frac{\pi\cos(\pi z) - \pi^2 z\sin(\pi z) - \pi\cos(\pi z)}{3\sin^2(\pi z)\cos(\pi z)\pi}$$

$$= \lim_{z\to 0} \pi^2 \frac{-z\pi}{\underbrace{3\sin(\pi z)}_{\to 1}\underbrace{\cos(\pi z)}_{\to 1}} = -\frac{\pi^2}{3}.$$

<u>Example.</u> Evaluate $\displaystyle\oint_{C_3(0)} \frac{1}{z^4+z^3-2z^2}\,dz$.

<u>Solution.</u> $z^4+z^3-2z^2 = z^2(z+2)(z-1) \Rightarrow \dfrac{1}{z^4+z^3-2z^2}$ has a pole of order 2 at 0, and has simple poles at -2 and 1.



Then

$$\oint_{C_3(0)} \frac{dz}{z^4+z^3-2z^2} = 2\pi i\left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=-2} f(z) + \operatorname*{Res}_{z=1} f(z)\right)$$

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{(z+2)(z-1)}\right) = \lim_{z\to 0}\left(\frac{1}{z^2+z-2}\right)'$$

$$= \lim_{z\to 0} \frac{-(2z+1)}{(z^2+z-2)^2} = -\frac{1}{4}.$$

$$\operatorname*{Res}_{z=-2} f(z) = \lim_{z\to -2} \frac{1}{z^2(z-1)} = -\frac{1}{12}, \text{ and}$$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1}\frac{1}{z^2(z+2)} = \frac{1}{3}. \text{ Thus}$$

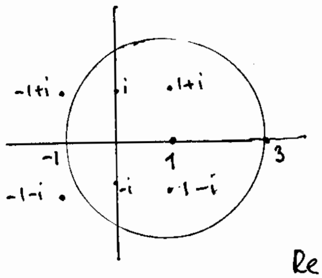
$$\oint_{C_3(0)} \frac{dz}{z^4+z^3-2z^2} = 2\pi i\left(-\frac{1}{4} - \frac{1}{12} + \frac{1}{3}\right) = 0.$$

<u>Example</u> Evaluate $\displaystyle\oint_{C_2(1)} (z^4+4)^{-1}\,dz$. A: $-\frac{\pi i}{4}$.

Solution Solving the equation $z^4 = -4$ (that is, finding the all 4^{th} roots of -4) we can easily show that

$$z^4 + 4 = (z-(1+i))(z-(1-i))(z-(-1+i))(z-(-1-i))$$

Thus, $\dfrac{1}{z^4+4}$ has simple poles at $z = 1 \pm i, -1 \pm i$, and



$$\oint_{C_2(1)} \frac{1}{z^4+4}\, dz = 2\pi i \left(\operatorname*{Res}_{z=1+i} \frac{1}{z^4+4} + \operatorname*{Res}_{z=1-i} \frac{1}{z^4+4} \right)$$

1^{st} way to calculate the residues:

$$\operatorname*{Res}_{z=1+i} \frac{1}{z^4+4} = \lim_{z \to 1+i} \frac{1}{(z-(1-i))(z-(-1+i))(z-(-1-i))} = \frac{1}{2i} \cdot \frac{1}{2} \cdot \frac{1}{2+2i} \cdot \frac{2-2i}{2-2i}$$

$$= \frac{2-2i}{32i} = \frac{-2-2i}{32}$$

$$= \frac{-1-i}{16}$$

$$\operatorname*{Res}_{z=1-i} \frac{1}{z^4+4} = \lim_{z \to 1-i} \frac{1}{(z-(1+i))(z-(-1+i))(z-(-1-i))} = \frac{1}{(-2i)} \cdot \frac{1}{(2-2i)} \cdot \frac{1}{2} \cdot \frac{2+2i}{2+2i} = \frac{2+2i}{-32i} \cdot \frac{i}{i} = \frac{-2+2i}{32}$$

$$= \frac{-1+i}{16}$$

$$\Rightarrow \quad \oint_{C_2(1)} \frac{dz}{z^4+4} = 2\pi i \left(\frac{-1-i}{16} + \frac{-1+i}{16} \right) = -\frac{\pi i}{4}.$$

2^{nd} way to calculate the residues:

$$\operatorname*{Res}_{z=1+i} \frac{1}{z^4+4} = \lim_{z \to 1+i} \frac{z-(1+i)}{z^4+4} \overset{L.R}{=} \lim_{z \to 1+i} \frac{1}{4z^3}$$

$$= \lim_{z \to 1+i} \frac{z}{4z^4} = \frac{1+i}{-16} = \frac{-1-i}{16}.$$

(Note that $(1+i)^4 + 4 = 0 \Rightarrow (1+i)^4 = -4$)

$$\operatorname*{Res}_{z=1-i} \frac{1}{z^4+4} = \lim_{z \to 1-i} \frac{z-(1-i)}{z^4+4} \overset{L.R}{=} \lim_{z \to 1-i} \frac{1}{4z^3}$$

$$= \lim_{z \to 1-i} \frac{z}{4z^4} = \frac{1-i}{-16} = \frac{-1+i}{16}.$$

and $\displaystyle \oint_{C_2(1)} \frac{1}{z^4+4}\, dz = -\frac{\pi i}{4}.$

Remark. The theory of residues can be used to expand the quotient of two polynomials into its partial fraction representation.

Example. Let $P(z)$ be a polynomial of degree at most 2. Show that if a, b, c are distinct complex numbers, then

$$f(z) = \frac{P(z)}{(z-a)(z-b)(z-c)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c}$$

where

$$A = \operatorname*{Res}_{z=a} f = \frac{P(a)}{(a-b)(a-c)} \qquad (*)$$

$$B = \operatorname*{Res}_{z=b} f = \frac{P(b)}{(b-a)(b-c)}, \text{ and } \qquad C = \operatorname*{Res}_{z=c} f = \frac{P(c)}{(c-a)(c-b)}$$

Solution We'll prove only $(*)$, the proof of the remaining part is similar.

We expand f in its Laurent series about the point a. The term $\frac{A}{z-a}$ is already a Laurent series about a, and $\frac{B}{z-b}$ and $\frac{C}{z-c}$ are analytic at a, so their Laurent series expansion about a is in fact Taylor series and does not contain any negative power of $z-a$. So $\operatorname*{Res}_{z=a} f = A$. And by calculation

$$\operatorname*{Res}_{z=a} f = \lim_{z\to a} (z-a) f(z) = \lim_{z \to a} \frac{P(z)}{(z-b)(z-c)} = \frac{P(a)}{(a-b)(a-c)}.$$

Example. Express $f(z) = \dfrac{3z+2}{z(z-1)(z-2)}$ in partial fractions

Solution $f(z) = \dfrac{A}{z} + \dfrac{B}{z-1} + \dfrac{C}{z-2}$, and $A = \dfrac{3.0+2}{(0-1)(0-2)} = 1$

$B = \dfrac{3.1+2}{1.(1-2)} = -5$ and $C = \dfrac{3.2+2}{(2)(2-1)} = 4$, so

$$\frac{3z+2}{z(z-1)(z-2)} = \frac{1}{z} - \frac{5}{z-1} + \frac{4}{z-2}$$

Remark If a repeated root occurs, then the process is similar, and one can easily show that if $P(z)$ has a degree of at most 2, then

$$f(z) = \frac{P(z)}{(z-a)^2(z-b)} = \frac{A}{(z-a)^2} + \frac{B}{z-a} + \frac{C}{z-b}$$

where $A = \operatorname{Res}_{z=a}\big((z-a)f(z)\big)$, $B = \operatorname{Res}_{z=a} f$, $C = \operatorname{Res}_{z=b} f$.

Example. Express $f(z) = \frac{z^2+3z+2}{z^2(z-1)}$ in partial fractions.

$$\operatorname{Res}_{z=0} z f(z) = \lim_{z\to 0} \frac{z^3+3z+2}{z-1} = -2$$

$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2+3z+2}{z-1}\right) = \lim_{z\to 0} \frac{(2z+3)(z-1)-(z^2+3z+2)}{(z-1)^2} = -5$$

$$\operatorname{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{z^2+3z+2}{z^2} = 6 \text{ . so}$$

$$\frac{z^2+3z+2}{z^2(z-1)} = -\frac{2}{z^2} - \frac{5}{z} + \frac{6}{z-1}.$$

Trigonometric Integrals

Consider the integral

$$\int_0^{2\pi} R(\sin\varphi, \cos\varphi)\, d\varphi$$

where $R(u,v)$ is a rational function. Let $z = e^{i\varphi}$, $0 \le \varphi \le 2\pi$ then $\sin\varphi = \frac{1}{2i}\left(z - \frac{1}{z}\right) = \frac{z^2-1}{2iz}$ and $\cos\varphi = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{z^2+1}{2z}$

and $dz = ie^{i\varphi}\, d\varphi$ and $d\varphi = \frac{dz}{iz}$, and so

$$\int_0^{2\pi} R(\sin\varphi, \cos\varphi)\, d\varphi = \oint_{C_1(0)} R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \frac{dz}{iz}.$$

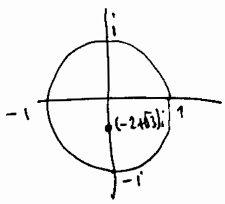
Example. Evaluate $\displaystyle I = \int_0^{2\pi} \frac{d\varphi}{2+\sin\varphi}$

Solution

$$I = \oint_{C_1(0)} \frac{1}{2+\frac{z^2-1}{2iz}} \frac{dz}{iz} = \oint_{C_1(0)} \frac{2}{z^2+4iz-1}\, dz$$

$$z^2+4iz-1=0 \iff z = \frac{-4i + (-16+4)^{\frac{1}{2}}}{2} = \frac{-4i \pm 2\sqrt{3}i}{2} = (-2\pm\sqrt{3})i$$

only $-2+\sqrt{3}i$ is inside the contour $C_1(0)$



$$\operatorname*{Res}_{z=(-2+\sqrt{3})i} \frac{2}{z^2+4iz-1} = \lim_{z\to(-2+\sqrt{3})i} \frac{2(z-(-2+\sqrt{3})i)}{z^2+4iz-1}$$

$$\overset{L.R}{=} \lim_{z\to(-2+\sqrt{3})i} \frac{2}{2z+4i} = \frac{2}{-4i+2\sqrt{3}i+4i} = \frac{-i}{\sqrt{3}}$$

and $\displaystyle I = 2\pi i\left(\frac{-i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$.

Example. Evaluate $\displaystyle I = \int_0^{2\pi} \frac{1}{1+3\cos^2\theta}\, d\theta$

Solution.

$$I = \oint_{C_1(0)} \frac{1}{1+3\left(\frac{z^2+1}{2z}\right)^2} \frac{dz}{iz} = \oint_{C_1(0)} \frac{-4iz}{4z^2+3(z^4+2z^2+1)}\, dz$$

$$= \oint_{C_1(0)} \frac{-4iz}{3z^4+10z^2+3}\, dz$$

let $z^2=w$, $\quad 3w^2+10w+3=0 \iff w = \dfrac{-10+(100-36)^{\frac{1}{2}}}{6} = \dfrac{-10\pm 8}{6} = -\dfrac{1}{3}$ or -3

Then $z = \pm\frac{i}{\sqrt{3}}$ or $\pm i\sqrt{3}$

Thus $\dfrac{-4iz}{3z^4+10z^2+3}$ has simple poles at $\pm\dfrac{i}{\sqrt{3}}$ and $\pm i\sqrt{3}$

But only $\pm\dfrac{i}{\sqrt{3}}$ are inside the contour $C_1(0)$

and
$$I = 2\pi i\left(\operatorname*{Res}_{z=\frac{i}{\sqrt{3}}} \frac{-4iz}{3z^4+10z^2+3} + \operatorname*{Res}_{z=-\frac{i}{\sqrt{3}}} \frac{-4iz}{3z^4+10z^2+3}\right)$$

$$\operatorname*{Res}_{z=\frac{i}{\sqrt{3}}} \frac{-4iz}{3z^4+10z^2+3} = \lim_{z\to\frac{i}{\sqrt{3}}} \frac{-4iz(z-\frac{i}{\sqrt{3}})}{3z^4+10z^2+3} \overset{L.R}{=} \lim_{z\to\frac{i}{\sqrt{3}}} \frac{-4i(z-\frac{i}{\sqrt{3}})-4iz}{12z^3+20z}$$

$$= \frac{-4i\cdot\frac{i}{\sqrt{3}}}{-\frac{4i}{\sqrt{3}}+\frac{20i}{\sqrt{3}}} = \frac{4}{16i} = -\frac{1}{4}i$$

similarly, $\operatorname*{Res}_{z=\frac{-i}{\sqrt{3}}} \frac{-4iz}{3z^4+10z^2+3} \overset{L.R}{=} \lim_{z\to-\frac{i}{\sqrt{3}}} \frac{-4i(z+\frac{i}{\sqrt{3}})-4iz}{12z^3+20z} = \frac{-\frac{4}{\sqrt{3}}}{\frac{4i}{\sqrt{3}}-\frac{20i}{\sqrt{3}}} = \frac{4}{16i} = -\frac{1}{4}i$

and $I = 2\pi i\left(-\frac{1}{4}i - \frac{1}{4}i\right) = \pi$.

Example. Evaluate $I = \int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta$.

Solution $z=e^{i\theta} \Rightarrow z^2 = e^{2i\theta} = \cos 2\theta + i\sin 2\theta$ & $z^{-2} = e^{-2i\theta} = \cos 2\theta - i\sin 2\theta$ and

$\cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) = \frac{z^4+1}{2z^2}$ and

$$I = \oint_{C_1(0)} \frac{\frac{z^4+1}{2z^2}}{5-4\frac{z^2+1}{2z}}\frac{dz}{iz} = \oint_{C_1(0)} \frac{-i(z^4+1)}{z^2(10z-4z^2-4)}\,dz = \oint_{C_1(0)} \frac{i(z^4+1)}{2z^2(2z^2-5z+2)}\,dz$$

$$= \oint_{C_1(0)} \frac{i(z^4+1)}{2z^2(2z-1)(z-2)}\,dz = 2\pi i\left(\operatorname*{Res}_{z=0} \frac{i(z^4+1)}{2z^2(2z-1)(z-2)} + \operatorname*{Res}_{z=\frac{1}{2}} \frac{i(z^4+1)}{2z^2(2z-1)(z-2)}\right)$$

$$\operatorname*{Res}_{z=0} \frac{i(z^4+1)}{2z^2(2z-1)(z-2)} = \lim_{z\to 0} \frac{d}{dz} \frac{i(z^4+1)}{2(2z^2-5z+2)} = \frac{i}{2}\lim_{z\to 0} \frac{4z^3(2z^2-5z+2)-(4z-5)(z^4+1)}{(2z^2-5z+2)^2}$$

$$= \frac{i}{2}\frac{5}{4} = \frac{5i}{8}$$

$$\operatorname*{Res}_{z=\frac{1}{2}} \frac{i(z^4+1)}{2z^2(2z-1)(z-2)} = \lim_{z\to\frac{1}{2}} \frac{i(z^4+1)}{2z^2\cdot 2\cdot(z-2)} = \frac{i\,\frac{17}{16}}{-\frac{3}{2}} = -\frac{17i}{24}$$

and $I = 2\pi i\left(\frac{5i}{8} - \frac{17i}{24}\right) = 2\pi i\,\frac{-2i}{24} = \frac{\pi}{6}$.

Improper integrals of rational functions

Let f be a continuous function of the real variable x on the interval $0 \le x < \infty$. The improper integral of f over $[0, \infty)$ is defined by

$$\int_0^\infty f(x)dx = \lim_{b \to \infty} \int_0^b f(x)dx$$

provided the limit exists. If f is defined for all real x, then the integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^\infty f(x)dx = \lim_{a \to -\infty} \int_a^0 f(x)dx + \lim_{b \to \infty} \int_0^b f(x)dx, \qquad (*)$$

provided both limits exist. If $\int_{-\infty}^\infty f(x)dx$ exists, we can find its value by taking a single limit

$$\int_{-\infty}^\infty f(x)dx = \lim_{R \to \infty} \int_{-R}^R f(x)dx. \qquad (**)$$

For some functions the limit on the right hand side of $(**)$ exists, but the limit on the right side of $(*)$ does not exist.

Let f be a continuous real-valued function for all x. The Cauchy principal value (P.V) of the integral $\int_{-\infty}^\infty f(x)dx$ is defined by

$$\text{P.V.} \int_{-\infty}^\infty f(x)dx = \lim_{R \to \infty} \int_{-R}^R f(x)dx,$$

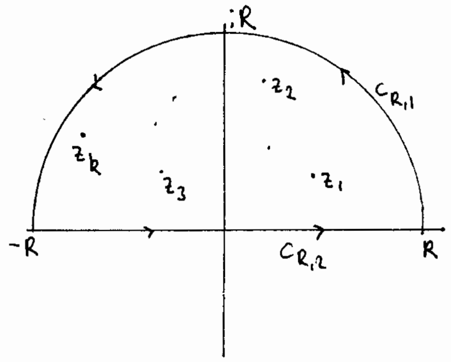
provided the limit exists.

Theorem Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+2$, then

$$\text{P.V.} \int_{-\infty}^\infty \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

where $z_1, z_2, \ldots, z_k$ are the poles of f that lie in the upper half-plane.

Proof. Let $R>0$ be so that all poles of f lies in the disk $|z|<R$. And $C_R = \underbrace{\{z\in\mathbb{C} \mid |z|=R,\ \operatorname{Im} z>0\}}_{C_{R,1}} \cup \underbrace{\{z\in\mathbb{R} \mid -R\le z\le R\}}_{C_{R,2}}$



Clearly,

$$\int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx = \int_{C_R} \frac{P(z)}{Q(z)}\,dz - \int_{C_{R,1}} \frac{P(z)}{Q(z)}\,dz$$

and

$$\int_{C_R} \frac{P(z)}{Q(z)}\,dz = \sum_{j=1}^{k} \operatorname*{Res}_{z=z_k} \frac{P(z)}{Q(z)}.$$

The proof is done if we show that $\displaystyle\int_{C_{R,1}} \frac{P(z)}{Q(z)}\,dz \to 0$ as $R\to\infty$.

Let $P(z) = a_m z^m + \dots + a_0$ and $Q(z) = b_n z^n + \dots + b_0$.

Then

$$\left|\frac{P(z)}{Q(z)}\right| = \frac{|z|^m \left(\left|a_m + \frac{a_{m-1}}{z} + \dots + \frac{a_0}{z^m}\right|\right)}{|z|^n \left(\left|b_n + \frac{b_{n-1}}{z} + \dots + \frac{b_0}{z^n}\right|\right)}$$

$$= \frac{1}{|z|}\underbrace{\left(\underbrace{\frac{1}{|z|^{n-m-1}}}_{\substack{\to 0 \\ \text{as } |z|\to\infty \\ n-m-1\ge 1}} \underbrace{\frac{\left|a_m + \frac{a_{m-1}}{z} + \dots + \frac{a_0}{z^m}\right|}{\left|b_n + \frac{b_{n-1}}{z} + \dots + \frac{b_0}{z^n}\right|}}_{\to \frac{|a_m|}{|b_n|} \text{ as } |z|\to\infty}\right)}_{\to 0 \text{ as } |z|\to\infty}$$

Thus $\forall \varepsilon>0, \exists R_0>0$ such that $\left|\frac{P(z)}{Q(z)}\right| < \frac{\varepsilon}{|z|}$ for $|z|>R_0$.

Therefore, using the ML-inequality, we get

$$\left|\int_{C_{R,1}} \frac{P(z)}{Q(z)}\,dz\right| \le \frac{\varepsilon}{R}\cdot\pi R = \pi\varepsilon \quad \forall R>R_0.$$

Since $\varepsilon>0$ is arbitrary, this means that

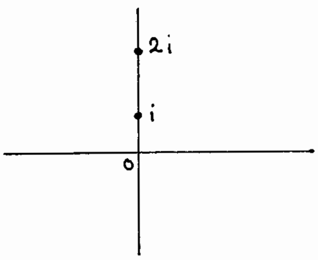
$$\lim_{R\to\infty} \int_{C_{R,1}} \frac{P(z)}{Q(z)}\,dz = 0. \quad \blacksquare$$

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$.

Solution $P(z)=1$, $Q(z) = (z^2+1)(z^2+4) = (z-i)(z+i)(z-2i)(z+2i)$.

$m=0$, $n=4 \Rightarrow n \geq m+2 \Rightarrow$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i \left(\operatorname*{Res}_{z=i} \frac{P(z)}{Q(z)} + \operatorname*{Res}_{z=2i} \frac{P(z)}{Q(z)} \right)$$



$\frac{P}{Q}$ has simple poles at i and $2i$, and

$$\operatorname*{Res}_{z=i} \frac{P}{Q} = \lim_{z\to i} \left((z+i)(z-2i)(z+2i)\right)^{-1} = (2i(-i)3i)^{-1} = \frac{-i}{6}$$

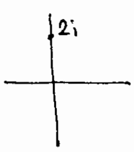
$$\operatorname*{Res}_{z=2i} \frac{P}{Q} = \lim_{z\to 2i} \left((z-i)(z+i)(z+2i)\right)^{-1} = (i(3i)4i)^{-1} = \frac{i}{12}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i \left(\frac{-i}{6} + \frac{i}{12} \right) = \frac{\pi}{6}.$$

Example. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$.

Solution. $P(z)=1$, $Q(z) = (z^2+4)^3 = (z-2i)^3(z+2i)^3$, $m=0$, $n=6 \Rightarrow n \geq m+2 \Rightarrow$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = 2\pi i \operatorname*{Res}_{z=2i} \frac{P}{Q}.$$



$\frac{P}{Q}$ has a pole of order 3 at $z=2i$, and

$$\operatorname*{Res}_{z=2i} \frac{P}{Q} = \frac{1}{2} \lim_{z\to 2i} \frac{d^2}{dz^2} \frac{1}{(z+2i)^3}$$

$$= \frac{1}{2} \lim_{z\to 2i} \frac{d}{dz} \left(-\frac{3}{(z+2i)^4} \right)$$

$$= \frac{1}{2} \lim_{z\to 2i} \frac{12}{(z+2i)^5} = \frac{6}{(4i)^5} = -\frac{6i}{1024} = -\frac{3i}{512}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = 2\pi i \left(-\frac{3i}{512} \right) = \frac{3\pi}{256}.$$

Improper integrals involving trigonometric functions

Theorem Let P and Q be polynomials with real coefficients of degree m and n respectively, where $n \geq m+1$ and $Q(x) \neq 0$, for all real x. If $\alpha > 0$ and

$$f(z) = \frac{e^{i\alpha z} P(z)}{Q(z)},$$

then
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = -2\pi \sum_{j=1}^{k} \operatorname{Im}\left(\operatorname*{Res}_{z=z_j} f\right), \text{ and}$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = 2\pi \sum_{j=1}^{k} \operatorname{Re}\left(\operatorname*{Res}_{z=z_j} f\right),$$

where $z_1, z_2, \dots, z_k$ are the poles of f that lie in the upper half-plane.

Proof. Postponed.

Example. Evaluate $\text{P.V.} \displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4}\, dx$.

Solution $P(z) = z$, $Q(z) = z^2+4 = (z-2i)(z+2i)$, $m=1$, $n=2 \Rightarrow n \geq m+1 \Rightarrow$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4}\, dx = 2\pi \operatorname{Re}\left(\operatorname*{Res}_{z=2i} f(z)\right) \quad \text{where}$$

$f(z) = \dfrac{e^{iz} z}{z^2+4}$. Clearly, f has a simple pole at $z=2i$,

and
$$\operatorname*{Res}_{z=2i} f = \lim_{z \to 2i} \frac{e^{iz} z}{z+2i} = \frac{e^{-2}\, 2i}{4i} = \frac{e^{-2}}{2} = \frac{1}{2e^2}$$

and
$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4}\, dx = \frac{\pi}{e^2}.$$

Example. Evaluate P.V. $\int_{-\infty}^{\infty} \frac{\cos x}{x^4+4}\,dx$.

Solution $P(z)=1$, $Q(z)=z^4+4=(z-(1+i))(z-(1-i))(z-(-1+i))(z-(-1-i))$,

$m=0$, $n=4$ $\Rightarrow$ $n \geq m+1$ and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\cos x}{x^4+4}\,dx = -2\pi \operatorname{Im}\left(\operatorname*{Res}_{z=1+i} f + \operatorname*{Res}_{z=-1+i} f\right) \quad \text{where}$$

$f(z) = \frac{e^{iz}}{z^4+4}$. f has simple poles at $1+i$ and $-1+i$, and

$$\operatorname*{Res}_{z=1+i} f = \lim_{z\to 1+i} \frac{e^{iz}(z-(1+i))}{z^4+4} \overset{L.R}{=} \lim_{z\to 1+i} \frac{ie^{iz}(z-(1+i))+e^{iz}}{4z^3}$$

$$= \lim_{z\to 1+i} \frac{ze^{iz}}{4z^4} = \frac{(1+i)e^{i(1+i)}}{-16}$$

$$= \frac{-(1+i)e^{i-1}}{16} = -\frac{(1+i)(\cos 1 + i\sin 1)}{16e}$$

$$= \frac{-(\cos 1 + i\sin 1 + i\cos 1 - \sin 1)}{16e}$$

$$\operatorname*{Res}_{z=-1+i} f = \lim_{z\to -1+i} \frac{e^{iz}(z-(-1+i))}{z^4+4} \overset{L.R}{=} \lim_{z\to -1+i} \frac{ie^{iz}(z-(-1+i))+e^{iz}}{4z^3}$$

$$= \lim_{z\to -1+i} \frac{ze^{iz}}{4z^4} = \frac{(-1+i)e^{i(-1+i)}}{-16}$$

$$= \frac{-(-1+i)e^{-i-1}}{16} = \frac{-(-1+i)(\cos 1 - i\sin 1)}{16e}$$

$$= \frac{-(-\cos 1 + i\cos 1 + i\sin 1 + \sin 1)}{16e}$$

$$\Rightarrow \operatorname*{Res}_{z=1+i} f + \operatorname*{Res}_{z=-1+i} f = \frac{-i(\sin 1+\cos 1)}{8e} \Rightarrow \operatorname{Im}\left(\operatorname*{Res}_{z=1+i} f + \operatorname*{Res}_{z=-1+i} f\right) = -\frac{(\sin 1+\cos 1)}{8e}, \text{ and}$$

$$\text{P.V} \int_{-\infty}^{\infty} \frac{\cos x}{x^4+4}\,dx = \frac{\pi(\sin 1+\cos 1)}{4e}.$$

Jordan's Lemma. Suppose that P and Q are polynomials of degree m and n respectively, where $n \geq m+1$. If $C_{R,1}$ is the upper semicircle $z = Re^{i\theta}$, for $0 \leq \theta \leq \pi$, then

$$\lim_{R\to\infty} \int_{C_{R,1}} \frac{e^{iz}P(z)}{Q(z)}\, dz = 0.$$

Proof. Since $n \geq m+1$, $\dfrac{|P(z)|}{|Q(z)|} \to 0$ as $|z| \to \infty$. Therefore, $\forall \varepsilon > 0$ $\exists R_0$ such that $\left|\dfrac{P(z)}{Q(z)}\right| < \varepsilon$ for $|z| > R_0$.

Then
$$\left| \int_{C_{R,1}} \frac{e^{iz}P(z)}{Q(z)}\, dz \right| = \left| \int_0^\pi e^{iRe^{i\theta}} \frac{P(Re^{i\theta})}{Q(Re^{i\theta})} Rie^{i\theta}\, d\theta \right|$$

$$\leq R\varepsilon \int_0^\pi \left| e^{iR(\cos\theta + i\sin\theta)} \right| d\theta$$

$$= R\varepsilon \int_0^\pi e^{-R\sin\theta}\, d\theta = 2R\varepsilon \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta \quad \text{for } R > R_0$$

On $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq \frac{2\theta}{\pi} \leq \sin\theta \Rightarrow$

$$\left| \int_{C_{R,1}} \frac{e^{iz}P(z)}{Q(z)}\, dz \right| \leq 2R\varepsilon \int_0^{\frac{\pi}{2}} e^{-\frac{2R\theta}{\pi}}\, d\theta = -\varepsilon\pi\, e^{-\frac{2R\theta}{\pi}} \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \varepsilon\pi(1 - e^{-R}) < \varepsilon$$

for $R > R_0$. Because $\varepsilon > 0$ is arbitrary, $\displaystyle\lim_{R\to\infty} \int_{C_{R,1}} \frac{e^{iz}P(z)}{Q(z)}\, dz = 0$.

Proof of the Theorem. Clearly,

$$\int_{-R}^{R} e^{i\alpha x} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f - \int_{C_{R,1}} e^{i\alpha z} \frac{P(z)}{Q(z)}\, dz$$

where $C_{R,1}$ is the upper semicircle $Re^{i\theta}$, for $0 \leq \theta \leq \pi$, and $f(z) = e^{i\alpha z}\dfrac{P(z)}{Q(z)}$. The change of variables $\zeta = \alpha z$ shows that the conclusion of Jordan's lemma holds for the integrand $e^{i\alpha z}\dfrac{P(z)}{Q(z)}$, and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{(\cos\alpha x + i\sin(\alpha x))P(x)}{Q(x)}\, dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f$$

$$= -2\pi \sum_{j=1}^{k} \operatorname{Im}\left(\operatorname*{Res}_{z=z_j} f\right) + 2\pi \sum_{j=1}^{k} \operatorname{Re}\left(\operatorname*{Res}_{z=z_j} f\right)$$

Equating the real and imaginary parts of this equation completes the proof.

Indented Contour Integrals

In this section, we will learn how to use residues to evaluate the Cauchy principal value of the integral of f over $(-\infty, \infty)$ when the integrand f has simple poles on the x-axis.

Theorem. Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials with real coefficients of degree m and n respectively, and $n \geq m+2$. If Q has simple zeros at the points $t_1, t_2, \ldots, t_\ell$ on the x-axis, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}_{z=z_j} f(z) + \pi i \sum_{j=1}^{\ell} \operatorname{Res}_{z=t_j} f(z),$$

where $z_1, z_2, \ldots, z_k$ are the poles of f that lie in the upper half-plane.

Proof. Postponed. ∎

Theorem. Let P and Q be polynomials of degree m and n respectively, where $n \geq m+1$, and let Q have simple zeros at the points $t_1, t_2, t_3, \ldots, t_\ell$ on the x-axis. If α is a positive real number and if $f(z) = e^{i\alpha z}\frac{P(z)}{Q(z)}$, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x\,dx = -2\pi \sum_{j=1}^{k} \operatorname{Im}\left(\operatorname{Res}_{z=z_j} f(z)\right) - \pi \sum_{j=1}^{\ell} \operatorname{Im}\left(\operatorname{Res}_{z=t_j} f(z)\right)$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x\,dx = 2\pi \sum_{j=1}^{k} \operatorname{Re}\left(\operatorname{Res}_{z=z_j} f(z)\right) + \pi \sum_{j=1}^{\ell} \operatorname{Im}\left(\operatorname{Res}_{z=t_j} f(z)\right).$$

where $z_1, z_2, \ldots, z_k$ are the poles of f that lie in the upper half-plane.

Proof. Postponed. ▨

Example. Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{x\,dx}{x^3-8}$.

Solution. $P(z)=z$, $Q(z)=z^3-8$, $m=1$, $n=3 \Rightarrow n \geq m+2$, and $Q(z)=(z-2)(z-(-1-i\sqrt{3}))(z-(-1+i\sqrt{3}))$

Thus, $f(z)=\frac{P(z)}{Q(z)}$ has simple pole at $z=2$ on the real axis, and simple pole at $z=-1+i\sqrt{3}$ in the upper half-plane, and

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} \frac{z}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})} = \frac{2}{(3+i\sqrt{3})(3-i\sqrt{3})} = \frac{2}{12} = \frac{1}{6},$$

$$\operatorname*{Res}_{z=-1+i\sqrt{3}} f(z) = \lim_{z\to -1+i\sqrt{3}} \frac{z}{(z-2)(z+1+i\sqrt{3})} = \frac{-1+i\sqrt{3}}{(-3+i\sqrt{3})2i\sqrt{3}} = \frac{-1+i\sqrt{3}}{-6(1+i\sqrt{3})}$$

$$= \frac{1-i\sqrt{3}}{6(1+i\sqrt{3})}\,\frac{(1-i\sqrt{3})}{(1-i\sqrt{3})} = \frac{1-2i\sqrt{3}-3}{24} = \frac{-1-i\sqrt{3}}{12}.$$

So, $$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\,dx}{x^3-8} = 2\pi i\left(\frac{-1-i\sqrt{3}}{12}\right) + \pi i \frac{1}{6} = \frac{\pi\sqrt{3}}{6}.$$

Example. Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)}\,dx$.

Solution. $P(z)=1$, $Q(z)=(z-1)(z^2+4)=(z-1)(z-2i)(z+2i)$, $m=0$, $n=3 \Rightarrow n \geq m+1$. Thus, $f(z)=e^{iz}\frac{P(z)}{Q(z)}$ has simple pole at $z=1$ on the real-axis, and simple pole at $z=2i$ in the upper half-plane, and

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{e^{iz}}{(z-2i)(z+2i)} = \frac{e^{i}}{(1-2i)(1+2i)} = \frac{\cos 1 + i\sin 1}{5}$$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z\to 2i} \frac{e^{iz}}{(z-1)(z+2i)} = \frac{e^{-2}}{(-1+2i)4i} = \frac{e^{-2}}{(-8-4i)}\,\frac{(-8+4i)}{(-8+4i)} = \frac{-8+4i}{80e^2} = \frac{-2+i}{20e^2}$$

So, $$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+4)}\,dx = 2\pi\left(\frac{-2}{20e^2}\right) + \pi\,\frac{\cos 1}{5} = \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right).$$

Lemma. Suppose that f has a simple pole at the point t_0 on the x-axis. If C_r is the contour $C_r: z = t_0 + re^{i\theta}$, for $0 \le \theta \le \pi$, then

$$\lim_{r\to 0} \int_{C_r} f(z)\,dz = i\pi \operatorname*{Res}_{z=t_0} f(z).$$

Proof. If f has a simple pole at $z = t_0$, then f has the form
$$f(z) = \frac{A}{z-t_0} + \sum_{k=0}^{\infty} c_k (z-t_0)^k = \frac{A}{z-t_0} + g(z), \text{ where}$$

$A = \operatorname*{Res}_{z=t_0} f(z)$ and g is a function analytic at t_0.

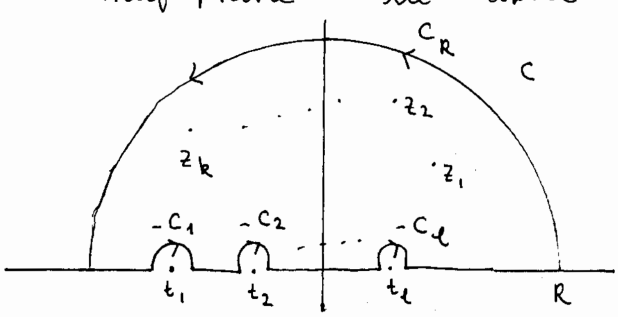
Then
$$\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{A}{re^{i\theta}}\, rie^{i\theta}\,d\theta + \int_0^{\pi} g(t_0 + re^{i\theta})\, rie^{i\theta}\,d\theta$$
$$= i\pi A + ir \int_0^{\pi} g(t_0 + re^{i\theta}) e^{i\theta}\,d\theta$$

Since g is continuous at t_0, $\exists r_0 > 0$, $M > 0$ such that $|g(t_0 + re^{i\theta})| \le M$ for $r \le r_0$, and

$$\left|\lim_{r\to 0} ir \int_0^{\pi} g(t_0 + re^{i\theta}) e^{i\theta}\,d\theta\right| \le \lim_{r\to 0} r \int_0^{\pi} M\,d\theta = \lim_{r\to 0} r\pi M = 0.$$

Thus,
$$\lim_{r\to 0} \int_{C_R} f(z)\,dz = i\pi A = i\pi \operatorname*{Res}_{z=t_0} f(z). \blacksquare$$

Proof of Theorems. Since f has only a finite number of poles, we can choose r small enough that the semicircles $C_j: z = t_j + re^{i\theta}$, $0 \le \theta \le \pi$ and $j = 1, 2, \dots, l$, are disjoint and the poles $z_1, z_2, \dots, z_k$ of f in the upper half-plane lie above them as shown below:



Let R be large enough so that the poles of f in the upper half-plane lie under the semicircle C_R,

$C_R: z = Re^{i\theta}$, $0 \le \theta \le \pi$, and the poles of f on the x-axis lie in the interval $-R \le x \le R$. Let C be the simple closed positively oriented contour that consists of C_R and $-C_1, \ldots, -C_\ell$ and the segments of the real axis that lie between the semicircles.

Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z), \text{ and so}$$

$$\int_{I_R} f(x)\,dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) + \sum_{j=1}^{\ell} \int_{C_j} f(z)\,dz - \int_{C_R} f(z)\,dz,$$

where $I_R = C - C_R - C_1 - C_2 - \cdots - C_\ell$. We can easily show (as before) that $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$.

If we let $R \to \infty$ and $r \to 0$ in $(*)$, and use the lemma we obtain

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z) + \pi i \sum_{j=1}^{\ell} \operatorname*{Res}_{z=t_j} f(z).$$

The proof of the remaining theorem is similar. ▨

<u>Example.</u> Evaluate $\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x(x-1)(x-2)}$

<u>Solution</u> Let $f(z) = \frac{1}{z(z-1)(z-2)}$. Clearly f satisfies the conditions of the theorem, and

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{1}{(z-1)(z-2)} = \frac{1}{2}, \quad \operatorname*{Res}_{z=1} f(z) = \lim_{z\to 1} \frac{1}{z(z-2)} = -1$$

and $\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} \frac{1}{z(z-1)} = \frac{1}{2}$. Thus $\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x(x-1)(x-2)} = \pi i\left(\frac{1}{2} - 1 + \frac{1}{2}\right) = 0.$

Integrands with branch points

We start with the following example.

<u>Example.</u> Evaluate $\int_0^\infty \frac{x^\rho}{1+x}\,dx$, $-1<\rho<0$.

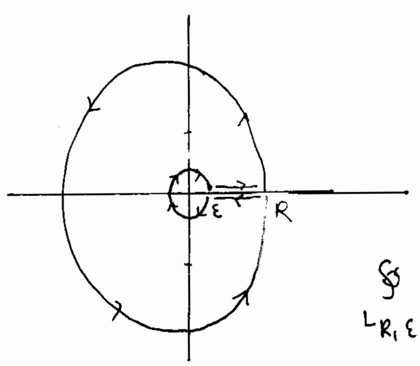
<u>Solution</u> Consider the region $\mathbb{C}\setminus[0,\infty)$ and the branch of z^ρ in this region defined by

$$z^\rho = |z|^\rho e^{i\rho\arg z}, \quad 0<\arg z<2\pi .$$

Extend this definition to the both sides of the positive ray in such a way:

$$z^\rho\big|_{\text{upperside}} = x^\rho \qquad z^\rho\big|_{\text{lower side}} = x^\rho e^{2\pi i\rho}$$

Consider $f(z)=\frac{z^\rho}{1+z}$ in $\mathbb{C}\setminus[0,\infty)$. Take a big $R>0$ and a small $\varepsilon>0$ and consider the region $G_{R,\varepsilon}$, $L_{R,\varepsilon}=\partial G_{R,\varepsilon}$



$$\oint_{L_{R,\varepsilon}} \frac{z^\rho}{1+z}\,dz = 2\pi i \operatorname*{Res}_{z=-1} f(z) = 2\pi i\, z^\rho\big|_{-1} = 2\pi i\, e^{\pi i\rho}.$$

$$\oint_{L_{R,\varepsilon}} \frac{z^\rho}{1+z}\,dz = \int_\varepsilon^R \frac{x^\rho}{1+x}\,dx + \int_{|z|=R} \frac{z^\rho}{1+z}\,dz - \int_\varepsilon^R \frac{x^\rho e^{2\pi i\rho}}{1+x}\,dx + \int_{|z|=\varepsilon}\frac{z^\rho}{1+z}\,dz .$$

$$\left|\int_{|z|=R}\frac{z^\rho}{1+z}\,dz\right| \le 2\pi R\cdot\frac{R^\rho}{R-1} \to 0 \quad \text{as} \quad R\to\infty$$

$$\Rightarrow$$

$$\left|\int_{|z|=\varepsilon}\frac{z^\rho}{1+z}\,dz\right| \le 2\pi\varepsilon\,\frac{\varepsilon^\rho}{1-\varepsilon} \to 0 \quad \text{as} \quad \varepsilon\to 0$$

$$(1-e^{2\pi i\rho})\int_0^\infty \frac{x^\rho}{1+x}\,dx = 2\pi i\, e^{\pi i\rho} \Rightarrow \int_0^\infty \frac{x^\rho}{1+x}\,dx = \frac{2\pi i\, e^{\pi i\rho}}{1-e^{2\pi i\rho}} = \frac{2\pi i}{e^{-\pi i\rho}-e^{\pi i\rho}}$$

$$= -\frac{\pi}{\sin\pi\rho}.$$

Using the methods we have used to solve the previous example, we can prove the following theorem.

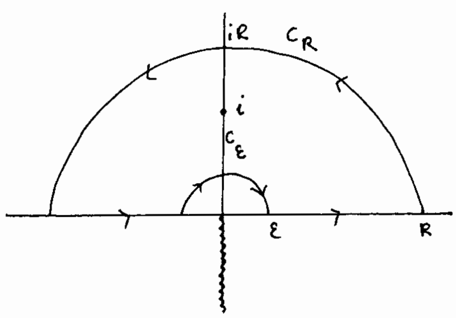
Theorem. Let P and Q be polynomials of degree m and n, respectively, where $n \geq m+2$. If $Q(x) \neq 0$ for $x>0$, Q has a zero of order at most 1 at the origin, and $f(z) = z^{\alpha} \frac{P(z)}{Q(z)}$ where $0<\alpha<1$, then

$$\text{P.V.} \int_0^{\infty} x^{\alpha} \frac{P(x)}{Q(x)}\, dx = \frac{2\pi i}{1-e^{i\alpha 2\pi}} \sum_{j=1}^{k} \operatorname*{Res}_{z=z_j} f(z)$$

where $z_1, z_2, \ldots, z_k$ are the nonzero poles of $\frac{P}{Q}$.

Example Evaluate $\text{P.V.} \int_0^{\infty} \frac{x^{\frac{1}{2}}}{1+x^2}\, dx$.

Solution Let $L_{R,\varepsilon}$ be the contour below, and



$$f(z) = \frac{z^{\frac{1}{2}}}{1+z^2} = \frac{e^{\frac{1}{2}\log z}}{1+z^2} = \frac{e^{\frac{1}{2}(\ln|z| + i\arg z)}}{1+z^2}$$

where $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

Then,

$$\int_{L_{R,\varepsilon}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z) = 2\pi i \lim_{z\to i} \frac{z^{\frac{1}{2}}}{1+z^2}(z-i)$$

$$= 2\pi i \lim_{z\to i} \frac{z^{\frac{1}{2}}}{(z-i)(z+i)}(z-i) = 2\pi i \left. \frac{z^{\frac{1}{2}}}{z+i} \right|_{z=i}$$

$$= 2\pi i \frac{e^{i\frac{\pi}{4}}}{2i} = \pi e^{i\frac{\pi}{4}}.$$

On the other hand,

$$\int_{L_{R,\varepsilon}} f(z)\,dz = \int_{\varepsilon}^{R} \frac{x^{\frac{1}{2}}}{1+x^2}\,dx + \int_{C_R} \frac{z^{\frac{1}{2}}}{1+z^2}\,dz + \int_{-R}^{-\varepsilon} \frac{e^{\frac{1}{2}(\ln|x| + i\pi)}}{1+x^2}\,dx + \int_{C_\varepsilon} \frac{z^{\frac{1}{2}}}{1+z^2}\,dz.$$

$$\left|\int_{C_R} \frac{z^{\frac{1}{2}}}{1+z^2}\,dz\right| \le \pi R \cdot \frac{R^{\frac{1}{2}}}{R^2-1} \to 0 \quad \text{as} \quad R\to\infty$$

$$\left|\int_{C_\varepsilon} \frac{z^{\frac{1}{2}}}{1+z^2}\,dz\right| \le \pi \varepsilon \cdot \frac{\varepsilon^{\frac{1}{2}}}{1-\varepsilon^2} \to 0 \quad \text{as} \quad \varepsilon\to 0$$

$$\int_{-R}^{-\varepsilon} \frac{e^{\frac{1}{2}(\ln|x|+i\pi)}}{1+x^2}\,dx = e^{\frac{i\pi}{2}} \int_{-R}^{-\varepsilon} \frac{(-x)^{\frac{1}{2}}}{1+x^2}\,dx = e^{\frac{i\pi}{2}} \int_{R}^{\varepsilon} \frac{t^{\frac{1}{2}}}{1+t^2}(-dt) = e^{\frac{i\pi}{2}} \int_{\varepsilon}^{R} \frac{t^{\frac{1}{2}}}{1+t^2}\,dt.$$

Therefore $\quad (1+e^{i\frac{\pi}{2}}) \int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,dx = \pi e^{i\frac{\pi}{4}} \Rightarrow$

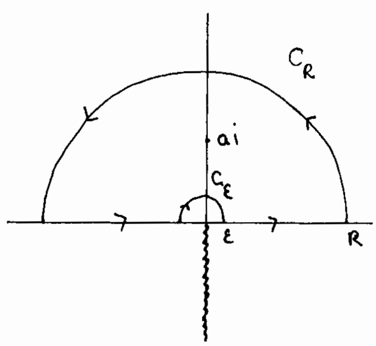
$$\int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\,dx = \frac{\pi\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}{1+i} = \frac{\pi\left(\frac{1}{\sqrt{2}} - \cancel{\frac{i}{\sqrt{2}}} + \cancel{\frac{i}{\sqrt{2}}} + \frac{1}{\sqrt{2}}\right)}{2}$$

$(1-i)$

$$= \frac{\pi}{\sqrt{2}}.$$

<u>Example.</u> Evaluate P.V. $\displaystyle\int_0^\infty \frac{\ln x}{x^2+a^2}\,dx$, $a>0$.

<u>Solution.</u> Let $L_{R,\varepsilon}$ be the contour below, and



$$f(z) = \frac{\log z}{z^2+a^2} = \frac{\ln|z| + i\arg(z)}{z^2+a^2}, \quad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

Then $\displaystyle\oint_{L_{R,\varepsilon}} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=ai} f(z)$

$$= 2\pi i \lim_{z\to ai} \frac{\log z}{(z+ai)(z-ai)}(z-ai)$$

$$= 2\pi i \left.\frac{\log z}{z+ai}\right|_{z=ai}$$

$$= 2\pi i\, \frac{\ln a + i\frac{\pi}{2}}{2ai} = \frac{\pi}{a}\left(\ln a + i\frac{\pi}{2}\right).$$

On the other hand,

$$\int_{L_{R,\varepsilon}} f(z)dz = \int_{\varepsilon}^{R} \frac{\ln x}{x^2+a^2}dx + \int_{C_R} \frac{\log z}{z^2+a^2}dz - \int_{R}^{\varepsilon} \frac{\log(-x)}{x^2+a^2}dx + \int_{C_\varepsilon} \frac{\log z}{z^2+a^2}dz.$$

$$\left| \int_{C_R} \frac{\log z}{z^2+a^2}dz \right| \le \pi R \, \frac{\ln R+\pi}{R^2-a^2} \to 0 \quad \text{as} \quad R\to\infty$$

$$\left| \int_{C_\varepsilon} \frac{\log z}{z^2+a^2}dz \right| \le \pi \varepsilon \cdot \frac{\ln \varepsilon+\pi}{a^2-\varepsilon^2} \to 0 \quad \text{as} \quad \varepsilon\to 0$$

use L'Hospital's rule.

and $\displaystyle -\int_{R}^{\varepsilon} \frac{\log(-x)}{x^2+a^2}dx = \int_{\varepsilon}^{R} \frac{\ln x + i\pi}{x^2+a^2}dx$, and so

$$\int_0^\infty \frac{\ln x}{x^2+a^2}dx + \int_0^\infty \frac{\ln x + i\pi}{x^2+a^2}dx = \frac{\pi}{a}\left(\ln a + i\frac{\pi}{2}\right) \quad \Rightarrow$$

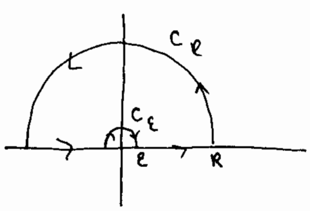
(by equating real and imaginary parts of the above equation)

$$2\int_0^\infty \frac{\ln x}{x^2+a^2}dx = \frac{\pi}{a}\ln a, \text{ and hence}$$

$$\int_0^\infty \frac{\ln x}{x^2+a^2} = \frac{\pi \ln a}{2a}.$$

<u>Example.</u> Evaluate $\displaystyle \int_{-\infty}^{\infty} \frac{\sin x}{x}dx.$

<u>Solution.</u> Let $L_{R,\varepsilon}$ be the contour below and $f(z) = \dfrac{e^{iz}}{z}$.



Then, $\displaystyle \int_{L_{R,\varepsilon}} f(z)dz = 0.$

On the other hand,

$$\int_{L_{R,\varepsilon}} f(z)dz = \int_{\varepsilon}^{R} \frac{e^{ix}}{x}dx + \int_{C_R} \frac{e^{iz}}{z}dz - \int_{R}^{\varepsilon} \frac{e^{-ix}}{-x}dx - \int_{C_\varepsilon} \frac{e^{iz}}{z}dz.$$

$$\int_{C_R} \frac{e^{iz}}{z}dz \to 0 \quad \text{as} \quad R\to\infty \quad (\text{See Jordan's Lemma})$$

$$\int_{C_\varepsilon} \frac{e^{iz}}{z}dz \to i\pi \operatorname*{Res}_{z=0} \frac{e^{iz}}{z} \quad \text{as} \quad \varepsilon\to 0$$ (see the Lemma in the section "Indented Contour Integrals".

and $\quad i\pi \operatorname{Res}_{z=0} \frac{e^{iz}}{z} = i\pi \lim_{z\to 0} \frac{e^{iz}}{z}\cdot z = i\pi$

and $\quad -\int_R^{\varepsilon} \frac{e^{-ix}}{-x}\,dx = -\int_{\varepsilon}^{R} \frac{e^{-ix}}{x}\,dx .$

Therefore $\quad 0 = \int_0^{\infty} \frac{e^{ix}}{x}\,dx - \int_0^{\infty} \frac{e^{-ix}}{x}\,dx - i\pi$

$$i\pi = \int_0^{\infty} \frac{\cos x + i\sin x}{x}\,dx - \int_0^{\infty} \frac{\cos x - i\sin x}{x}\,dx \Rightarrow$$

$$i\pi = 2i \int_0^{\infty} \frac{\sin x}{x}\,dx = i \int_{-\infty}^{\infty} \underbrace{\frac{\sin x}{x}}_{\text{even function}}\,dx \Rightarrow \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi .$$

The Argument Principle and Rouché's Theorem.

A function f is said to be <u>meromorphic</u> in a domain $\mathcal{D}$ provided the only singularities of f are isolated poles and removable singularities.

A meromorphic function can have only finitely many zeros, counting multiplicities, in a bounded connected domain unless it is identically zero and have only finitely many poles, counting multiplicities, in a bounded connected domain, because its zeros and poles are isolated.

<u>Theorem</u> Suppose that f is meromorphic in the simply connected domain $\mathcal{D}$ and that C is a simple closed positively oriented contour in $\mathcal{D}$ such that f has no zeros or poles for $z \in C$. Then,

$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\,dz = Z_f - P_f ,$$

where Z_f is the number of zeros of f that lie inside C and P_f is the number of poles of f that lie inside C.

<u>Proof.</u> Let $a_1, a_2, \ldots, a_k$ be the zeros of f inside C counted according to multiplicity and let $b_1, b_2, \ldots, b_\ell$ be the poles of f inside C counted according to multiplicity. Then f has the representation

$$f(z) = \frac{(z-a_1)(z-a_2)\cdots(z-a_k)}{(z-b_1)(z-b_2)\cdots(z-b_\ell)}\, g(z)$$

where g is analytic and nonzero on C and inside C. A straightforward calculation shows that

$$\frac{f'(z)}{f(z)} = \frac{1}{z-a_1} + \frac{1}{z-a_2} + \cdots + \frac{1}{z-a_k} - \frac{1}{z-b_1} - \frac{1}{z-b_2} - \cdots - \frac{1}{z-b_\ell} + \frac{g'(z)}{g(z)}$$

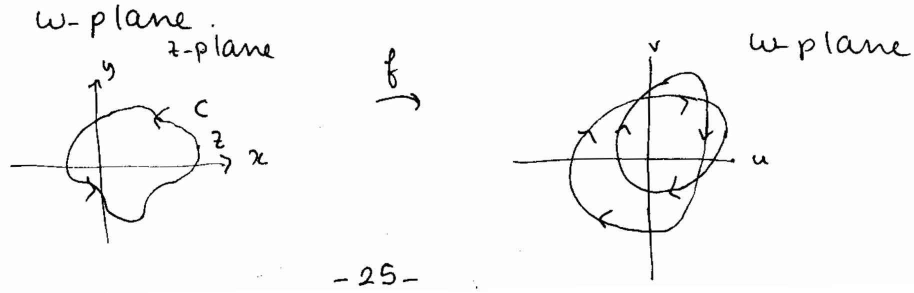
Since $\oint\limits_C \frac{dz}{z-a_j} = \oint\limits_C \frac{dz}{z-b_i} = 2\pi i$ for $1 \le j \le k$, $1 \le i \le \ell$,

and $\frac{g'(z)}{g(z)}$ is analytic inside and on C, we get

$$\oint\limits_C \frac{f'(z)}{f(z)}\, dz = 2\pi i k - 2\pi i \ell = 2\pi i (Z_f - P_f). \blacksquare$$

Now, let f be meromorphic in the domain interior to a simple closed contour C and that it is analytic and nonzero on C:

The image Γ of C under the transformation $w = f(z)$ is a closed contour, not necessarily simple, in the w-plane.



As a point z traverses C in the positive direction, its images w traverses Γ in a particular direction that determines the orientation of Γ. Since f has no zeros on C, Γ does not pass through the origin in the w-plane.

Let w and w_0 be points on Γ where w_0 is fixed and ϕ_0 is a value of $\arg w_0$. Then, let $\arg w$ vary continuously, starting with the value ϕ_0, as the point w begins at the point w_0 and traverses Γ once in the direction of orientation assigned to it by the mapping $w = f(z)$.

When w returns to the point w_0, where it started, $\arg w$ assumes a particular value of $\arg w_0$, which we denote by ϕ_1. Thus the change in $\arg w$ as w describes Γ once in its direction of orientation is $\phi_1 - \phi_0$. This change is, of course, independent of the point w_0 chosen to determine it. Since $w = f(z)$, the number $\phi_1 - \phi_0$ is, in fact, the change in the argument of $f(z)$ as z describes C once in the positive direction, starting with a point z_0, and we write

$$\Delta_C \arg f(z) = \phi_1 - \phi_0$$

The value of $\Delta_C \arg f(z)$ is evidently an integer multiple of 2π, and the integer $\frac{1}{2\pi} \Delta_C \arg f(z)$ represents the number of times the point w winds around the origin in the w-plane.
For this reason, this integer is sometimes called the <u>winding number</u> of Γ with respect to the origin.

it is positive if Γ winds around the origin in the counterclockwise direction and negative if it winds clockwise around that point. The winding number is always zero when Γ does not enclose the origin.

<u>Theorem (The argument principle)</u>

Suppose that

i) a function $f(z)$ is meromorphic in the domain interior to a positively oriented simple closed contour C.

ii) $f(z)$ is analytic and nonzero on C.

iii) counting multiplicities, Z_f is the number of zeros and P_f is the number of poles of $f(z)$ inside C. Then,

$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z_f - P_f .$$

<u>Proof</u>. It is enough to prove that

$$\oint_C \frac{f'(z)}{f(z)}\,dz = i\,\Delta_C \arg f(z).$$

Let $z = z(t)$, $a \leq t \leq b$ be a parametrization of C, so that

$$\int_C \frac{f'(z)}{f(z)}\,dz = \int_a^b \frac{f'(z(t))\,z'(t)}{f(z(t))}\,dt .$$

Since under the transformation $w = f(z)$, the image Γ of C never passes through the origin in the w-plane, the image of any point $z = z(t)$ on C can be expressed in exponential form as $w = \rho(t)e^{i\phi(t)}$. Thus

$$f(z(t)) = \rho(t)\,e^{i\phi(t)}, \quad a \leq t \leq b,$$

and, along each of the smooth arcs making up the contour Γ, it follows that

$$f'(z(t))z'(t) = \frac{d}{dt}(f(z(t))) = \frac{d}{dt}(\rho(t)e^{i\phi(t)})$$

$$= \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t)$$

so, we can write

$$\oint_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{\rho'(t)}{\rho(t)}dt + i\int_a^b \phi'(t)\,dt$$

$$= \ln\rho(t)\Big|_a^b + i\phi(t)\Big|_a^b$$

But $\rho(b)-\rho(a)=0$ and $\phi(b)-\phi(a)=\Delta_C \arg f(z)$

Hence
$$\oint_C \frac{f'(z)}{f(z)}dz = i\Delta_C \arg f(z). \blacksquare$$

<u>Theorem</u> Suppose that f is meromorphic in the simply connected domain $\mathcal{D}$. If C is a simple closed positively oriented contour in $\mathcal{D}$ such that for $z\in C$, $f(z)\neq 0$ and $f(z)\neq\infty$, then

$$W(f(C),a) = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)-a}dz$$

known as the winding number of $f(C)$ about a, counts the number of times the curve $f(C)$ winds around the point a. If $a=0$,

$$W(f(C),a) = \frac{1}{2\pi}\Delta_C \arg f(z).$$

<u>Proof</u>. Exercise!

<u>Example</u> If $f(z)=\frac{1}{z^2}$ and $C=\{z:|z|=1\}$. Find $\Delta_C \arg f(z)$.

<u>Solution</u>.
$$\frac{1}{2\pi}\Delta_C \arg f(z) = Z_f - P_f = -2 .$$

direct verification: $f(e^{i\theta}) = e^{-2i\theta}$, $0 \le \theta \le 2\pi$, turns around the origin, staying on the unit circle, two times in the negative direction.

<u>Example</u>. Let $f(z) = z^2+z$, $C = \{z : |z| = 2\}$. Find $\Delta_C \arg f(z)$.

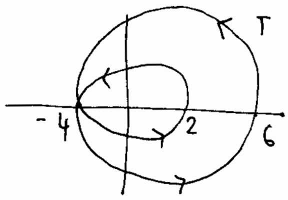
<u>Solution</u>. $f(z) = z(z+1)$

$$\frac{1}{2\pi} \Delta_C \arg f(z) = Z_f - P_f = 2.$$

Direct verification:

$f(2e^{i\theta}) = (4\cos 2\theta + 2\cos\theta) + i(4\sin 2\theta + 2\sin\theta)$, $0 \le \theta \le 2\pi$

has the graph



turns around the origin two times in the positive direction.

<u>Theorem (Rouché)</u> Suppose that

i) two functions f and g are analytic inside and on a simple closed contour C;

ii) $|f(z)| > |g(z)|$ at each point on C. Then f and $f+g$ have the same number of zeros, counting multiplicities, inside C.

<u>Proof</u>. $|f(z)| > |g(z)| \ge 0$ on C, and

$$|f(z)+g(z)| > |f(z)| - |g(z)| > 0 \quad \text{on } C$$

$\Rightarrow$ f and $f+g$ has no poles and zeros on C

Then by the argument principle,

$$Z_f = \frac{1}{2\pi} \Delta_C \arg f(z) \quad \text{and} \quad Z_{f+g} = \frac{1}{2\pi} \Delta_C \arg (f(z)+g(z)).$$

Since $\quad \Delta_C \arg[f(z)+g(z)] = \Delta_C \arg\left\{ f(z)\left[1+\frac{g(z)}{f(z)}\right]\right\}$

$$= \Delta_C \arg f(z) + \Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right],$$

it is clear that $\quad Z_{f+g} = Z_f + \frac{1}{2\pi}\Delta_C \arg F(z) \quad$ where $F(z) = 1+\frac{g(z)}{f(z)}$

But $\quad |F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1$ and this means that, under the transformation $w = F(z)$, the image of C lies in the open disk $|w-1|<1$. That image does not then enclose the origin $w=0$. Hence $\Delta_C \arg F(z) = 0$, and so $Z_{f+g} = Z_f$. ■

<u>Example.</u> Determine the number of roots of the equation $z^7 - 4z^3 + z - 1 = 0$ inside the circle $|z| = 1$

<u>Solution</u> Let $f(z) = -4z^3$ and $g(z) = z^7+z-1$, on C $|f(z)| = 4$ and $|g(z)| \le 1+1+1 = 3$ and so $|f(z)| > |g(z)|$ on C. So

$$Z_{f+g} = Z_f = 3.$$

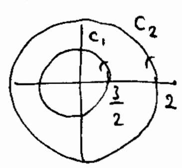
<u>Example.</u> Show that all four roots of the polynomial $p(z) = z^4 - 7z - 1$ lie in the disk $D_2(0) = \{z : |z| < 2\}$.

<u>Solution.</u> Let $f(z) = z^4$, $g(z) = -7z-1$, on C $|f(z)| = 16$ and $|g(z)| \le 7.2+1 = 15$, so $|f(z)| > |g(z)|$ on C and

$$Z_p = Z_{f+g} = Z_f = 4.$$

<u>Example.</u> Show that the function $p(z) = z^5 + 15z + 1$ has precisely four zeros inside the annulus $\frac{3}{2} < |z| < 2$.

<u>Solution</u> Zeros inside C_2: $f(z) = z^5$, $g(z) = 15z+1 \Rightarrow$ on C_2



$|f(z)| = 32 > |g(z)| \le 15.2+1 = 31 \Rightarrow$ Inside C_2

$$Z_f = Z_{f+g} = 5.$$

zeros inside C_1: $f(z) = 15z$, $g(z) = z^5+1 \Rightarrow$ on C_1 $|f(z)| = \frac{45}{2} = 22.5$

$|g(z)| \le \left(\frac{3}{2}\right)^5 + 1 = \frac{275}{32} = 8.59375 \Rightarrow Z_f = Z_{f+g} = 1 \Rightarrow$ Zeros between C_1 and C_2 is $5-1 = 4$.

Using residues to evaluate sums of series

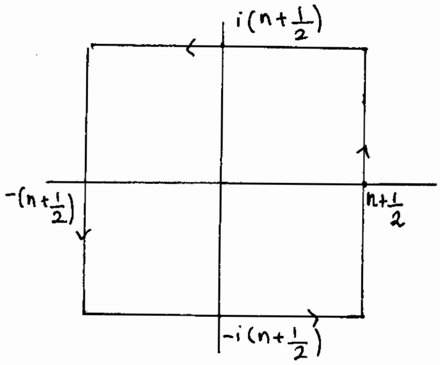
We will consider the series of types

$$\sum_{k=-\infty}^{\infty} R(k) \quad , \quad \sum_{k=-\infty}^{\infty} (-1)^k R(k)$$

where $R(x)$ is a rational function such that the series are absolutely convergent. (For example, $R(x) = \frac{P(x)}{Q(x)}$, $\deg(P(x)) \leq \deg(Q(x))$)

Example. Evaluate $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2}$, $\alpha \notin \mathbb{Z}$

Solution. Let $f(z) = \frac{1}{(z+\alpha)^2} \cot \pi z$ and L_n be the contour shown below



Clearly f has a double pole at $z=-\alpha$ and simple poles at $z=k$, $k \in \mathbb{Z}$

Clearly,

$$\operatorname*{Res}_{z=k} \frac{1}{(z+\alpha)^2} \cot \pi z = \lim_{z \to k} \frac{1}{(z+\alpha)^2} \frac{\cos \pi z}{\sin \pi z} (z-k)$$

$$= \frac{\cos \pi k}{(k+\alpha)^2} \lim_{z \to k} \frac{z-k}{\sin \pi z}$$

$$\overset{L.R}{=} \frac{\cos \pi k}{(k+\alpha)^2} \cdot \frac{1}{\pi \cos \pi k}$$

$$= \frac{1}{\pi} \frac{1}{(k+\alpha)^2}$$

Then by the residue theorem,

$$\oint_{L_n} f(z)dz = 2\pi i \left(\sum_{k=-n}^{n} \operatorname*{Res}_{z=k} f(z) + \operatorname*{Res}_{z=-\alpha} f(z) \right)$$

$$= 2\pi i \left(\frac{1}{\pi} \sum_{k=-n}^{n} \frac{1}{(k+\alpha)^2} + \operatorname*{Res}_{z=-\alpha} f(z) \right) \qquad (*)$$

Before going further, let us prove the following lemma:

<u>Lemma 1</u> There exists a constant $C>0$ which does not depend on n such that

$$|\cot \pi z| \leqslant C \qquad \text{for all } z \in L_n \text{ and } n = 1, 2, \ldots .$$

<u>Proof.</u> On vertical sides of L_n: $x = \pm(n+\frac{1}{2})$, $-(n+\frac{1}{2}) \leq y \leq (n+\frac{1}{2})$, and

$$|\cot \pi z| = \frac{|e^{i\pi z} + e^{-i\pi z}|}{|e^{i\pi z} - e^{-i\pi z}|} = \frac{|e^{\pm i\pi(n+\frac{1}{2})}e^{-\pi y} + e^{\mp i\pi(n+\frac{1}{2})}e^{\pi y}|}{|e^{\pm \pi(n+\frac{1}{2})}e^{-\pi y} - e^{\mp i\pi(n+\frac{1}{2})}e^{\pi y}|}$$

$$= \frac{|e^{-\pi y} + e^{\mp \pi i(2n+1)}e^{\pi y}|}{|e^{-\pi y} - e^{\mp \pi i(2n+1)}e^{\pi y}|} = \frac{e^{\pi|y|} - e^{-\pi|y|}}{e^{\pi|y|} + e^{-\pi|y|}} < 1$$

On horizontal sides of L_n: $-(n+\frac{1}{2}) \leq x \leq (n+\frac{1}{2})$, $y = \pm(n+\frac{1}{2})$, and

$$|\cot \pi z| = \frac{|e^{i\pi x}e^{\mp\pi(n+\frac{1}{2})} + e^{-i\pi x}e^{\pm\pi(n+\frac{1}{2})}|}{|e^{i\pi x}e^{\mp\pi(n+\frac{1}{2})} - e^{-i\pi x}e^{\pm\pi(n+\frac{1}{2})}|}$$

$$\leqslant \underbrace{\frac{e^{\pi(n+\frac{1}{2})} + e^{-\pi(n+\frac{1}{2})}}{e^{\pi(n+\frac{1}{2})} - e^{-\pi(n+\frac{1}{2})}}}_{\downarrow} < C$$

1 as $n \to \infty$, and never 0

hence it is bounded. ∎

Now, letting $n \to \infty$ in $(*)$, we get

$$\lim_{n\to\infty} \oint_{L_n} \frac{\cot \pi z}{(z+\alpha)^2}\, dz = 2\pi i \left(\frac{1}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2} + \operatorname{Res}_{z=-\alpha} \frac{\cot \pi z}{(z+\alpha)^2} \right)$$

Using the ML-inequality, we obtain

$$\left| \oint_{L_n} \frac{\cot \pi z}{(z+\alpha)^2}\, dz \right| \leq \quad 4(2n+1) \frac{C}{\left(n+\frac{1}{2} - |\alpha|\right)^2} \to 0 \quad \text{as} \quad n \to \infty$$

Therefore,
$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+\alpha)^2} = -\pi \operatorname*{Res}_{z=-\alpha} \frac{\cot \pi z}{(z+\alpha)^2}.$$

Example. Evaluate $\displaystyle\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+\alpha)^2}$, $\alpha \notin \mathbb{Z}$.

Solution Let $f(z) = \dfrac{1}{(z+\alpha)^2 \sin \pi z}$ and L_n be the same contour used in the previous example. Using the same method and the analogous lemma given below, we can get

$$\oint_{L_n} \frac{dz}{(z+\alpha)^2 \sin \pi z} = 2\pi i \left(\sum_{k=-n}^{n} \operatorname*{Res}_{z=k} f(z) + \operatorname*{Res}_{z=-\alpha} f(z) \right)$$

$$= 2\pi i \left(\frac{1}{\pi} \sum_{k=-n}^{n} \frac{(-1)^k}{(k+\alpha)^2} + \operatorname*{Res}_{z=-\alpha} f(z) \right)$$

and $\displaystyle\lim_{n\to\infty} \oint_{L_n} \frac{dz}{(z+\alpha)^2 \sin \pi z} = 0$, and hence

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(k+\alpha)^2} = -\pi \operatorname*{Res}_{z=-\alpha} \frac{1}{(z+\alpha)^2 \sin \pi z}.$$

Lemma 2 There exists $C > 0$ not depending on n such that $\left| \dfrac{1}{\sin \pi z} \right| \leq C$ for all $z \in L_n$ and $n = 1, 2, \ldots$.

Proof. Exercise! ☒

Example Evaluate $\sum_{k=1}^{\infty} \frac{1}{k^2}$.

Solution Clearly, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2}$

Let $f(z) = \frac{\cot \pi z}{z^2}$, and L_n be the same contour used in previous examples.

Then

$$\oint_{L_n} \frac{\cot \pi z}{z^2} \; 2\pi i \left[\operatorname*{Res}_{z=0} \frac{\cot \pi z}{z^2} + \sum_{\substack{k=-n \\ k\neq 0}}^{n} \operatorname*{Res}_{z=k} \frac{\cot \pi z}{z^2} \right]$$

If $k \neq 0$, f has a simple pole at $z=k$, and

$$\operatorname*{Res}_{z=k} f(z) = \frac{1}{\pi k^2}$$

Using the ML-inequality and Lemma 1, we get

$$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{k^2} = -\pi \operatorname*{Res}_{z=0} \frac{\cot \pi z}{z^2}, \text{ and hence}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{\pi}{2} \operatorname*{Res}_{z=0} \frac{\cot \pi z}{z^2}.$$

Clearly,

$$\frac{\cot \pi z}{z^2} = \frac{\cos \pi z}{z^2 \sin \pi z} = \frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \frac{\pi^6 z^6}{6!} + \cdots}{z^2\left(\pi z - \frac{\pi^3 z^3}{3!} + \frac{\pi^5 z^5}{5!} - \frac{\pi^7 z^7}{7!} + \cdots\right)}$$

$$= \frac{1}{\pi z^3} \; \frac{1 - \frac{\pi^2 z^2}{2!} + \frac{\pi^4 z^4}{4!} - \frac{\pi^6 z^6}{6!} + \cdots}{1 - \frac{\pi^2 z^2}{3!} + \frac{\pi^4 z^4}{5!} - \frac{\pi^6 z^6}{7!} + \cdots}$$

$$= \frac{1}{\pi z^3} \left(1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \right)$$

where $\operatorname*{Res}_{z=0} \frac{\cot \pi z}{z^2} = \frac{a_2}{\pi}$, and a_2 can be calculated from the relation

$$1-\frac{\pi^2 z^2}{2!}+\frac{\pi^4 z^4}{4!}-\frac{\pi^6 z^6}{6!}+\dots=\left(1-\frac{\pi^2 z^2}{3!}+\frac{\pi^4 z^4}{5!}-\frac{\pi^6 z^6}{7!}+\dots\right)(1+a_1 z+a_2 z^2+a_3 z^3+\dots)$$

clearly, $a_1=0$ and $a_2-\frac{\pi^2}{3!}=-\frac{\pi^2}{2!} \Rightarrow a_2=-\pi^2\left(\frac{1}{2}-\frac{1}{6}\right)$

$$=-\frac{\pi^2}{3}$$

and,
$$\sum_{k=1}^{\infty}\frac{1}{k^2}=-\frac{\pi}{2}\cdot\frac{1}{\pi}\cdot\left(-\frac{\pi^2}{3}\right)=\frac{\pi^2}{6}.$$

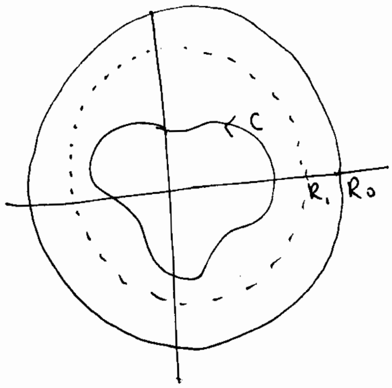
Using a single residue

If the function f in Cauchy's residue theorem, is, in addition, analytic at each point in the finite plane exterior to C, it is sometimes more efficient to evaluate the integral of f around C by finding a single residue of a certain related function. We present the method as a theorem (this result arises in the theory of residues at infinity which we will develop later)

Theorem If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2}f\left(\frac{1}{z}\right)\right).$$

Proof. Let R_1 be large enough so that the contour C is interior to $|z|=R_1$. Then if C_0 denotes a positively oriented circle $|z|=R_0$, where $R_0>R_1$,



we know from Laurent's theorem that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad , \quad R_1 < |z| < \infty \qquad (*)$$

where

$$c_n = \frac{1}{2\pi i} \oint_{C_0} \frac{f(z)}{z^{n+1}}\, dz \quad , \; n \in \mathbb{Z}$$

Putting $n=-1$, we get $\int_{C_0} f(z)dz = 2\pi i c_{-1}$.

(Observe that, since the condition of validity with representation $(*)$ is not of the type $0 < |z| < R_2$, the coefficient c_{-1} is not the residue of f at $z=0$, which may not even be a a singular point of f.) If we replace z by $\frac{1}{z}$ in $(*)$ and its condition of validity, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n}, \qquad 0 < |z| < \frac{1}{R_1},$$

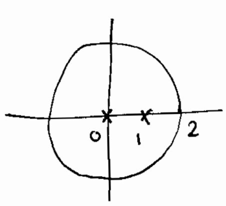
and hence that

$$c_{-1} = \operatorname*{Res}_{z=0} \left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right) \quad \text{and} \quad \int_{C_0} f(z) = 2\pi i \operatorname*{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right).$$

Finally, since f is analytic throughout the closed region bounded by C and C_0, the principal of deformation of contours yields the desired result. ∎

<u>Example</u>. Evaluate $\displaystyle\oint_{|z|=2} \frac{5z-2}{z(z-1)}\, dz$.

<u>Solution</u>



the integrand has simple poles at $z=0$ and $z=1$, and no other singularities in $\mathbb{C}$, then by the last theorem

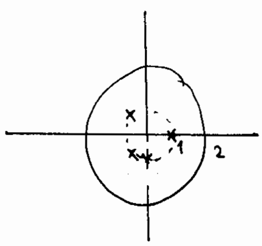
$$\oint_{|z|=2} \frac{5z-2}{z(z-1)}\,dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2}\left(\frac{\frac{5}{z}-2}{\frac{1}{z}\left(\frac{1}{z}-1\right)}\right)\right]$$

$$= 2\pi i \operatorname*{Res}_{z=0}\left[\frac{5-2z}{z(1-z)}\right]$$

$$= 2\pi i \lim_{z\to 0} \frac{5-2z}{1-z} = 10\pi i.$$

<u>Example.</u> Evaluate $I = \oint_{|z|=2} \frac{z^5}{1-z^3}\,dz.$

<u>Solution</u>



$$I = 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^2}\,\frac{\frac{1}{z^5}}{1-\frac{1}{z^3}}$$

$$= 2\pi i \operatorname*{Res}_{z=0} \frac{1}{z^4(z^3-1)}$$

$$= 2\pi i \operatorname*{Res}_{z=0} -\frac{1}{z^4(1-z^3)}$$

$$= 2\pi i \operatorname*{Res}_{z=0}\left[-\frac{1}{z^4}\left(1+z^3+z^6+z^9+\cdots\right)\right]$$

$$= -2\pi i.$$

<u>Isolated singularities at ∞ point</u>

If f is analytic in a neighborhood of ∞, we say that ∞ is an <u>isolated singularity</u> of f.

Assume, ∞ is an isolated singularity of f. Then f is analytic in $\{z : |z| > R\}$ for some $R > 0$. Applying Laurent series expansion theorem, we have

$$f(z) = \sum_{k=-\infty}^{\infty} c_k z^k = \underbrace{\sum_{k=1}^{\infty} c_k z^k}_{\text{principal part}} + \underbrace{\sum_{k=0}^{\infty} \frac{c_{-k}}{z^k}}_{\text{analytic part}}, \quad \text{for } |z| > R.$$

Classification of singularities at ∞.

1. If $c_k = 0$, $\forall k = 1, 2, 3, \ldots$ (i.e, all coefficients of the principal part vanish), then we say that ∞ is a removable singularity.

2. If the set of nonvanishing coefficients of the principal part is nonempty and finite, then we say that ∞ is a pole.

3. If the set of nonvanishing coefficients of the principal part is infinite, then we say that ∞ is an essential singularity.

Example. $f(z) = \frac{1}{z}$ has removable singularity at ∞
$f(z) = z$ " a pole (of order 1) at ∞
$f(z) = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ has essential singularity at ∞.

Behavior at singularities of each kind

Theorem. ∞ is a removable singularity if and only if f is bounded in some neighborhood of ∞.

Theorem. ∞ is a pole if and only if $\lim_{z \to \infty} f(z) = \infty$.

We say that f has a zero at ∞ of order m if $f(z) = \frac{\varphi(z)}{z^m}$, where φ is analytic at ∞ and $\varphi(\infty) \neq 0$.

we say that f has a <u>pole</u> at ∞ <u>of order m</u> if $c_m \neq 0$, $c_{m+1} = c_{m+2} = \cdots 0$

(that is $\quad f(z) = \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m$)

<u>Theorem</u>. f has a pole at ∞ of order m if and only if $f(z) = z^m \varphi(z)$, φ is analytic at ∞ and $\varphi(\infty) \neq 0$.

<u>Theorem</u>. f has a pole at ∞ of order m if and only if $\frac{1}{f}$ has a zero of order m at ∞.

<u>Residue at ∞ point.</u>

Let f be analytic in a neighborhood of ∞, then

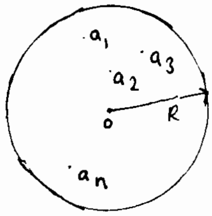
$$\operatorname*{Res}_{z=\infty} = \lim_{R\to\infty} \frac{1}{2\pi i} \oint_{|z|=R} f(z)\,dz .$$

<u>Note</u>: $\oint_{|z|=R}$ does not depend on R for R being large enough.

<u>2^{nd} Residue Theorem</u> Let f be a function analytic in $\mathbb{C}$ except a finitely many isolated singularities $a_1, a_2, \cdots, a_n$. Then

$$\sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z) + \operatorname*{Res}_{z=\infty} f(z) = 0.$$

<u>Proof</u>.



Take R so large that $a_1, \ldots, a_n \in \{z : |z| < R\}$

By the first residue theorem

$$\oint_{|z|=R} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=a_k} f(z)$$

$$\parallel$$

$$-2\pi i \operatorname*{Res}_{z=\infty} f(z)$$

■

Theorem. Let $f(z)=\sum_{k=-\infty}^{\infty} c_k z^k$ be a Laurent series expansion of $f(z)$ at ∞. Then

$$\operatorname*{Res}_{z=\infty} f(z) = -c_{-1}.$$

Proof.

$$\oint_{|z|=R} f(z)\,dz = \sum_{k=-\infty}^{\infty} c_k \oint_{|z|=R} z^k\,dz = -2\pi i\, c_{-1}.$$

Note It is possible that $\operatorname*{Res}_{z=\infty} f(z)\neq 0$ if even $f(z)$ has a removable singularity at ∞.

Example. $f(z)=\frac{1}{z}$, $\operatorname*{Res}_{z=\infty} f(z) = -1$.

Theorem If f is analytic in a neighborhood of ∞ and $\lim_{z\to\infty} f(z)=0$, then

$$\operatorname*{Res}_{z=-\infty} f(z) = -\lim_{z\to\infty} z f(z)$$

Proof. If $\lim_{z\to\infty} f(z)=0$, then $f(z)=\sum_{k=1}^{\infty}\frac{c_{-k}}{z^k}$ and $z f(z) = c_{-1} + \sum_{k=2}^{\infty}\frac{c_{-k}}{z^{k-1}} \to c_{-1}$ as $z\to\infty$.

Example. Evaluate $I=\oint_{|z|=2}\frac{z^{16}\sin\frac{1}{z}}{(z^5+1)^3(z-3)}\,dz$.

Solution. Inside $|z|=2$, f has poles of order 3 at $z_k=e^{\frac{2k+1}{5}\pi i}$, $k=0,1,2,3,4$. and essential singularity at 0.

By the first residue theorem $I=2\pi i\left(\sum_{k=0}^{4}\operatorname*{Res}_{z=z_k} f(z)+\operatorname*{Res}_{z=0} f(z)\right)$

By the second residue theorem

$$\sum_{k=0}^{4}\operatorname*{Res}_{z=z_k} f(z)+\operatorname*{Res}_{z=0} f(z)+\operatorname*{Res}_{z=3} f(z)+\operatorname*{Res}_{z=\infty} f(z)=0,$$ and

so

$$I = -2\pi i\left(\operatorname*{Res}_{z=3} f(z)+\operatorname*{Res}_{z=\infty} f(z)\right)$$

f has simple pole at 3 and

$$\operatorname*{Res}_{z=3} f(z) = \lim_{z\to 3} \frac{z^{16}\sin\frac{1}{z}}{(z^5+1)^3} = \frac{3^{16}\sin\frac{1}{3}}{(3^5+1)^3}.$$

Note that
$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z^{16}}{(z^5+1)^3(z-3)} \lim_{z\to\infty} \sin\frac{1}{z}$$

$$= \lim_{z\to\infty} \sin\frac{1}{z} = \lim_{z\to 0} \sin z = 0.$$

Hence
$$\operatorname*{Res}_{z=\infty} f(z) = -\lim_{z\to\infty} z f(z) = -\lim_{z\to\infty} \frac{z^{17}\sin\frac{1}{z}}{(z^5+1)(z-3)}$$

$$= -\lim_{z\to\infty} \frac{z^{16}}{(z^5+1)(z-3)} \lim_{z\to\infty} z\sin\frac{1}{z}$$

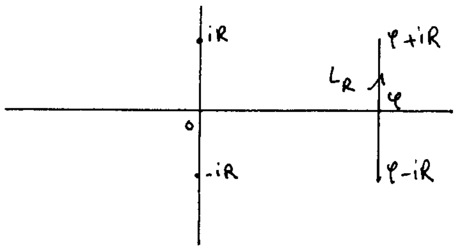
$$= -\lim_{z\to\infty} \frac{\sin\frac{1}{z}}{\frac{1}{z}} = -\lim_{z\to 0} \frac{\sin z}{z} = -1.$$

Therefore,
$$I = -2\pi i\left(\frac{3^{16}\sin\frac{1}{3}}{(3^5+1)^3} - 1\right)$$

Inverse Laplace transforms

Suppose that a function F is analytic in $\mathbb{C}$ except for a finite number of isolated singularities.

Let L_R be a vertical line segment from $\gamma - iR$ to $\gamma + iR$ where γ is positive and large enough that the singularities of F all lie to the left of that segment



Let f be the function defined by

$$f(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{L_R} e^{st} F(s)\,ds, \quad t>0$$

provided this limit exists.

Sometimes, this can be written as

$$f(t) = \frac{1}{2\pi i} \text{ P.V.} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s)\,ds, \quad t > 0 \qquad (*)$$

It can be shown that under some certain conditions, f is the <u>inverse Laplace transform</u> of $F(s)$. That is, if $F(s)$ is the <u>Laplace transform</u> of $f(t)$, defined by the equation

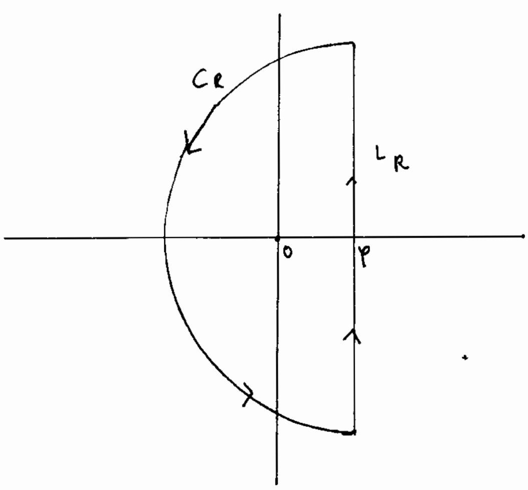
$$F(s) = \int_0^\infty e^{-st} f(t)\,dt,$$

then f is get back by $(*)$

Laplace transforms and their inverses are important in solving both ordinary and partial differential equations.

We will show how to evaluate $(*)$ using residues.

Let s_n, $n = 1, 2, \dots, N$ be the singularities of $F(s)$ and $R_0 = \max\{|s_1|, |s_2|, \dots, |s_N|\}$, and C_R be the semicircle $\gamma + Re^{i\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$ where $R > \gamma + R_0$.



Hence all the singularities of F lie inside the region bounded by C_R and L_R. Then

$$\int_{L_R} e^{st} F(s)\,ds = 2\pi i \sum_{n=1}^{N} \operatorname{Res}_{s = s_n} \left(e^{st} F(s) \right) - \int_{C_R} e^{st} F(s)\,ds \qquad (**)$$

<u>Lemma.</u> If there is a positive constant M_R such that $|F(s)| \le M_R$ for all $s \in C_R$ and $M_R \to 0$ as $R \to \infty$, then

$$\int_{C_R} e^{st} F(s)\,ds \to 0 \quad \text{as} \quad R \to \infty$$

Proof.

$$\int_{C_R} e^{st}F(s)ds = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{t(\varphi+Re^{i\theta})} F(\varphi+Re^{i\theta})Rie^{i\theta}\,d\theta \Rightarrow$$

$$\left|\int_{C_R} e^{st}F(s)ds\right| \le M_R R e^{t\varphi} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{tR\cos\theta}\,d\theta$$

$$= M_R R e^{t\varphi} \int_0^{\pi} e^{tR\cos(\theta+\frac{\pi}{2})}\,d\theta$$

$$= M_R R e^{t\varphi} \int_0^{\pi} e^{-tR\sin\theta}\,d\theta$$

(Ex!) (see the proof of Jordan's lemma)

$$< \frac{e^{t\varphi}M_R\pi}{t} \to 0 \quad \text{as } R\to\infty \quad ■$$

Then, letting $R\to\infty$ in $(**)$, we get $f(t) = \sum_{n=1}^{N} \operatorname{Res}\left(e^{st}F(s)\right)$, $t>0$.

Example. Find $f(t)$ if $F(s) = \frac{s}{(s^2+a^2)^2}$, $a>0$.

Solution. $e^{st}F(s)$ has double poles at $s=\pm ai$.

Clearly,

$$\operatorname*{Res}_{s=ai} e^{st}F(s) = \lim_{s\to ai} \frac{d}{ds}\left(\frac{e^{st}s}{(s^2+a^2)^2}(s-ai)^2\right) = \lim_{s\to ai}\frac{d}{ds}\frac{e^{st}s}{(s+ai)^2}$$

$$= \lim_{s\to ai} \frac{(e^{st}ts+e^{st})(s+ai)^2 - 2(s+ai)e^{st}s}{(s+ai)^4}$$

$$= \frac{(e^{ait}tai+e^{ait})(2ai)^2 - 2(2ai)e^{ait}ai}{(2ai)^4} = \frac{e^{ait}ait}{(2ai)^2}$$

$$\operatorname*{Res}_{s=-ai} e^{st}F(s) = \lim_{s\to -ai} \frac{d}{ds}\left(\frac{e^{st}s}{(s^2+a^2)^2}(s+ai)^2\right) = \lim_{s\to -ai}\frac{d}{ds}\frac{e^{st}s}{(s-ai)^2}$$

$$= \lim_{s\to -ai} \frac{(e^{st}ts+e^{st})(s-ai)^2 - 2(s-ai)e^{st}s}{(s-ai)^4}$$

$$= \frac{(e^{-ait}(-ait)+e^{-ait})(-2ai)^2 - 2(-2ai)e^{-ait}(-ai)}{(-2ai)^4} = \frac{e^{-ait}(-ait)}{(-2ai)^2}$$

and $$\operatorname*{Res}_{s=ai} e^{st}F(s) + \operatorname*{Res}_{s=-ai} e^{st}F(s) = \frac{e^{ait}\,ait}{(2ai)^2} + \frac{e^{-ait}(-ait)}{(-2ai)^2}$$

$$= \frac{ait}{2ai}\,\frac{(e^{ait}-e^{-ait})}{2ai} = \frac{t}{2a}\sin at.$$

If we show that $|F(s)| \le M_R \to 0$ as $R\to\infty$ on C_R, then $f(t) = \frac{t}{2a}\sin at$.

Clearly, $$\left|\frac{s}{(s^2+a^2)^2}\right| \le \frac{|\gamma+Re^{i\theta}|}{\big||(\gamma+Re^{i\theta}|-a^2\big|^2} \le \frac{\gamma+R}{\left((R-\gamma)^2-a^2\right)^2} \to 0 \quad \text{as } R\to\infty.$$

Thus, $f(t) = \frac{t}{2a}\sin at$.

<u>Remark</u> In many applications of Laplace transforms, the function $F(s)$ is analytic for all values of s in the finite plane except for an infinite set of isolated singular points s_n, $n=1,2,\dots$ that lie to the left of some vertical line $\operatorname{Re} s=\gamma$. Often the method just described for finding $f(t)$ can be modified in such a way that the finite sum is replaced by an infinite series of residues

$$f(t) = \sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n} \left(e^{st}F(s)\right), \quad t>0$$

The basic modification is to replace the vertical line segments L_R by vertical line segments L_N, $N=1,2,\dots$ from $s=\gamma-ib_N$ to $s=\gamma+ib_N$. The circular arcs C_R are then replaced by contours C_N, $N=1,2,\dots$ from $\gamma+ib_N$ to $\gamma-ib_N$ such that for each N, the sum L_N+C_N is a simple closed contour enclosing the singular points $s_1, s_2, \dots, s_N$. Once it is shown that

$$\lim_{N\to\infty} \int_{C_N} e^{st}F(s)\,ds = 0,$$ it follows that

$$f(t) = \sum_{n=1}^{\infty} \operatorname*{Res}_{s=s_n} \left(e^{st}F(s)\right).$$

Since it is often quite tedious to establish

$$\lim_{N\to\infty} \int_{C_N} e^{st} F(s)\,ds = 0$$

in any case, we shall accept in the example below that involve an infinite number of singularities. Thus our result will be only formal.

<u>Example.</u> Find $f(t)$ when $F(s) = \dfrac{\tanh s}{s^2}$.

<u>Solution.</u> $F(s) = \dfrac{\tanh s}{s^2} = \dfrac{1}{s^2}\dfrac{\sinh s}{\cosh s} \Rightarrow F(s)e^{st}$ has singularities

at $s=0$ and $\dfrac{e^s+e^{-s}}{2}=0 \Leftrightarrow e^{2s}=-1 \Leftrightarrow 2s=\pm(2n-1)\pi i \Leftrightarrow s_{\pm n} = \pm\dfrac{(2n-1)}{2}\pi i, n\in\mathbb{N}.$

It can be easily checked that each singularity is a simple pole.

Clearly,
$$\operatorname*{Res}_{s=0} e^{st}F(s) = \lim_{s\to 0} \frac{\sinh s}{s}\frac{e^{st}}{\cosh s} = \lim_{s\to 0}\frac{\sinh s}{s}$$
$$\overset{L.R}{=} \lim_{s\to 0} \cosh s = 1.$$

and
$$\operatorname*{Res}_{s=\pm s_n} e^{st}F(s) = \lim_{s\to \pm s_n} \frac{1}{s^2}\frac{\sinh s}{\cosh s} e^{st}(s\mp s_n)$$
$$= \frac{1}{s_n^2}\sinh(\pm s_n)\, e^{\pm s_n t} \lim_{s\to\pm s_n}\frac{(s\mp s_n)}{\cosh s}$$
$$\overset{L.R}{=} \frac{1}{s_n^2}\sinh(\pm s_n)\, e^{\pm s_n t}\frac{1}{\sinh(\pm s_n)} = \frac{e^{\pm s_n t}}{s_n^2}$$

Thus
$$\operatorname*{Res}_{s=s_n} e^{st}F(s) + \operatorname*{Res}_{s=-s_n} e^{st}F(s) = \left(\frac{2}{(2n-1)\pi i}\right)^2 \left(e^{\left(\frac{2n-1}{2}\right)\pi i t} + e^{-\left(\frac{2n-1}{2}\right)\pi i t}\right)$$
$$= -\frac{8}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi t}{2}$$

and so
$$f(t) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi t}{2}.$$

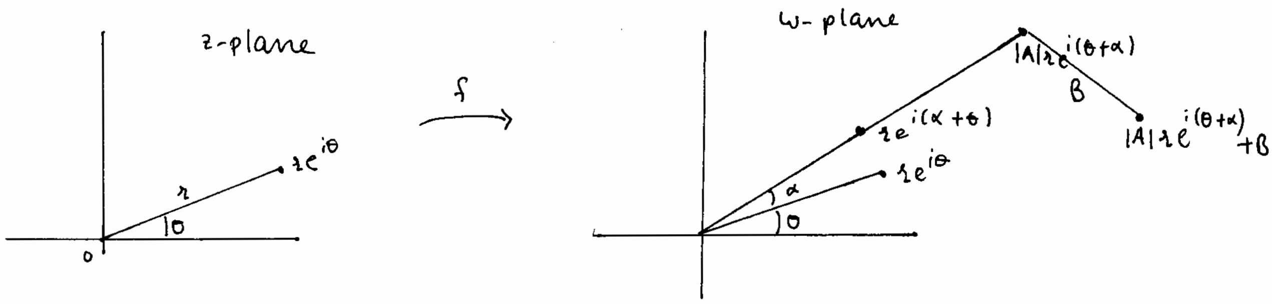
Conformal Mappings

Basic properties of conformal mappings.

Let $f(z) = Az+B$ be a linear mapping of $\mathbb{C}$ onto $\mathbb{C}$ where $A = |A|e^{i\alpha}$. Then

$$w = f(re^{i\theta}) = |A|e^{i\alpha} re^{i\theta} + B$$
$$= |A|r\, e^{i(\alpha+\theta)} + B$$

rotates the plane by the angle α, followed by a magnification by the factor $|A|$, followed by a translation by the vector B.



Hence, the mapping $f(z) = Az+B$ preserves the angles.

Now, let f be an analytic function in the domain $\mathcal{D}$ and let z_0 be a point in $\mathcal{D}$. If $f'(z_0) \neq 0$, we can express f in the form

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \eta(z)(z-z_0)$$

where $\eta(z) \to 0$ as $z \to z_0$.

So, if z is near z_0, then the transformation $w = f(z)$ has the linear approximation

$$S(z) = A + B(z-z_0) = Bz + A - Bz_0$$

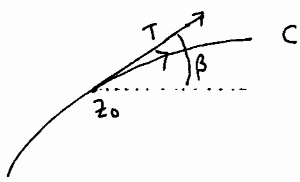
where $A = f(z_0)$ and $B = f'(z_0)$. Because $\eta(z) \to 0$ when $z \to z_0$, for points near z_0 the transformation $w = f(z)$ has an effect much like the linear mapping $w = S(z)$.

The effect of S is a rotation of the plane by the angle $\operatorname{Arg} f'(z_0)$, followed by a magnification by the factor $|f'(z_0)|$ and translation by the vector $f(z_0) - f'(z_0)z_0$.

So, $w = S(z)$ preserves the angles at the point z_0.

We'll now show that the mapping $w = f(z)$ also preserves angles at z_0.

Let $C: z(t) = x(t) + iy(t)$, $-1 \le t \le 1$ be a smooth curve passing through the point $z(0) = z_0$, and T be the vector tangent to C at z_0 given by $T = z'(0)$ and β be the angle of inclination of T with respect to the positive x-axis



The image of C under the mapping $w = f(z)$ is the curve K given by

$$K: w(t) = u(x(t), y(t)) + i\, v(x(t), y(t)).$$

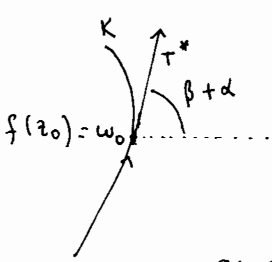
By the chain rule, we can show that a vector T^* tangent to K at $w_0 = f(z_0)$ is given by

$$T^* = w'(0) = f'(z_0)\, z'(0)$$

and so the angle of inclination of T^* with respect to the positive u-axis is

$$\gamma = \operatorname{Arg} f'(z_0) + \operatorname{Arg}(z'(0)) = \alpha + \beta$$

where $\alpha = \operatorname{Arg} f'(z_0)$

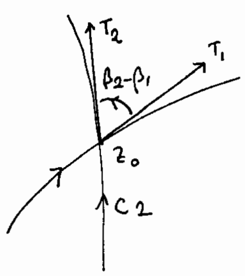


Therefore, the effect of the transformation $w = f(z)$ is to rotate the angle of inclination of the tangent vector T at z_0 through the angle $\alpha = \operatorname{Arg} f'(z_0)$ to obtain the angle of inclination of the tangent vector T^* at w_0.

A mapping $w = f(z)$ is said to be angle preserving, or conformal at z_0, if it preserves the angles between oriented curves in magnitude as well as in orientation.

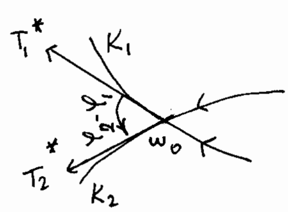
Theorem. Let f be an analytic function in the domain $\mathcal{D}$, and let z_0 be a point in $\mathcal{D}$. If $f'(z_0) \neq 0$, then f is conformal at z_0.

Proof. Let C_1 and C_2 be two smooth curves passing through z_0 with tangents T_1 and T_2, respectively.



Let β_1 and β_2 be the angles of inclination of T_1 and T_2, respectively. Then images are K_1 and K_2 that passes through $w_0 = f(z_0)$ have tangents T_1^* and T_2^*, respectively.

Then the angles of inclination φ_1 and φ_2 of T_1^* and T_2^* are given by $\varphi_1 = \alpha + \beta_1$ and $\varphi_2 = \alpha + \beta_2$ where $\alpha = \operatorname{Arg} f'(z_0)$. Hence $\varphi_2 - \varphi_1 = \beta_2 - \beta_2$. That is, the angle $\varphi_2 - \varphi_1$ from K_1 to K_2 is the same in magnitude and orientation as the angle $\beta_2 - \beta_1$ from C_1 to C_2. Therefore, the mapping $w = f(z)$ is conformal at z_0. ∎



Example. Show that the mapping $w = f(z) = \cos z$ is conformal at the points $z_1 = i$ and $z_2 = 1$, and determine the angle of rotation at given points.

Solution. $f'(z) = -\sin z$ and $f'(i) = -\sin i = -\frac{e^{ii} - e^{-ii}}{2i} = -\frac{e^{-1} - e^{1}}{2i} \Rightarrow$ $f'(i) = -i\,\frac{e^{1} - e^{-1}}{2} = -i \sinh 1 \neq 0$ and $f'(1) = -\sin 1 \neq 0$.
Since $\operatorname{Arg} f'(i) = -\frac{\pi}{2}$ and $\operatorname{Arg} f'(1) = \pi$, the angle of rotations at given points are $-\frac{\pi}{2}$ and π.

Let f be a nonconstant analytic function. If $f'(z_0) = 0$, then z_0 is called a critical point of f, and the mapping $w = f(z)$ is not conformal at z_0. The following theorem shows what happens at a critical point.

Theorem. Let f be analytic at z_0. If $f'(z_0) = 0$, $f''(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0$ and $f^{(k)}(z_0) \neq 0$, then the mapping $w = f(z)$ magnifies angles at the vertex z_0 by a factor k.

Proof. Clearly, f has the Taylor series expansion

$$f(z) = f(z_0) + a_k (z - z_0)^k + a_{k+1}(z - z_0)^{k+1} + \cdots,$$

or equivalently,

$$f(z) - f(z_0) = (z - z_0)^k \, g(z)$$

where g is analytic at z_0 and $g(z_0) = a_k \neq 0$.
Consequently, $\operatorname{Arg}(w - w_0) = k \operatorname{Arg}(z - z_0) + \operatorname{Arg} g(z)$
Thus, if C is a smooth curve that passes through z_0 and $z \to z_0$ along C, then $w \to w_0$

along the image curve K. The angle of inclinations of the tangent T to C and T^* to K, respectively, are then given by $\beta = \lim_{z \to z_0} \operatorname{Arg}(z - z_0)$ and $\varphi = \lim_{w \to w_0} \operatorname{Arg}(w - w_0)$

Then $\varphi = k\beta + \delta$ where $\delta = \operatorname{Arg}(g(z_0)) = \operatorname{Arg} a_k$.

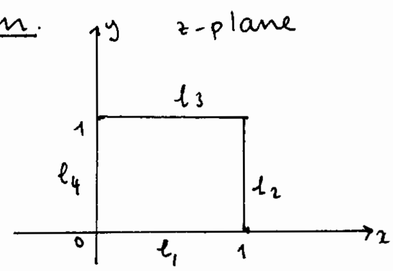
If C_1 and C_2 are two smooth curves that pass through z_0 and K_1 and K_2 are their images, then we have

$$\Delta\varphi = \varphi_2 - \varphi_1 = k(\beta_2 - \beta_1) = k\Delta\beta.$$

That is, the angle $\Delta\varphi$ from K_1 to K_2 is k times as large as the angle $\Delta\beta$ from C_1 to C_2. ∎

<u>Example</u>. Show that the mapping $w = f(z) = z^2$ maps the unit square $S = \{x+iy : 0 < x < 1,\ 0 < y < 1\}$ onto the region in the upper half-plane $\operatorname{Im} w > 0$, which lies under the parabolas $u = 1 - \frac{1}{4}v^2$ and $u = -1 + \frac{1}{4}v^2$

<u>Solution</u>.



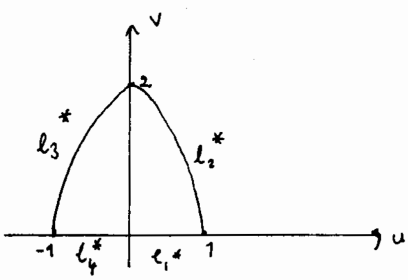
$z^2 = (re^{i\theta})^2 = r^2 e^{2i\theta}$ doubles the angles and squares the norm. Thus, it is easy to find the images of l_1 and l_4.

If $z \in l_2$, then $z = 1 + iy$, $0 \le y \le 1$ and $u + iv = z^2 = 1 - y^2 + 2iy$

$\Rightarrow u = 1 - y^2 = 1 - \left(\frac{2y}{2}\right)^2 = 1 - \left(\frac{v}{2}\right)^2 = 1 - \frac{v^2}{4}$, $0 \le v \le 2$.

If $z \in l_3$, then $z = x + i$, $0 \le x \le 1$ and $u + iv = z^2 = x^2 - 1 + 2ix \Rightarrow$

$u = x^2 - 1 = \left(\frac{2x}{2}\right)^2 - 1 = \left(\frac{v}{2}\right)^2 - 1 = \frac{v^2}{4} - 1 = -1 + \frac{v^2}{4}$, $0 \le v \le 1$



Note that, the angles at 1 and -1 are preserved but the angle at 0 is doubled.

Another property of a conformal mapping $w=f(z)$ is obtained by considering the modulus of $f'(z_0)$. If z_1 is near z_0, we have the approximation

$$w_1-w_0 = f(z_1)-f(z_0) \approx f'(z_0)(z_1-z_0).$$

Then, the distance $|w_1-w_0|$ between the images of points z_1 and z_0 is given approximately by $|f'(z_0)||z_1-z_0|$. Therefore, we say that the transformation $w=f(z)$ changes small distances near z_0 by the scale factor $|f'(z_0)|$. For example, the scale factor of the transformation $w=f(z)=z^2$ near the point $z_0=1+i$ is $|f'(1+i)| = |2(1+i)| = 2\sqrt{2}$.

We also need to say a few things about the inverse transformation $z=g(w)$ of a conformal mapping $w=f(z)$ near a point z_0, where $f'(z_0)\neq 0$. A complete justification of the following assertions relies on theorems studied in advanced calculus. We express the mapping $w=f(z)$ in the coordinate form

$$u=u(x,y) \quad \text{and} \quad v=v(x,y) \qquad (*)$$

The mapping in $(*)$ represents a transformation from the xy-plane into the uv-plane, and the Jacobian determinant, $J(x,y)$, is defined by $J(x,y) = \begin{vmatrix} u_x(x,y) & u_y(x,y) \\ v_x(x,y) & v_y(x,y) \end{vmatrix}$

The transformation in $(*)$ has a local inverse, provided $J(x,y)\neq 0$. Expanding the determinant and using the Cauchy-Riemann equations, we obtain $J(x_0,y_0) = u_x^2+v_x^2\big|_{(x_0,y_0)} = |f'(z_0)|^2 \neq 0$. Consequently, $f'(z_0)\neq 0$ imply that a local inverse $z=g(w)$ exists in a neighborhood of the point w_0. The derivative of g at w_0 is given by the familiar expression.

$$g'(w_0) = \lim_{w\to w_0} \frac{g(w)-g(w_0)}{w-w_0} = \lim_{z\to z_0} \frac{z-z_0}{f(z)-f(z_0)} = \frac{1}{f'(z_0)} = \frac{1}{f'(g(w_0))}$$

Bilinear transformations

The mapping

$$w = S(z) = \frac{az+b}{cz+d}, \quad \text{where} \quad ad-bc \neq 0$$

is called a <u>bilinear transformation</u>, a <u>Möbius transformation</u>, or a <u>linear fractional transformation</u>.

We can think of S as a mapping of $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ into $\hat{\mathbb{C}}$ by defining

$$S(\infty) = \lim_{z\to\infty} S(z) = \frac{a}{c} \quad \text{and} \quad S\left(-\frac{d}{c}\right) = \infty$$

Clearly, $w = \dfrac{az+b}{cz+d} \Rightarrow czw + dw = az + b \Rightarrow z(cw-a) = b - dw \Rightarrow$

$z = \dfrac{-dw+b}{cw-a}$, and so $S^{-1}(w) = \dfrac{-dw+b}{cw-a}$ gives a formula

for the inverse of S and hence shows that S is a one-to-one and onto mapping of $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$.

If $c=0$, S reduces to a linear map.

If $c \neq 0$,

$$S(z) = \frac{az+b}{cz+d} = \frac{a(cz+d)+bc-ad}{c(cz+d)} = \frac{a}{c} + \frac{bc-ad}{c}\,\frac{1}{cz+d}.$$

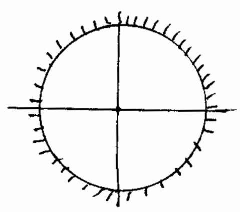
(Note that the condition $ad-bc\neq 0$ avoid that S reduces to a constant)

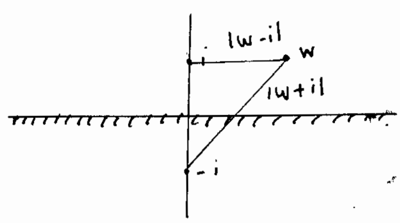
Thus, any bilinear mapping is a composition of the linear map $\xi = cz+d$, followed by the reciprocal transformation $\zeta = \dfrac{1}{\xi}$, followed by the linear map $w = \dfrac{a}{c} + \dfrac{bc-ad}{c}\zeta$.

We have seen before that linear maps transforms circles to circles, lines to lines; and the reciprocal maps transforms lines to lines or circles, circles to circles or lines. Therefore, we conclude that a bilinear transformation maps the class of circles and lines to itself.

<u>Example.</u> Show that $w = \frac{i(1-z)}{1+z}$ maps the unit disk $|z|<1$ one-to-one and onto the upper half plane $\text{Im}(w)>0$.

<u>Solution</u> clearly, $w = \frac{i(1-z)}{1+z} \Rightarrow w + wz = i - iz \Rightarrow$ $z(w+i) = 1-w \Rightarrow z = \frac{-w+i}{w+i}$. Image of $\{z : |z|<1\}$ is $\{w : \text{Im}\, w > 0\}$ if and only if $\left|\frac{-w+i}{w+i}\right| < 1$. That is, $|-w+i| < |w+i|$ or equivalently $|w-i| < |w+i|$. That is, the distance between w and i is less than the distance between w and $-i$. Clearly the set of such w's are exactly the set of points that lie in the upper half-plane.





<u>Remark.</u> The general formula for a bilinear transformation seems to involve four independent coefficients a, b, c, d. If $ad-bc \neq 0$ then either $a \neq 0$ or $c \neq 0$, and we can write either

$$S(z) = \frac{z + \frac{b}{a}}{\frac{cz}{a} + \frac{d}{a}} \quad \text{or} \quad S(z) = \frac{\frac{az}{c} + \frac{b}{c}}{z + \frac{d}{c}}, \text{ respectively}$$

Thus, we can determine a bilinear transformation

uniquely by three distinct image values
$S(z_1)=w_1$, $S(z_2)=w_2$ and $S(z_3)=w_3$.

More precisely,

<u>Theorem</u> There exists a unique bilinear transformation that maps three distinct points, z_1, z_2 and z_3, onto three distinct points, w_1, w_2, and w_3, respectively. An implicit formula for the mapping is given by

$$\frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w_1}$$

<u>Proof</u>. Exercise!

<u>Example</u>. Find a bilinear transformation that maps $-i, 1, i$ onto $-1, 0, 1$, respectively.

<u>Solution</u> put $z_1=-i$, $z_2=1$, $z_3=i$ and $w_1=-1$, $w_2=0$, $w_3=1$ in the previous theorem.

$$\frac{z+i}{z-i} \cdot \frac{1-i}{1+i} = \frac{w+1}{w-1} \cdot \frac{-1}{1} \Rightarrow$$

$$\frac{z-iz+i+1}{z+iz-i+1} = \frac{-w-1}{w-1} \Rightarrow$$

$$zw-\cancel{z}-i\cancel{zw}+iz+i\cancel{w}-i+w-\cancel{1} = -wz-\cancel{z}-i\cancel{zw}-iz+i\cancel{w}+i-w-\cancel{1} \Rightarrow$$

$$w(2z+2) = -2iz+2i \Rightarrow w = \frac{-iz+i}{z+1}.$$

<u>Example.</u> Find a bilinear transformation that maps $-2, -1-i, 0$ onto $-1, 0, 1$, respectively.

<u>Solution</u> $$\frac{z+2}{z} \cdot \frac{-1-i}{1-i} = \frac{w+1}{w-1} \cdot \frac{-1}{1} \Rightarrow$$

$$\frac{-z-iz-2-2i}{z-iz} = \frac{-w-1}{w-1} \Rightarrow$$

$$-\cancel{zw}+z-izw+\cancel{iz}-2w+2-2iw+2i = -\cancel{zw}-z+izw+\cancel{iz} \Rightarrow$$

$$w(-2iz-2-2i) = -2z-2-2i \Rightarrow$$

$$w = \frac{z+1+i}{iz+1+i}.$$

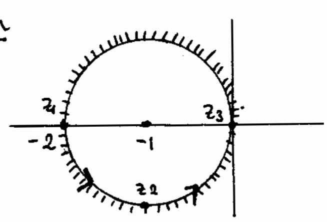
<u>Remark</u> Let $\mathcal{D}$ be a region bounded by either a circle or a straight line C, and z_1, z_2, z_3 be three distinct points that lie on C such that when we move along C from z_1 to z_3 through z_2, the region $\mathcal{D}$ stays on the left and similarly, let G be a region bounded by either a circle or a straight line K, and w_1, w_2, w_3 be three distinct points that lie on K such that when we move along K from w_1 to w_3 through w_2, the region G stays on the left. Then, the bilinear transformation

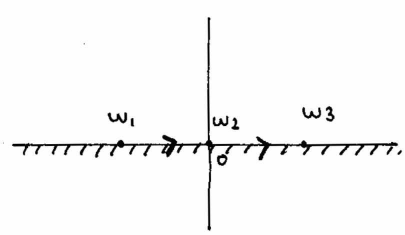
$$\frac{z-z_1}{z-z_3} \cdot \frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3} \cdot \frac{w_2-w_3}{w_2-w},$$

maps $\mathcal{D}$ to G in a one-to-one and onto manner.

<u>Example</u>. Find a bilinear map that maps $\mathcal{D} = \{z: |z+1| < 1\}$ onto the upper half plane.

<u>Solution</u>





The bilinear map which sends $-2, -1-i, 0$ onto $-1, 0, 1$ respectively, satisfies the desired property.
Hence by the previous example

$w = \dfrac{z+1+i}{iz+1+i}$ maps $D = \{z : |z+1| < 1\}$ onto the upper half plane.

<u>Corollary</u> (The implicit formula with a point at infinity)

Case i) $z_3 = \infty$

$$\frac{z - z_1}{z_2 - z_1} = \frac{w - w_1}{w - w_3} \cdot \frac{w_2 - w_3}{w_2 - w_1}$$

(z_1, z_2, ∞ are mapped onto w_1, w_2, w_3, respectively)

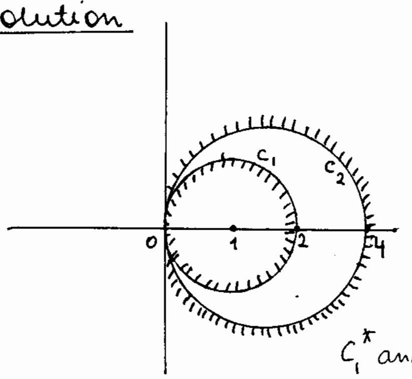
Case ii) $w_3 = \infty$

$$\frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} = \frac{w - w_1}{w_2 - w_1}$$

(z_1, z_2, z_3 are mapped onto w_1, w_2, ∞ respectively)

<u>Example</u> Show that the mapping $S(z) = \dfrac{-iz + 4i}{z}$ maps the region that lies inside the disk $|z-2| < 2$ and outside the circle $|z-1| = 1$ onto a horizontal strip.

<u>Solution</u>

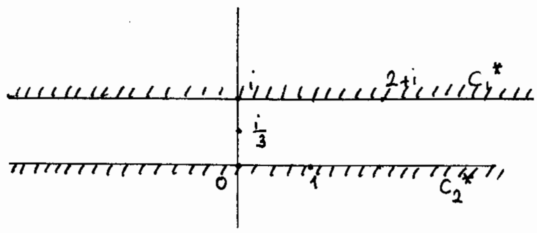


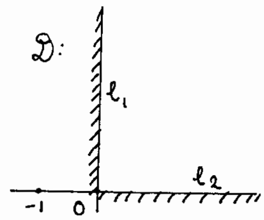
$C_1 \ni 0$, and $C_2 \ni 0 \Rightarrow$ image of C_1 and C_2 are lines in w-plane. Hence images of two nonzero points on C_1 and C_2 are enough to find the images C_1^* and C_2^* of C_1 and C_2. $2 \in C_1 \Rightarrow S(2) = \frac{2i}{2} = i \in C_1^*$

$S(1+i) = \dfrac{-i+1+4i}{1+i} = \dfrac{1+3i}{1+i}\,(1-i) = \dfrac{1-i+3i+3}{2} = 2+i \in C_1^* \Rightarrow C_1^*$ is the line which passes through i and $2+i$.

Similarly, $4 \in C_2 \Rightarrow S(4) = 0 \in C_2^*$ and $2+2i \in C_2 \Rightarrow$
$S(2+2i) = \dfrac{-2i+2+4i}{2+2i} = 1 \in C_2^*$ and hence C_2^* is the line which passes through 0 and 1.

And $S(3) = \frac{i}{3}$ lies in the image set.



Example. Find the image of $\mathcal{D}$: 

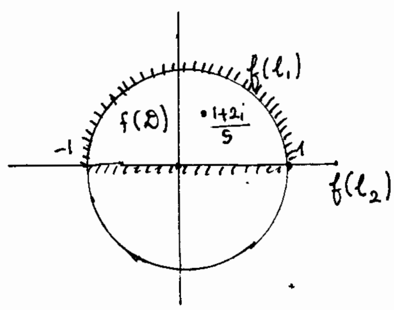
under the transformation $w = f(z) = \dfrac{z-1}{z+1}$.

Solution $-1 \to \infty \Rightarrow f(\ell_1)$ is a circle and $f(\ell_2)$ is a line

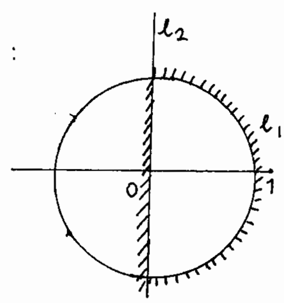
$0 \in \ell_1 \Rightarrow -1 \in f(\ell_1)$ and $\infty \in \ell_1 \Rightarrow 1 \in f(\ell_1)$

$f(\ell_2) = \mathbb{R}$ (that is f sends real numbers to real numbers)

$\ell_1 \perp \ell_2 \Rightarrow$ (by conformality) $f(\ell_1) \perp f(\ell_2)$



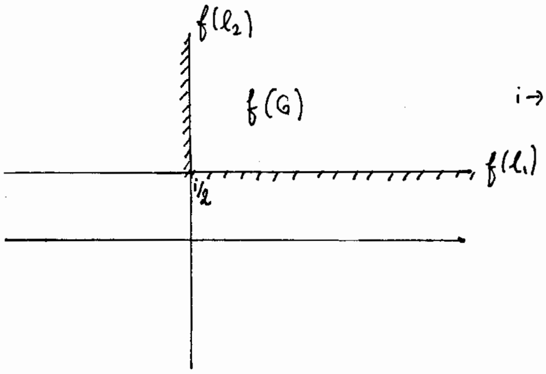
$1+i \in \mathcal{D} \Rightarrow f(1+i) = \dfrac{i}{2+i}_{(2-i)} = \dfrac{1+2i}{5} \in f(\mathcal{D})$

Example. Find the image of G: 

under the transformation $w = f(z) = \dfrac{1}{z-i}$

Solution $\quad i \to \infty \Rightarrow f(\ell_2)$ is a straight line

$0 \to i\,,\ \infty \to 0 \Rightarrow f(\ell_2) = i\mathbb{R}$ (the imaginary axis)



$i \to \infty \Rightarrow f(\ell_1)$ is a straight line

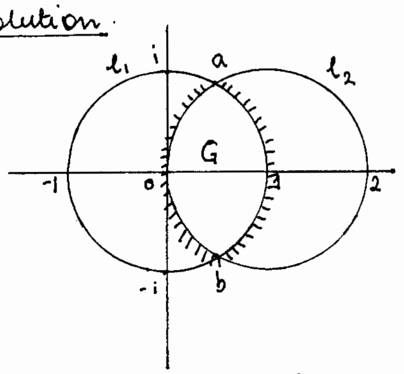
$\ell_1 \perp \ell_2 \Rightarrow f(\ell_1) \perp f(\ell_2)$

$-i \in \ell_1 \Rightarrow f(-i) = \frac{i}{2} \in f(\ell_1)$

Example Let $G_1 = \{z : |z| < 1\}$ and $G_2 = \{z : |z-1| < 1\}$ and $G = G_1 \cap G_2$ as shown below. Find the image of G under the transformation

$$w = f(z) = \frac{z-a}{z-b}$$

Solution.

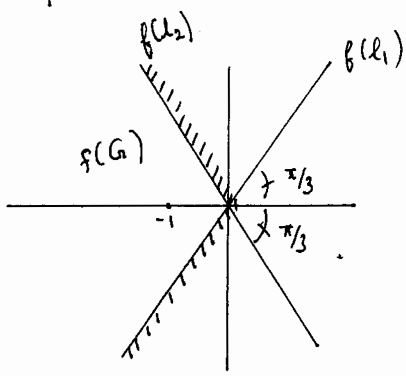


Clearly, $a = e^{i\frac{\pi}{3}}$, $b = e^{-i\frac{\pi}{3}}$

$b \to \infty \Rightarrow f(\ell_1)$ and $f(\ell_2)$ are straight lines

$0 \in \ell_2 \Rightarrow f(0) = \dfrac{-e^{i\frac{\pi}{3}}}{-e^{-i\pi/3}} = e^{i\frac{2\pi}{3}} \in f(\ell_2)$

$a \in \ell_2 \Rightarrow f(a) = 0 \in f(\ell_2) \cap f(\ell_1)$



$-1 \in \ell_1 \Rightarrow f(-1) = \dfrac{-1-\frac{1}{2}-\frac{i\sqrt{3}}{2}}{-1-\frac{1}{2}+\frac{i\sqrt{3}}{2}} = \dfrac{-\frac{3}{2}-i\frac{\sqrt{3}}{2}}{-\frac{3}{2}+\frac{i\sqrt{3}}{2}}$

$= \dfrac{\frac{9}{4} - \frac{2i3\sqrt{3}}{4} - \frac{3}{4}}{\frac{9}{4}+\frac{3}{4}} = \dfrac{6-6i\sqrt{3}}{12}$

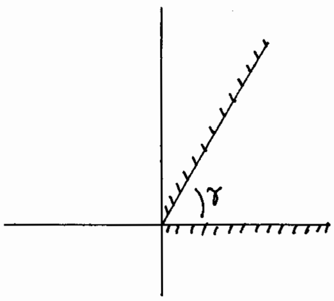
$= \frac{1}{2} - \frac{i\sqrt{3}}{2} \in f(\ell_1)$

$f\left(\frac{1}{2}\right) = \dfrac{\frac{1}{2}-\frac{1}{2}-\frac{i\sqrt{3}}{2}}{\frac{1}{2}-\frac{1}{2}+\frac{i\sqrt{3}}{2}} = -1 \in f(G)$

Mapping by $w=z^n$, $n>0$

Let us consider the branch of z^n in $A_\gamma = \{z : 0 < \arg z < \gamma\}$, $\gamma \le 2\pi$ defined by

$$z^n = |z|^n e^{in\arg z} \quad \text{where} \quad 0 < \arg z < \gamma, \quad \gamma \le 2\pi$$



Under which condition on γ the function z^n is univalent in A_γ.

Take $z_1 \neq z_2$, $z_1, z_2 \in A_\gamma$ and assume

$$z_1^n = z_2^n \iff |z_1|^n e^{in\arg z_1} = |z_2| e^{in\arg z_2}$$

$$\iff |z_1|^n = |z_2|^n \quad \text{and} \quad n\arg z_1 = n\arg z_2 + 2k\pi$$

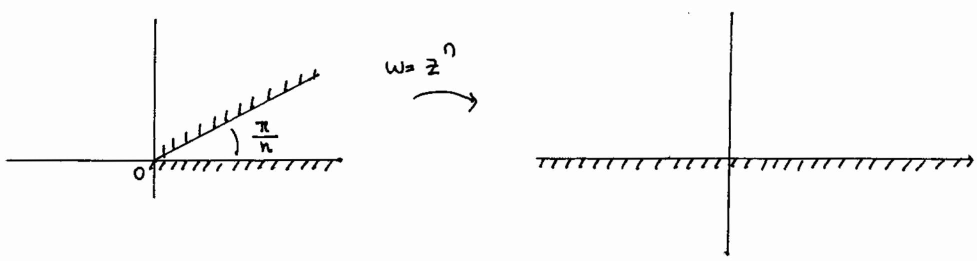
$$\iff |z_1| = |z_2| \quad \text{and} \quad \arg z_1 = \arg z_2 + \frac{2k\pi}{n} \quad \text{for some } k \in \mathbb{Z}$$

If $z_1 \neq z_2$ then $k \neq 0$ and $\gamma > |\arg z_1 - \arg z_2| \geq \frac{2\pi}{n} \Rightarrow$ $\gamma > \frac{2\pi}{n}$. Therefore, we have

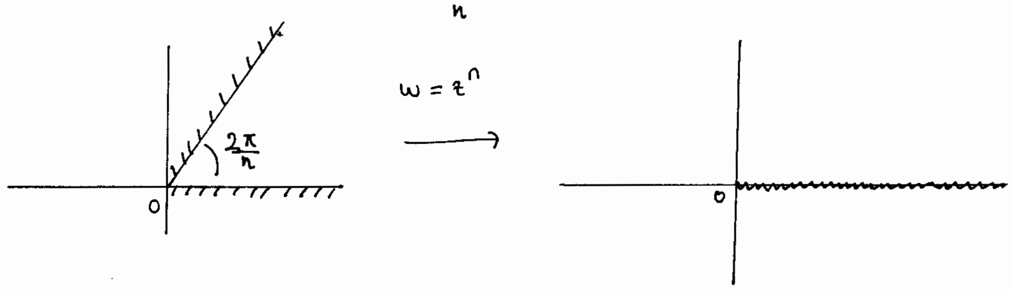
<u>Theorem</u> If $\gamma \le \frac{2\pi}{n}$, then z^n is one-to-one in A_γ.

and the image of A_γ under the mapping $w=z^n$ is $A_{n\gamma}$.

<u>Corollary</u>. 1. $w=z^n$ maps $A_{\frac{\pi}{n}}$ onto the upper half plane

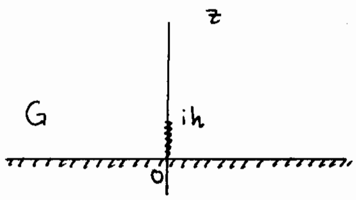


2. $w=z^n$ maps $A_{\frac{2\pi}{n}}$ onto the cut plane $\mathbb{C}\setminus[0,\infty)$

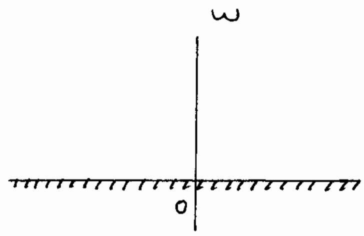


<u>Remark</u> Image of rays are rays and image of circular arc are circular arcs under the mapping z^n.

Example. Map

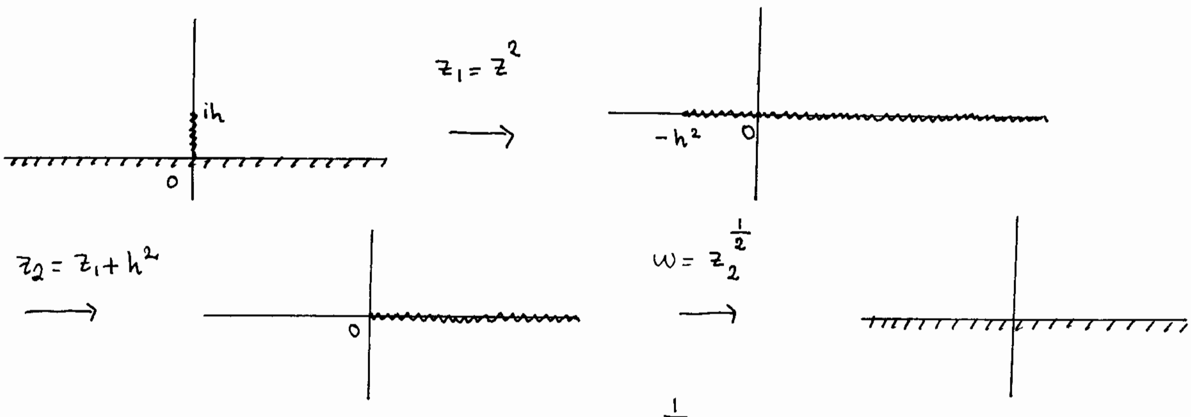


onto



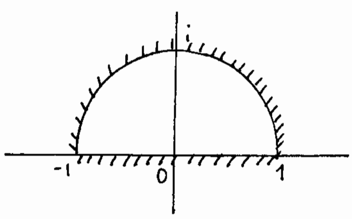
$(G = \{z : \operatorname{Im} z > 0\} \setminus \{z : \operatorname{Re} z = 0,\ 0 \le \operatorname{Im} z \le h\})$

Solution

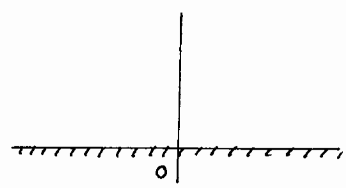


$$w = (z^2 + h^2)^{\frac{1}{2}}.$$

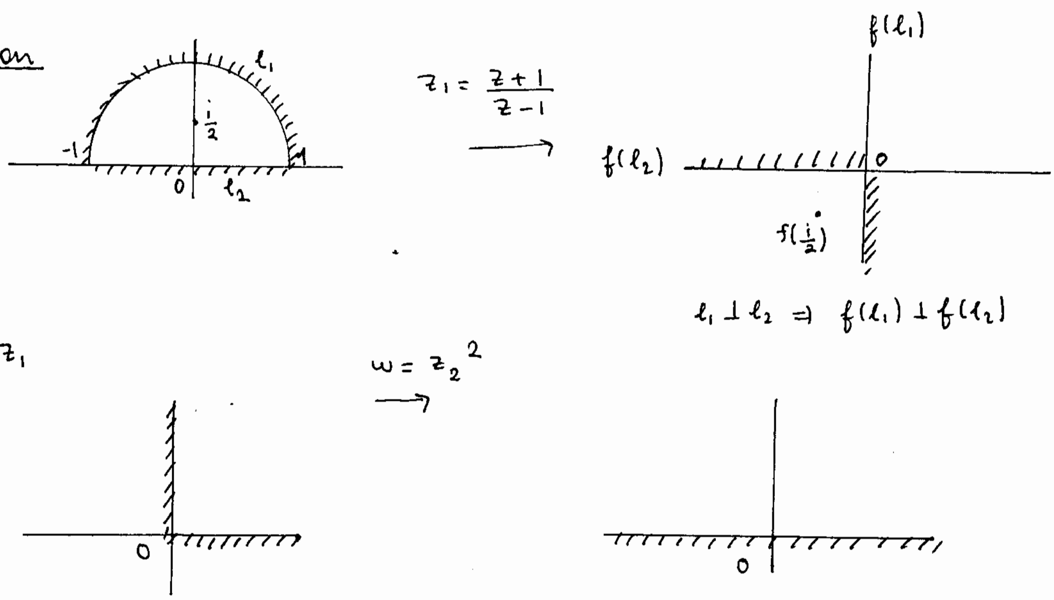
Example. Map



onto



Solution

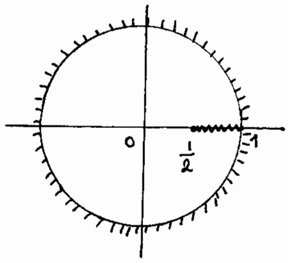


$-1 \to 0$
$1 \to \infty$
$f(l_2) = l_2$
$f(\frac{i}{2}) = \frac{-3-4i}{5}$

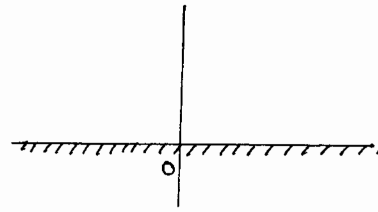
$l_1 \perp l_2 \Rightarrow f(l_1) \perp f(l_2)$

$$w = \left(\frac{z+1}{z-1}\right)^2.$$

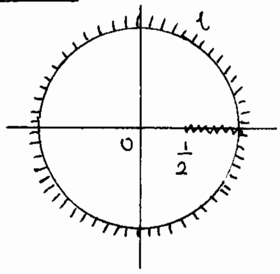
Example map



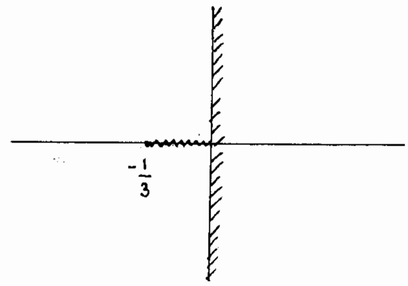
onto



Solution.



$z_1 = \dfrac{z-1}{z+1}$ $\longrightarrow$



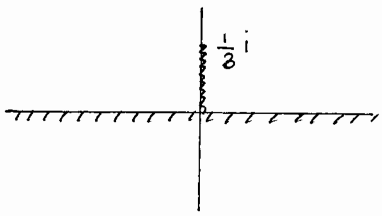
$1 \to 0$

$-1 \to \infty$

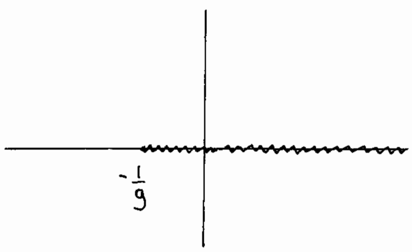
$\mathbb{R} \to \mathbb{R}$

$l \perp \mathbb{R} \Rightarrow f(l) \perp \mathbb{R}$

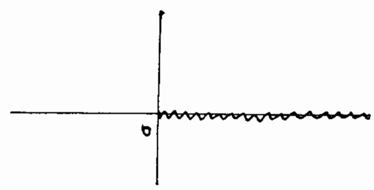
$z_2 = e^{-\frac{i\pi}{2}} = -i z_1$ $\longrightarrow$



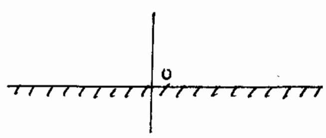
$z_3 = z_2^{\,2}$ $\longrightarrow$



$z_4 = z_3 + \frac{1}{9}$



$z_5 = z_4^{1/2}$ $\longrightarrow$



$$w = \left(-\left(\frac{z-1}{z+1}\right)^2 + \frac{1}{9}\right)^{\frac{1}{2}}$$

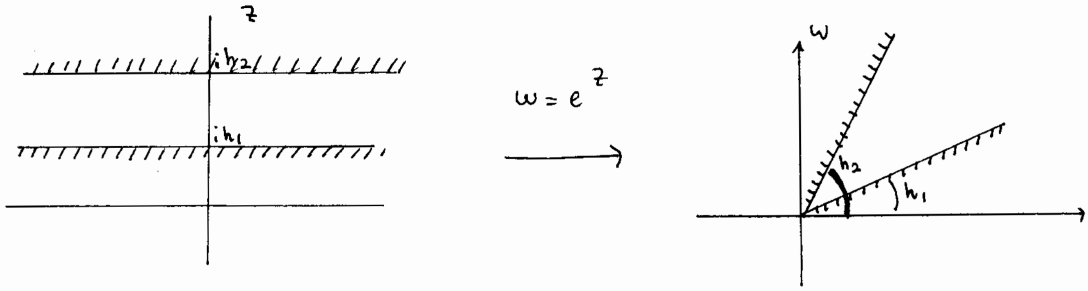
mapping by e^z

$z_1 = x_1 + iy_1 \neq z_2 = x_2 + iy_2$ but $e^{z_1} = e^{z_2} \Leftrightarrow e^{x_1} e^{iy_1} = e^{x_2} e^{iy_2} \Leftrightarrow$

$x_1 = x_2$ and $y_1 = y_2 + 2k\pi$, $k \in \mathbb{Z}\setminus\{0\} \Leftrightarrow |y_1 - y_2| \geq 2\pi$.

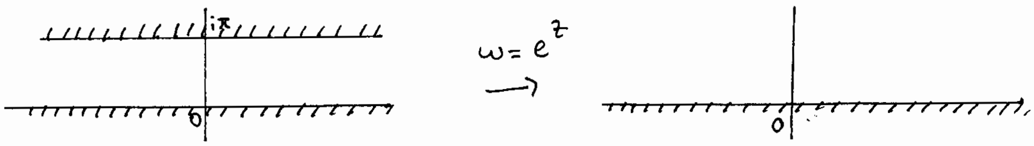
Therefore, e^z is one-to-one in any region which does not contain any pair of points z_1, z_2 such that $\operatorname{Re} z_1 = \operatorname{Re} z_2$, $|\operatorname{Im} z_1 - \operatorname{Im} z_2| \geq 2\pi$.

__Theorem__ e^z maps the horizontal strip $\{z : h_1 < \operatorname{Im} z < h_2\}$ with $h_2 - h_1 \leq 2\pi$ onto the angle $\{w : h_1 < \arg w < h_2\}$

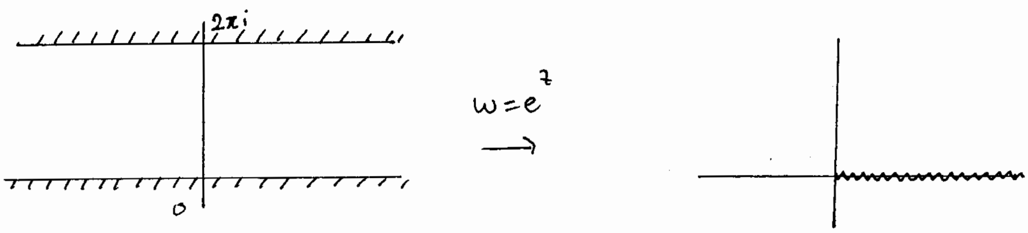


__Example__

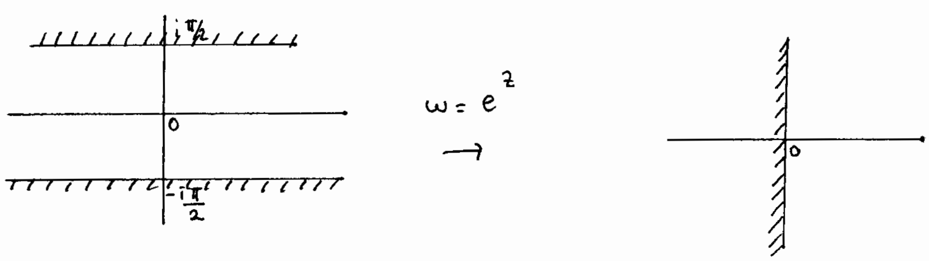
1. e^z maps $\{z : 0 < \operatorname{Im} z < \pi\}$ onto $\{w : \operatorname{Im} w > 0\}$



2. e^z maps $\{z : 0 < \operatorname{Im} z < 2\pi\}$ onto $\mathbb{C} \setminus \{w : 0 \leq w < \infty\}$

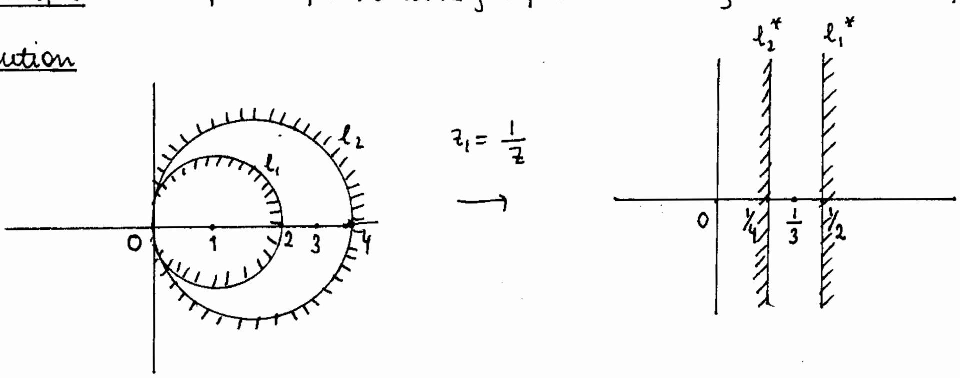


3. e^z maps $\{z : |\operatorname{Im} z| < \frac{\pi}{2}\}$ onto $\{w : \operatorname{Re} w > 0\}$



__Example__ Map $\{z : |z-2| < 2\} \setminus \{z : |z-1| \leq 1\}$ onto $\{w : \operatorname{Im} w > 0\}$

__Solution__



$\mathbb{R} \to \mathbb{R}$

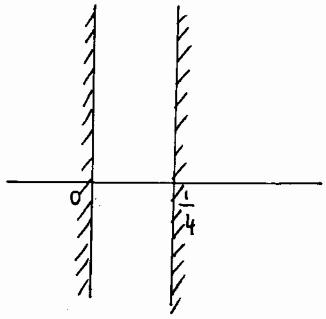
$\ell_1 \perp \mathbb{R} \Rightarrow \ell_1^* \perp \mathbb{R}$

$\ell_2 \perp \mathbb{R} \Rightarrow \ell_2^* \perp \mathbb{R}$

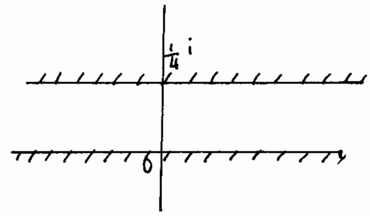
$2 \in \ell_1 \Rightarrow \frac{1}{2} \in \ell_1^*$

$4 \in \ell_4 \Rightarrow \frac{1}{4} \in \ell_2^*$

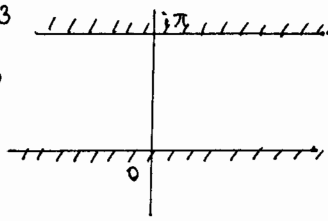
$z_2 = z_1 - \frac{1}{4}$ $\longrightarrow$



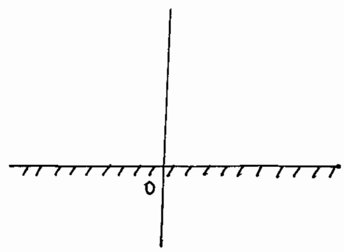
$z_3 = e^{i\frac{\pi}{2}} z_2 = i z_2$ $\longrightarrow$



$z_4 = 4\pi z_3$ $\longrightarrow$



$w = e^{z_4}$ $\longrightarrow$

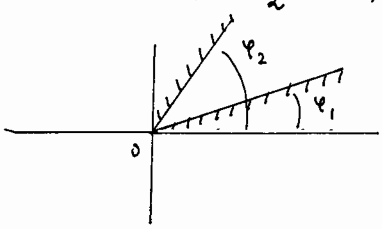


Mapping by $\log z$

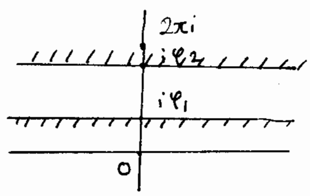
we shall consider the branch of $\log z$ in $\mathbb{C} \setminus \{z : 0 \leq z < \infty\}$ defined by $\log z = \ln|z| + i \arg z$, $\quad 0 < \arg z < 2\pi$

Clearly, $\quad z_1 \neq z_2 \Rightarrow \log z_1 \neq \log z_2$

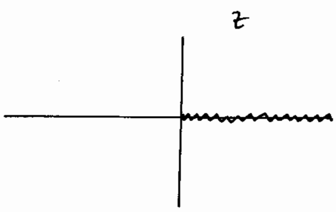
and



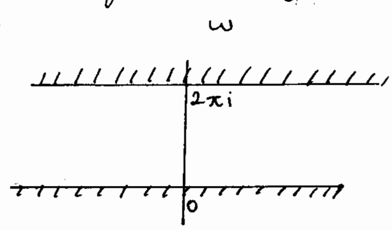
is mapped onto



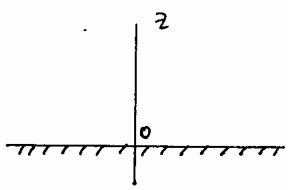
Corollary 1. $\mathbb{C} \setminus \{z : 0 \leq z < \infty\}$ is mapped by $w = \log z$ onto $\{w : 0 < \operatorname{Im} w < 2\pi\}$



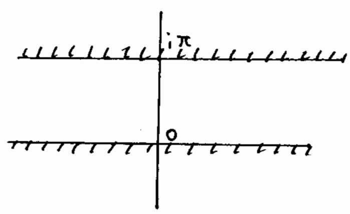
$\xrightarrow{w = \log z}$



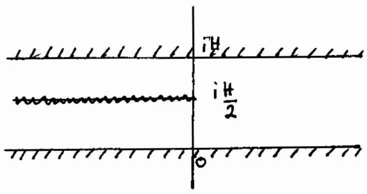
2. $\{z : \operatorname{Im} z > 0\}$ is mapped by $w = \log z$ onto $\{w : 0 < \operatorname{Im} w < \pi\}$



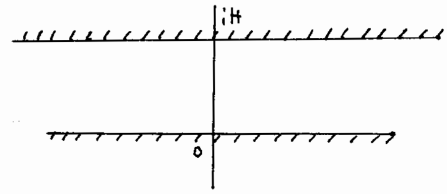
$\xrightarrow{w = \log z}$



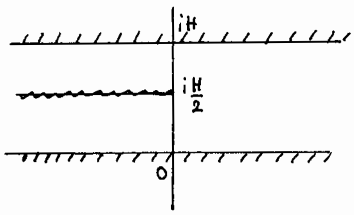
Example. map



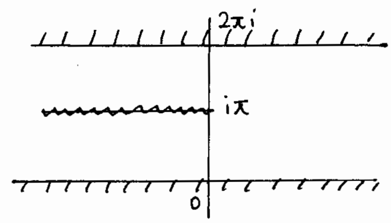
onto



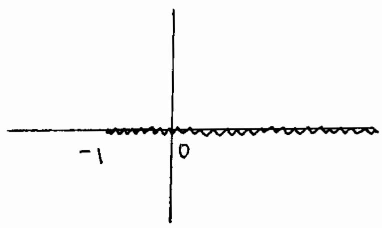
Solution.



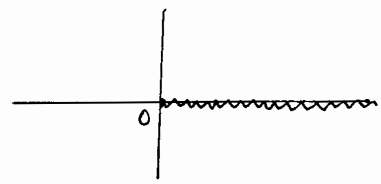
$$z_1 = \frac{2\pi}{H} z \longrightarrow$$



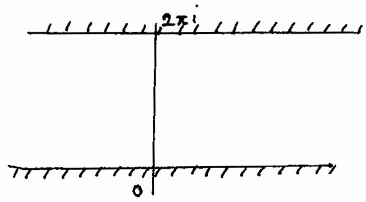
$$z_2 = e^{z_1} \longrightarrow$$



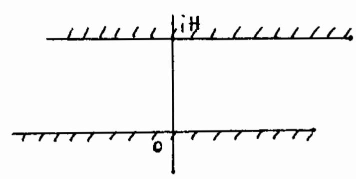
$$z_3 = z_2 + 1 \longrightarrow$$



$$z_4 = \log z_3 \longrightarrow$$



$$w = \frac{H}{2\pi} z_4$$



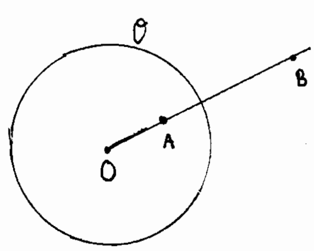
$$w = \frac{H}{2\pi} \log \left(e^{\frac{2\pi}{H} z} + 1 \right)$$

Bilinear transformations (Revisited)

Points A and B are called symmetrical with respect to the circle $\mathcal{O}$ if

i) both A and B are situated on the same ray with the starting point O (center of the circle)

ii) $OA \cdot OB = R^2$ where R is the radius of the circle $\mathcal{O}$.

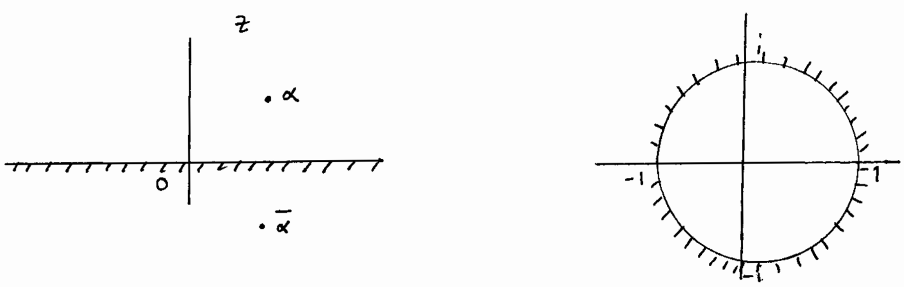


By definition 0 and ∞ are symmetrical with respect to $\mathcal{O}$.

Theorem (Preservation of symmetry under bilinear transformation) Let $\mathcal{C}$ be a generalized circle (a line or a circle). Let z_1, z_2 be two points symmetric with respect to $\mathcal{C}$. Let $w = B(z)$ be a bilinear transformation. Then, $w_1 = B(z_1)$ and $w_2 = B(z_2)$ are symmetrical with respect to $B(\mathcal{C})$.

Proof. Omitted. ∎

General form of bilinear transformation mapping the upper half plane onto the unit disk.



Let $w = \dfrac{az+b}{cz+d}$ be a transformation mapping the upper half plane onto the unit disk. Then there is a point α which is mapped to 0 and so $\bar{\alpha}$ is mapped to ∞. Then

$$a\alpha + b = 0 \quad \text{and} \quad c\bar{\alpha} + d = 0 \quad \text{and hence}$$

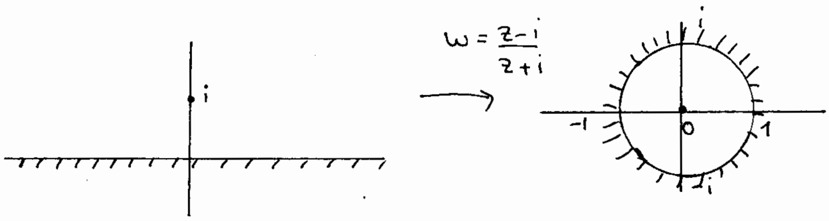
$w = \dfrac{a}{c}\,\dfrac{z-\alpha}{z-\bar{\alpha}}$. Since the image of the real line must be the unit circle,

$$\left|\frac{a}{c}\,\frac{z-\alpha}{z-\bar{\alpha}}\right| \doteq \left|\frac{a}{c}\right| = 1 \text{ whenever } z \in \mathbb{R} \text{ and hence}$$

$\dfrac{a}{c} = e^{i\gamma}$ for some $\gamma \in \mathbb{R}$ and $w = e^{i\gamma}\dfrac{z-\alpha}{z-\bar{\alpha}}$. Thus, we have the following theorem.

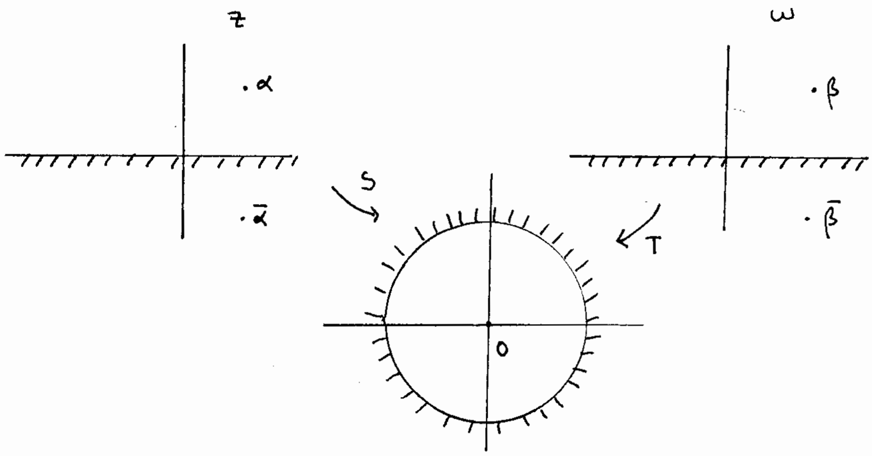
Theorem The general form of bilinear transformation which maps the upper half plane onto the unit disk is $w = e^{i\gamma}\dfrac{z-\alpha}{z-\bar{\alpha}}$ where $\gamma \in \mathbb{R}$, $\operatorname{Im}\alpha > 0$.

<u>Example.</u> $w = \frac{z-i}{z+i}$ maps the upper half plane to the unit disk $(\alpha = i, \gamma = 0)$



in particular, i is mapped to 0.

<u>General form of bilinear transformation which maps the upper half plane onto itself.</u>



$S(z) = e^{i\gamma_1} \frac{z-\alpha}{z-\bar{\alpha}}$ maps the upper half plane onto the unit disk and α to 0.

$T(w) = e^{i\gamma_2} \frac{w-\beta}{w-\bar{\beta}}$ maps the upper half plane onto the unit disk and β to 0

Thus, $w = T^{-1}S(z)$ maps the upper half plane onto itself and α to β. Therefore,

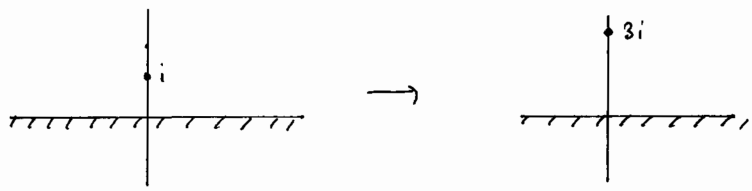
$$T(w) = T(T^{-1}(S(z)) = S(z), \text{ and}$$

$$e^{i\gamma_1} \frac{z-\alpha}{z-\bar{\alpha}} = e^{i\gamma_2} \frac{w-\beta}{w-\bar{\beta}} \Rightarrow \frac{z-\alpha}{z-\bar{\alpha}} = e^{i\gamma} \frac{w-\beta}{w-\bar{\beta}} \text{ for some}$$

$\gamma \in \mathbb{R}$.

<u>Example</u> Find a bilinear transformation which maps the upper half plane onto itself and i to $3i$.

Solution.



Taking (for example) $\gamma = \frac{\pi}{2}$ we get

$$\frac{z-i}{z+i} = i\,\frac{w-3i}{w+3i} = \frac{iw+3}{w+3i} \implies$$

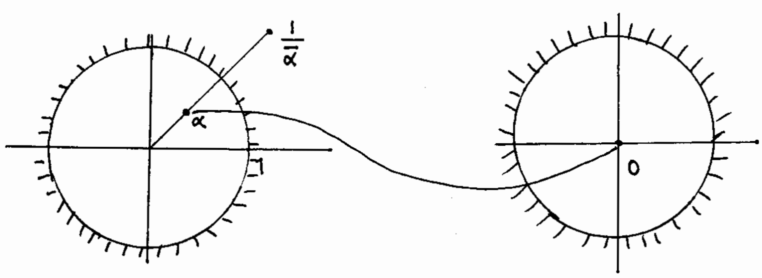
$$zw + 3iz - iw + 3 = izw - w + 3z + 3i \implies$$

$$w(z - i - iz + 1) = -3iz - 3 + 3z + 3i \implies$$

$$w = 3\,\frac{(1-i)z - 1 + i}{(1-i)z + 1 - i} = 3\,\frac{z-1}{z+1}.$$

Thus, $w = 3\,\frac{z-1}{z+1}$ maps the upper half plane onto the upper half plane and i to $3i$ (Check!)

General form of bilinear transformation which maps the unit disk onto itself.



α is mapped to the origin

Note that α and $\frac{1}{\bar\alpha}$ are symmetrical with respect to the unit circle and so $\frac{1}{\bar\alpha}$ is mapped to ∞

if $w = \frac{az+b}{cz+d}$ is the desired map, we have

$$a\alpha + b = 0 \qquad\qquad \text{and so} \qquad w = \frac{a}{c}\,\frac{z-\alpha}{z - \frac{1}{\bar\alpha}} = -\frac{a\bar\alpha}{c}\,\frac{z-\alpha}{1-\bar\alpha z}$$
$$c\,\frac{1}{\bar\alpha} + d = 0$$

Since the unit circle is mapped on the unit circle, $\left|-\frac{a\bar\alpha}{c}\right|\left|\frac{e^{i\theta}-\alpha}{1-\bar\alpha e^{i\theta}}\right| = 1$ for any $\theta \in \mathbb{R}$

Since $\left|\dfrac{e^{i\theta}-\alpha}{1-\bar{\alpha}e^{i\theta}}\right| = \left|\dfrac{e^{i\theta}-\alpha}{e^{-i\theta}-\bar{\alpha}}\right| = 1$, we must have $\left|\dfrac{-a\bar{\alpha}}{c}\right| = e^{i\gamma}$

for some $\gamma \in \mathbb{R}$, and hence $w = e^{i\gamma}\dfrac{z-\alpha}{1-\bar{\alpha}z}$.

Thus, we have proved the following theorem.

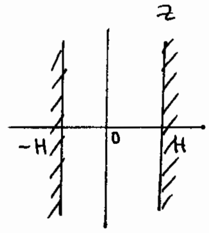
<u>Theorem</u> The general form of the bilinear transformation which maps the unit disk onto itself is $w = e^{i\gamma}\dfrac{z-\alpha}{1-\bar{\alpha}z}$, where $\gamma \in \mathbb{R}$ and $|\alpha|<1$. ■

$S(z) = e^{i\gamma_1}\dfrac{z-\alpha}{1-\bar{\alpha}z}$ maps $\alpha \to 0$

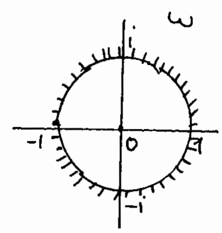
$T(w) = e^{i\gamma_2}\dfrac{z-\beta}{1-\bar{\beta}w}$ maps $\beta \to 0$

Then $\dfrac{z-\alpha}{1-\bar{\alpha}z} = e^{i\gamma}\dfrac{w-\beta}{1-\bar{\beta}w}$ maps the unit disk onto itself and α to β.

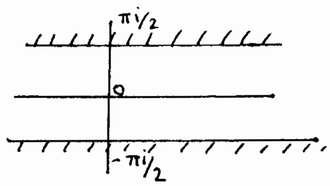
<u>Example</u>. Map



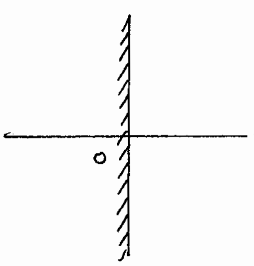
onto



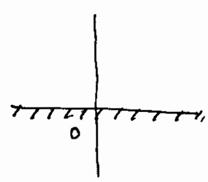
<u>Solution</u> $z_1 = iz\dfrac{\pi}{2H} \to$



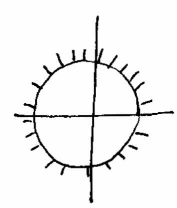
$\xrightarrow{z_2 = e^{z_1}}$



$z_3 = iz_2$



$\xrightarrow{w = \frac{z_3-i}{z_3+i}}$



$$w = \frac{z_2-1}{z_2+1} = \frac{e^{iz\frac{\pi}{2H}}-1}{e^{iz\frac{\pi}{2H}}+1} = \frac{e^{iz\frac{\pi}{4H}}-e^{-iz\frac{\pi}{4H}}}{e^{iz\frac{\pi}{4H}}+e^{-iz\frac{\pi}{4H}}} = i\tan\left(\frac{\pi z}{4H}\right)$$

Zhukovski's transformation

Zhukovski's transformation is defined by

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

clearly, if $z_1 \neq z_2$ and $\frac{1}{2}\left(z_1 + \frac{1}{z_1}\right) = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$, then $z_1 - z_2 = \frac{1}{z_2} - \frac{1}{z_1} = \frac{z_1 - z_2}{z_1 z_2}$ and so $z_1 z_2 = 1$.

Therefore, Zhukovski's transformation is one-to-one in any region G satisfying the condition:

"G does not contain both z and $\frac{1}{z}$".

For example, Zhukovski's transformation is one-to-one on

(i) $G = \{z : |z| < 1\}$

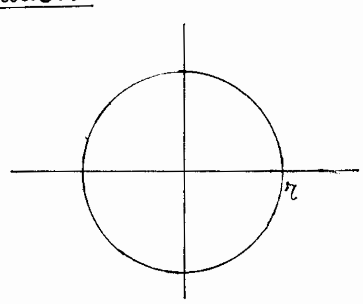
(ii) $G = \{z : |z| > 1\}$

(iii) $G = \{z : \operatorname{Im} z > 0\}$

(iv) $G = \{z : \operatorname{Im} z < 0\}$

<u>Example</u> Find the image of the circle $z = re^{i\varphi}$, $0 \leq \varphi \leq 2\pi$ under the Zhukovski's transformation.

<u>Solution</u>



$$w = \frac{1}{2}\left(re^{i\varphi} + \frac{1}{r}e^{-i\varphi}\right)$$

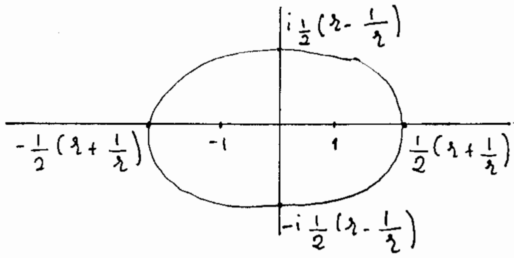
$w = u + iv$ where

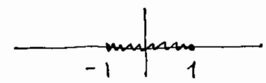
$u = \frac{1}{2}\left(r + \frac{1}{r}\right)\cos\varphi$, $0 \leq \varphi \leq 2\pi$

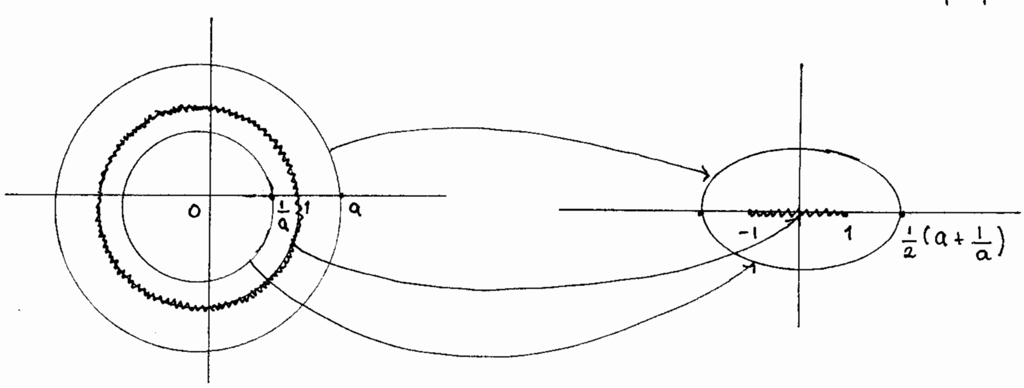
$v = \frac{1}{2}\left(r - \frac{1}{r}\right)\sin\varphi$

If $r \neq 1$, then $$\frac{u^2}{\left[\frac{1}{2}\left(r+\frac{1}{r}\right)\right]^2} + \frac{v^2}{\left[\frac{1}{2}\left(r-\frac{1}{r}\right)\right]^2} = 1$$

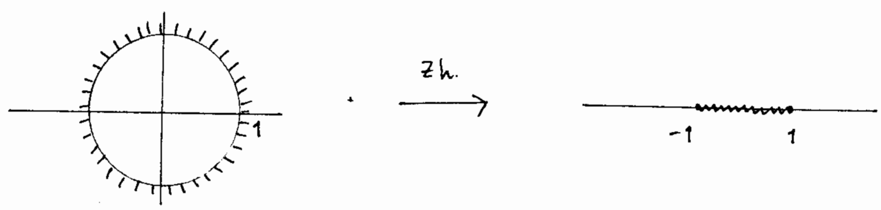
which defines an ellipse in the uv-plane

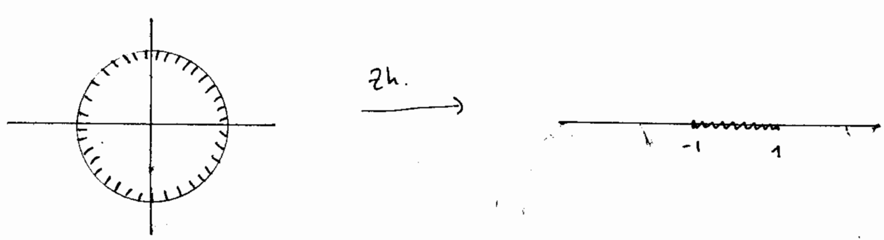


if $r=1$, then $u = \cos\varphi$ and $v=0$ and this defines a line segment in the uv-plane 

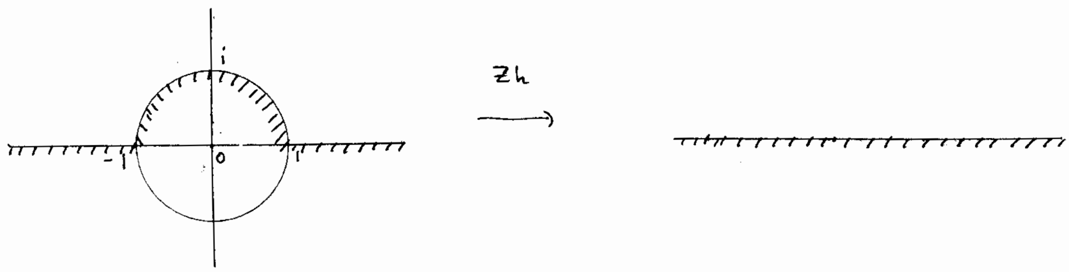


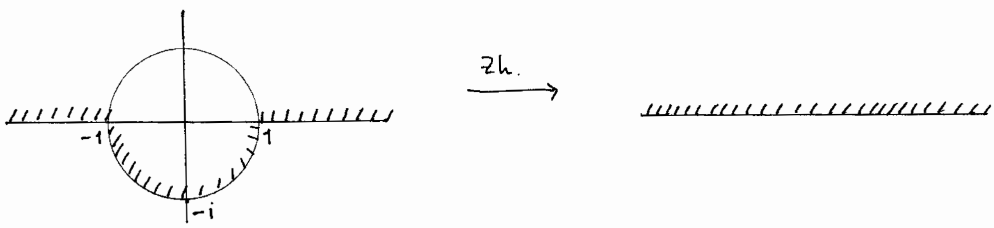
<u>Theorem.</u> Zhukovski's transformation maps both $\{z : |z| < 1\}$ and $\{z : |z| > 1\}$ onto $\mathbb{C} \setminus [-1,1]$:



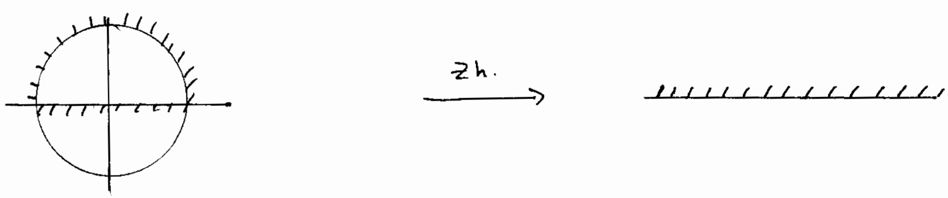


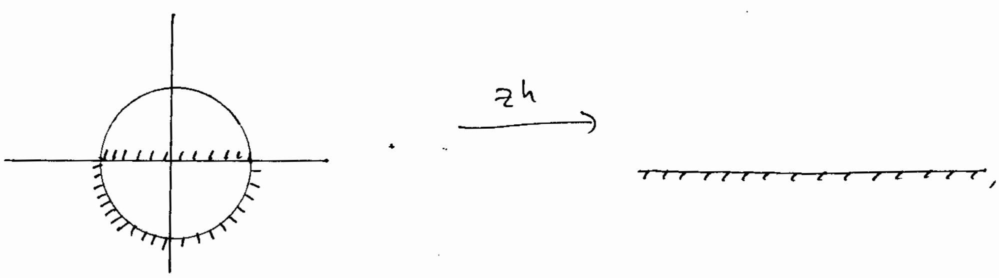
<u>Theorem</u> Zhukovski's transformation maps $G_1 = \{z : |z|>1,\ \operatorname{Im} z > 0\}$ onto $\{w : \operatorname{Im} w > 0\}$ and $G_2 = \{z : |z|>1,\ \operatorname{Im} z<0\}$ onto $\{w : \operatorname{Im} w < 0\}$.



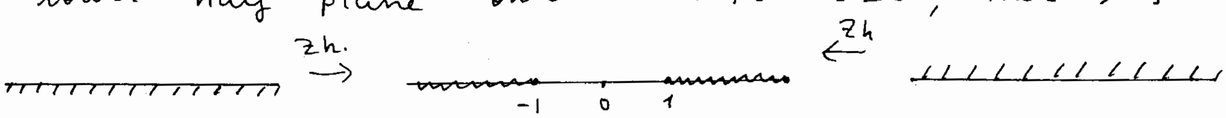


<u>Theorem.</u> Zhukovski's transformation maps $G_3 = \{z : |z|<1,\ \operatorname{Im} z > 0\}$ onto $\{w : \operatorname{Im} w < 0\}$ and $G_4 = \{z : |z|<1,\ \operatorname{Im} z < 0\}$ onto $\{w : \operatorname{Im} w > 0\}$:





<u>Theorem</u> Zhukovski's transformation maps the upper half plane and the lower half plane onto $\mathbb{C} \setminus \{z : \operatorname{Im} z = 0,\ |\operatorname{Re} z| \geq 1\}$

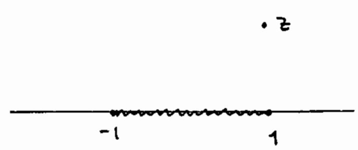


Inverse Zhukovski's transformation

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right) \Rightarrow \quad 2wz = z^2+1 \Rightarrow \; z^2 - 2wz + 1 = 0 \Rightarrow$$

$$z = \frac{2w + (4w^2-4)^{\frac{1}{2}}}{2} = w + (w^2-1)^{\frac{1}{2}} .$$

Consider $\sqrt{z^2-1}$ in $\mathbb{C} \setminus [-1,1]$. Let us show that $\exists$ a branch of $\sqrt{z^2-1}$ in $\mathbb{C} \setminus [-1,1]$



$$z^2-1 = (z-1)(z+1)$$

$$= |z-1|\, e^{i\arg(z-1)} |z+1|\, e^{i\arg(z+1)}$$

$$\sqrt{z^2-1} = \sqrt{|z-1||z+1|} \quad e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}$$

we choose both $\arg(z-1)$ and $\arg(z+1) = 0$ for $z>1$

For example

$$\sqrt{z^2-1}\,\Big|_{z=2} = \sqrt{3} \qquad \text{and} \qquad \sqrt{z^2-1}\,\Big|_{z=-2} = -\sqrt{3}$$

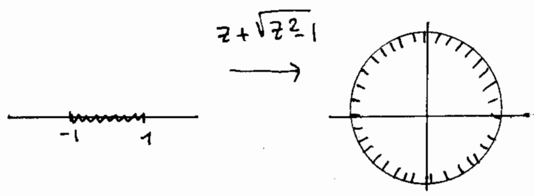
Therefore there are two inverse Zhukovski's transformation

$$w = z + \sqrt{z^2-1} \qquad \text{for} \qquad z \in \mathbb{C}\setminus[-1,1]$$

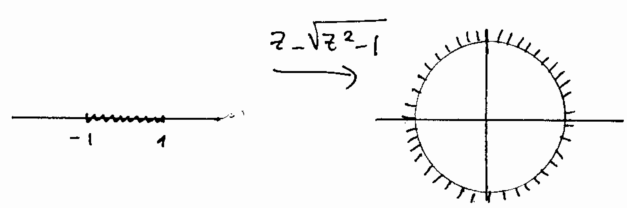
$$w = z - \sqrt{z^2-1}$$

where $\sqrt{z^2-1} = \sqrt{|z-1||z+1|} \; e^{i\frac{\mathrm{Arg}(z-1)+\mathrm{Arg}(z+1)}{2}}$ as defined previously,

Theorem $w = z + \sqrt{z^2-1}$ maps $\mathbb{C} \setminus [-1,1]$ onto $\{w : |w| > 1\}$ and $w = z - \sqrt{z^2-1}$ maps $\mathbb{C}\setminus[-1,1]$ onto $\{w : |w| < 1\}$.

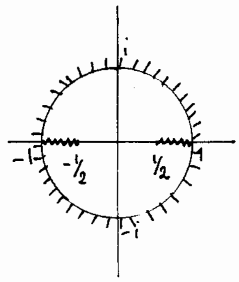


and

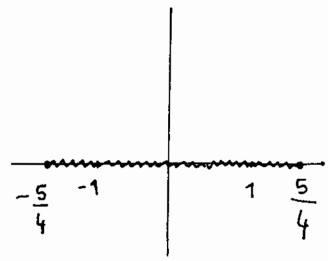


Example. Map $\{z : |z|<1\} \setminus \{z : \operatorname{Im} z = 0,\ \frac{1}{2} \leq |\operatorname{Re} z| \leq 1\}$ onto $\{w : |w|<1\}$

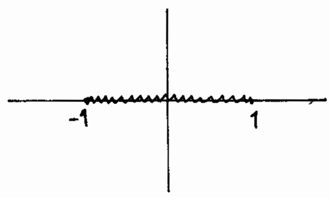
Solution



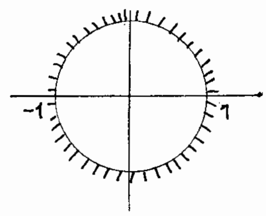
$z_1 = \frac{1}{2}\left(z + \frac{1}{z}\right)$



$z_2 = \frac{4}{5} z_1$

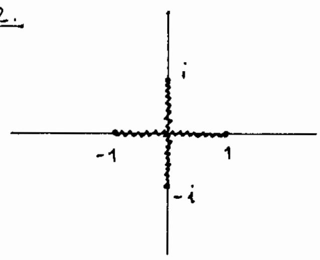


$w = z_2 - \sqrt{z_2^{\,2} - 1}$

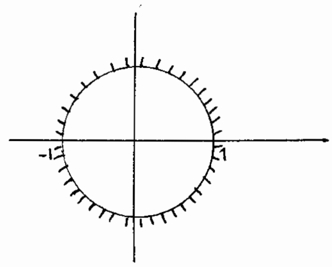


Example.

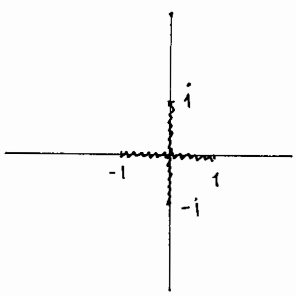
Map



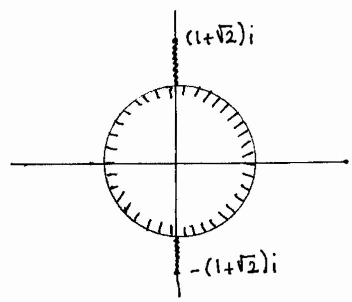
onto



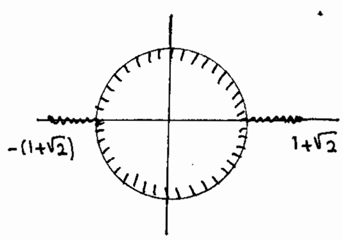
Solution.



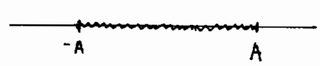
$z_1 = z + \sqrt{z^2 - 1}$



$z_2 = i z_1$

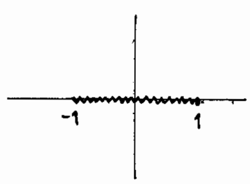


$z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)$

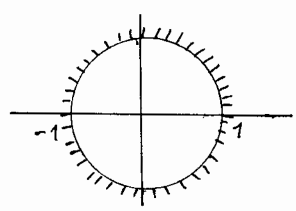


where $A = \frac{1}{2}\left(1+\sqrt{2} + \frac{1}{1+\sqrt{2}}\right)$

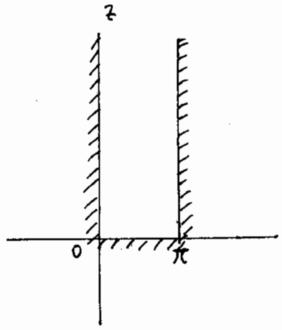
$z_4 = \frac{z_3}{A}$



$w = z_4 - \sqrt{z_4^{\,2} - 1}$

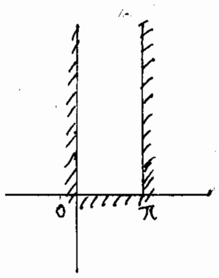


Example. Map

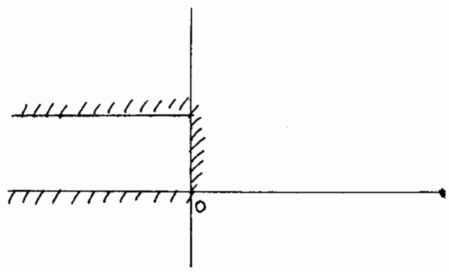


onto

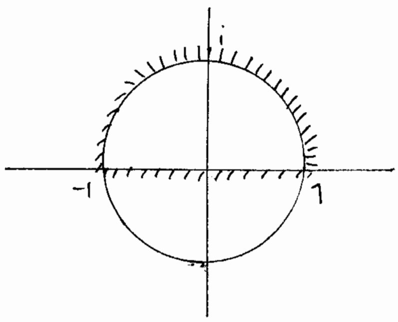
Solution



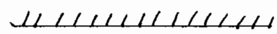
$\xrightarrow{z_1 = iz}$



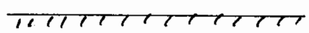
$\xrightarrow{z_2 = e^{z_1}}$



$\xrightarrow{z_3 = \frac{1}{2}\left(z_2 + \frac{1}{z_2}\right)}$

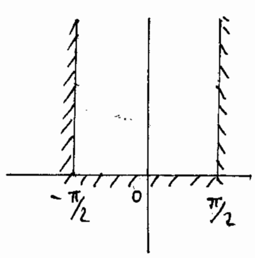


$\xrightarrow{w = -z_3}$

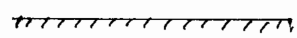


$$w = -\frac{1}{2}\left(e^{iz} + e^{-iz}\right) = -\cos z$$

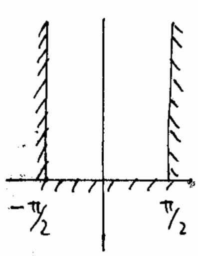
Example. Map



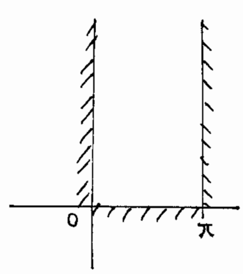
onto



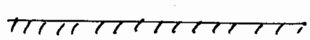
Solution.



$\xrightarrow{z_1 = z + \frac{\pi}{2}}$



$\xrightarrow{w = -\cos z_1}$



$$w = -\cos\left(z + \frac{\pi}{2}\right) = -\cos z \cos\frac{\pi}{2} + \sin z \sin\frac{\pi}{2}$$
$$= \sin z .$$

The Schwarz - Christoffel Transformation.

Remind that, if the contour C has the parametrization $z(t) = x(t)+iy(t)$, then a vector τ tangent to C at the point z_0 is $\tau = z'(t_0) = x'(t_0)+iy'(t_0)$. The image of C is a contour K given by $w = u(x(t), y(t)) + iv(x(t), y(t))$, and a vector T tangent to K at the point $w_0 = f(z_0)$ is $T = w'(t_0) = f'(z_0)z'(t_0)$ If the angle of inclination of τ is $\beta = \operatorname{Arg} z'(t_0)$, then the angle of inclination of T is $\operatorname{Arg} T = \operatorname{Arg}(f'(z_0)\, z'(t_0)) = \operatorname{Arg} f'(z_0)+\beta$. Hence the angle of inclination of the tangent τ to C at z_0 is rotated through the angle $\operatorname{Arg} f'(z_0)$ to obtain the angle of inclination of the tangent T to K at the point w_0.

Many applications involving conformal mappings require the construction of a one-to-one mapping from the upper half plane $\operatorname{Im} z>0$ onto a domain G in the w plane where the boundary consists of straight-line segments. Let's consider the case where G is the interior of a polygon P with vertices $w_1, w_2, \ldots, w_n$ specified in the positive sense. We want to find a function $w = f(z)$ with the property

$$w_k = f(x_k), \quad \text{for} \quad k = 1, 2, \ldots, n-1, \quad \text{and} \qquad (*)$$
$$w_n = f(\infty), \quad \text{where} \quad x_1 < x_2 < \cdots < x_{n-1} < \infty$$

Theorem (Schwarz-Christoffel) Let P be a polygon in the w plane with vertices $w_1, w_2, \cdots, w_n$ and exterior angles α_k, where $-\pi < \alpha_k < \pi$. There exists a one-to-one conformal mapping $w = f(z)$ from the upper half plane $\operatorname{Im} z > 0$ onto G that satisfies the boundary conditions $(*)$. The derivative $f'(z)$ is

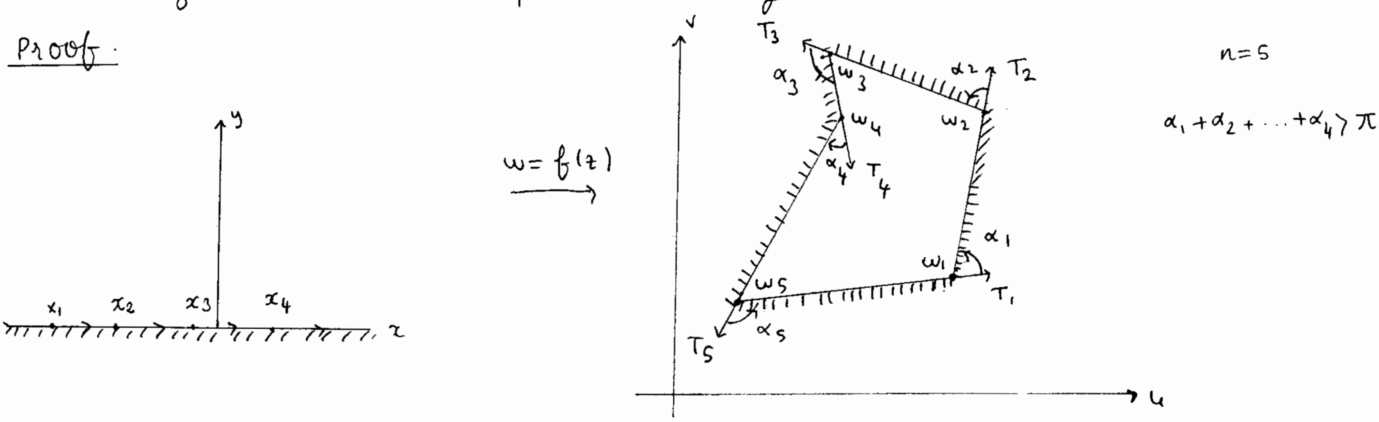
$$f'(z) = A(z-x_1)^{-\frac{\alpha_1}{\pi}}(z-x_2)^{-\frac{\alpha_2}{\pi}} \cdots (z-x_{n-1})^{-\frac{\alpha_{n-1}}{\pi}},$$

and the function f can be expressed as an indefinite integral

$$f(z) = B + A \int (z-x_1)^{-\frac{\alpha_1}{\pi}}(z-x_2)^{-\frac{\alpha_2}{\pi}} \cdots (z-x_{n-1})^{-\frac{\alpha_{n-1}}{\pi}} dz,$$

where A and B are suitably chosen constants. Two of the points $\{x_k\}$ may be chosen arbitrarily, and the constants A and B determine the size and position of P.

Proof:



The proof relies on finding how much the tangent $r_j = 1 + 0i$ (which always points to the right) at the point $(x,0)$ must be rotated by the mapping $w = f(z)$ so that the line segment $x_{j-1} < x < x_j$ is mapped onto the edge of P that lies between the points $w_{j-1} = f(x_{j-1})$ and $w_j = f(x_j)$. The amount of rotation is determined by $\operatorname{Arg} f'(x)$. For values of x that lie in the interval $x_{j-1} < x < x_j$, the amount of rotation is

$$\operatorname{Arg} f'(x) = \operatorname{Arg} A - \frac{1}{\pi}\left[\alpha_1 \operatorname{Arg}(x-x_1) + \alpha_2 \operatorname{Arg}(x-x_2) + \cdots + \alpha_{n-1}\operatorname{Arg}(x-x_{n-1})\right].$$

Because $\operatorname{Arg}(x-x_k) = 0$, for $1 \le k < j$, and $\operatorname{Arg}(x-x_k) = \pi$, for $j \le k \le n-1$, we can write this equation as

$$\operatorname{Arg} f'(x) = \operatorname{Arg} A - \alpha_j - \alpha_{j+1} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the tangent vector T_j to the polygon P at the point $w = f(x)$ for $x_{j-1} < x < x_j$ is

$$\gamma_j = \operatorname{Arg} A - \alpha_j - \alpha_{j+1} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the vector T_{j+1} tangent to the polygon P at the point $w = f(x)$, for $x_j < x < x_{j+1}$, is

$$\gamma_{j+1} = \operatorname{Arg} A - \alpha_{j+1} - \alpha_{j+2} - \cdots - \alpha_{n-1}.$$

The angle of inclination of the vector tangent to the polygon P jumps abruptly by the amount α_j as the point $w = f(x)$ moves along the side $\widehat{w_{j-1}w_j}$ through the vertex w_j to the side $\widehat{w_j w_{j+1}}$. Therefore, the exterior angle to the polygon P at the vertex w_j is given by the angle α_j and satisfies the inequality $-\pi < \alpha_j < \pi$, for $j = 1, 2, \ldots, n-1$. Since the sum of the exterior angles of a polygon equals 2π, we have $\alpha_n = 2\pi - \alpha_1 - \alpha_2 - \cdots - \alpha_{n-1}$ and only $n-1$ angles need to be specified. If the case $\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1} \le \pi$ occurs, then $\alpha_n > \pi$, and the vertices $w_1, w_2, \ldots, w_n$ cannot form a closed polygon. For this case, f determines a mapping from the upper half plane $\operatorname{Im} z > 0$ onto an infinite region in the w-plane, where the vertex w_n is at infinity (as shown below for $n=5$) ▨

The following list of indefinite integrals is often used to evaluate $f(z) = B + A\int (z-x_1)^{-\frac{\alpha_1}{\pi}} \cdots (z-x_{n-1})^{-\frac{\alpha_{n-1}}{\pi}} dz$,

1. $$\int \frac{dz}{(z^2-1)^{\frac{1}{2}}} = i \arcsin z$$

2. $$\int \frac{dz}{(z^2-1)^{\frac{1}{2}}} = \log\left(z + (z^2-1)^{\frac{1}{2}}\right) - \frac{i\pi}{2}$$

3. $$\int \frac{dz}{z^2+1} = \arctan z$$

4. $$\int \frac{dz}{z^2+1} = \frac{i}{2} \log \frac{i+z}{i-z}$$

5. $$\int \frac{dz}{z(z^2-1)^{\frac{1}{2}}} = -\arcsin \frac{1}{z}$$

6. $$\int \frac{dz}{z(z^2-1)^{\frac{1}{2}}} = i \log\left[\frac{1}{z} + \left(\frac{1}{z^2} - 1\right)^{\frac{1}{2}}\right]$$

7. $$\int \frac{dz}{z(z+1)^{\frac{1}{2}}} = -2 \operatorname{arctanh} (z+1)^{\frac{1}{2}}.$$

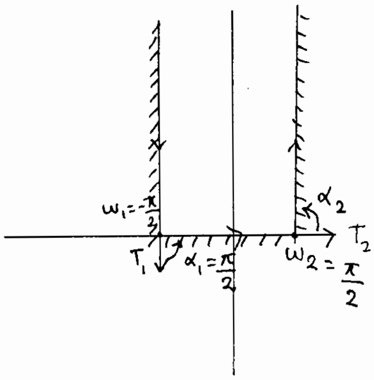
8. $$\int \frac{dz}{z(z+1)^{\frac{1}{2}}} = \log \frac{1-(z+1)^{\frac{1}{2}}}{1+(z+1)^{\frac{1}{2}}}$$

9. $$\int (1-z^2)^{\frac{1}{2}} dz = \frac{1}{2}\left[z(1-z^2)^{\frac{1}{2}} + \arcsin z\right].$$

10. $$\int (1-z^2)^{\frac{1}{2}} dz = \frac{i}{2}\left[z(z^2-1)^{\frac{1}{2}} + \log\left(z + (z^2-1)^{\frac{1}{2}}\right)\right].$$

<u>Example</u>. Use the Schwarz-Christoffel formula to verify that the function $w = f(z) = \operatorname{Arcsin} z$ maps the upper half-plane $\operatorname{Im} z > 0$ onto the semi-infinite strip $-\frac{\pi}{2} < u < \frac{\pi}{2}$, $v > 0$.

Solution Choose $x_1=-1$, $x_2=1$, $w_1=-\frac{\pi}{2}$, and $w_2=\frac{\pi}{2}$, then $\alpha_1=\frac{\pi}{2}$ and $\alpha_2=\frac{\pi}{2}$, and so



$$f(z)=A\int (z+1)^{-\frac{\pi/2}{\pi}}(z-1)^{-\frac{\pi/2}{\pi}}dz+B$$

$$=A\int \frac{1}{\sqrt{z^2-1}}dz+B$$

$$=A\,i\,\text{Arcsin}\,z+B$$

Since $f(-1)=-\frac{\pi}{2}$ and $f(1)=\frac{\pi}{2}$, we have

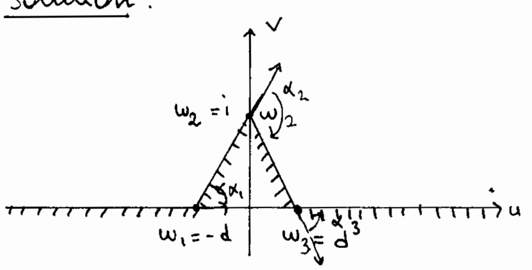
$$-\frac{\pi}{2}=A\,\frac{-i\pi}{2}+B \quad\text{and}\quad \frac{\pi}{2}=A\,\frac{i\pi}{2}+B$$

which we can solve to obtain $B=0$ and $A=-i$.

Hence $f(z)=\text{Arcsin}\,z$.

Example Verify that $w=f(z)=(z^2-1)^{\frac{1}{2}}$ maps the upper half plane $\text{Im}\,z>0$ onto the upper half-plane $\text{Im}\,w>0$ slit along the segment from 0 to i.

Solution.



If we choose $x_1=-1$, $x_2=0$, $x_3=1$, $w_1=-d$, $w_2=i$, and $w_3=d$, then the formula

$$g'(z)=A(z+1)^{-\frac{\alpha_1}{\pi}}\,z^{-\frac{\alpha_2}{\pi}}(z-1)^{-\frac{\alpha_3}{\pi}}$$

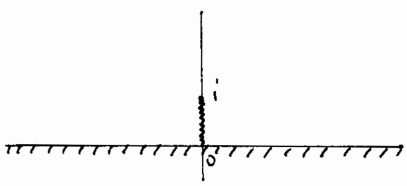
will determine a mapping $w=g(z)$ from the upper half-plane $\text{Im}\,z>0$ onto the portion of the upper half-plane $\text{Im}\,w>0$ that line outside the triangle with vertices $\pm d$, i as indicated in the figure above. If $d\to 0$, then $w_1\to 0$, $w_3\to 0$, $\alpha_1\to\frac{\pi}{2}$, $\alpha_2\to-\pi$ and $\alpha_3\to\frac{\pi}{2}$.

The limiting formula for the derivative $g'(z)$ becomes

$$f'(z) = A(z+1)^{-\frac{1}{2}}\, z\,(z-1)^{-\frac{1}{2}},$$

which will determine a mapping $w = f(z)$ from the upper half-plane $\operatorname{Im} z > 0$ onto the upper half-plane $\operatorname{Im} w > 0$ slit from 0 to i as indicated in the figure below. Clearly,

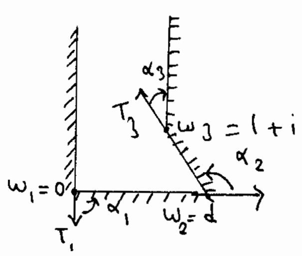
$$f(z) = A\int \frac{z}{\sqrt{z^2-1}} = A\sqrt{z^2-1} + B$$



and the boundary values $f(\pm 1) = 0$ and $f(0) = i$ lead to the solution $f(z) = \sqrt{z^2-1}$.

Example. Show that the function $w = \frac{1}{\pi}\operatorname{Arcsin} z + \frac{i}{\pi}\operatorname{Arcsin}\frac{1}{z} + \frac{1+i}{2}$ maps the upper half-plane $\operatorname{Im} z > 0$ onto the right-angle channel in the first quadrant bounded by the coordinate axes and the rays $x \geq 1,\ y = 1$ and $y \geq 1,\ x = 1$.

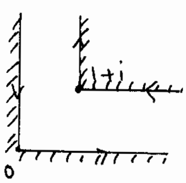
Solution Choose $x_1 = -1,\ x_2 = 0,\ x_3 = 1,\quad w_1 = 0,\ w_2 = d$, and $w_3 = 1+i$ then $g(z) = A\int (z+1)^{-\frac{\alpha_1}{\pi}} z^{-\frac{\alpha_2}{\pi}} (z-1)^{-\frac{\alpha_3}{\pi}}\, dz + B$ will determine a mapping of the upper half plane onto the domain indicated in the figure below:



with $\alpha_1 = \frac{\pi}{2}$, we let $d \to \infty$, then $\alpha_2 = \pi$ and $\alpha_3 \to -\frac{\pi}{2}$ and the function

$$f(z) = A\int \frac{(z-1)^{\frac{1}{2}}}{z\,(z+1)^{\frac{1}{2}}}\, dz + B = A\int \frac{z-1}{z\sqrt{z^2-1}}\, dz + B$$

determines a map from the upper half-plane onto the desired channel. Clearly



$$f(z) = A\int \frac{dz}{\sqrt{z^2-1}} - A\int \frac{dz}{z\sqrt{z^2-1}} + B = iA\operatorname{Arcsin} z - A\operatorname{Arcsin}\frac{1}{z} + B.$$

The result follows if we use the relations $f(-1) = 0$ and $f(1) = 1+i$ to find A and B (Exercise!)