MCS 352 2009-2010 Spring
Exercise Set IX

1. Prove that the following series converge uniformly on the sets indicated.

   (a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} z^k$ on $\overline{D}_1(0) = \{z : |z| \leq 1\}$.

   (b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{(z^2-1)^k}$ on $\{z : |z| \geq 2\}$.

   (c) $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{z^{2k}+1}$ on $\overline{D}_r(0)$, where $0 < r < 1$.

2. By starting with the series for the complex cosine, choose an appropriate contour and use the termwise integration to obtain the series for the complex sine.

3. Suppose that the sequences of functions $\{f_n\}$ and $\{g_n\}$ converge uniformly on the set T.

   (a) Show that the sequence $\{f_n + g_n\}$ converges uniformly on the set T.

   (b) Show by example that it is not necessarily the case that $\{f_n g_n\}$ converges uniformly on the set T.

4. On what portion of $D_1(0)$ does the sequence $\{nz^n\}_{n=1}^{\infty}$ converge, and on what portion does it converge uniformly?

5. Consider the function $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} n^{-z}$, where $n^{-z} = e^{-z \ln n}$.

   (a) Show that $\zeta(z)$ converges uniformly on the set $A = \{z : \mathrm{Re}(z) \geq 2\}$.

   (b) Let D be a closed disk contained in $\{z : \mathrm{Re}(z) > 1\}$. Show that $\zeta(z)$ converges uniformly on D.

6. By computing derivatives, find the Maclaurin series for each function and state where it is valid.

   (a) $\sinh z$.

   (b) $\cosh z$.

   (c) $\mathrm{Log}(1+z)$.

7. Using methods other than computing derivatives, find the Maclaurin series for

   (a) $\cos^3 z$. *Hint*: Use the trigonometric identity $4\cos^3 z = \cos 3z + 3\cos z$.

   (b) $\mathrm{Arctan}\, z$. *Hint*: Choose an appropriate contour and integrate the Maclaurin series for $\dfrac{1}{1+z^2}$.

   (c) $f(z) = (z^2+1)\sin z$.

   (d) $f(z) = e^z \cos z$. *Hint*: $\cos z = \dfrac{1}{2}(e^{iz} + e^{iz})$, so $f(z) = \dfrac{1}{2}e^{(1+i)z} + \dfrac{1}{2}e^{(1-i)z}$. Now use the Maclaurin series for e^z.

8. Find the Taylor series centered at $a = 1$ and state where it converges for

   (a) $f(z) = \dfrac{1-z}{z-2}$.

(b) $f(z) = \dfrac{1-z}{z-3}$.

9. Let $f(z) = \dfrac{\sin z}{z}$ and set $f(0) = 1$.

   (a) Explain why f is analytic at $z = 0$.

   (b) Find the Maclaurin series for $f(z)$.

   (c) Find the Maclaurin series for $g(z) = \displaystyle\int_C f(\zeta)\, d\zeta$, where C is the straight-line segment from 0 to z.

10. Show that $f(z) = \dfrac{1}{1-z}$ has its Taylor series representation about the point $\alpha = i$ given by

$$f(z) = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}},$$

for all $z \in D_{\sqrt{2}}(i) = \{z : |z-i| < \sqrt{2}\}$.

11. Let $f(z) = (1+z)^\beta = e^{\beta \operatorname{Log}(1+z)}$ be the principal branch of $(1+z)^\beta$, where β is a fixed complex number. Establish the validity for $z \in D_1(0)$ of the binomial expansion

$$\begin{aligned}(1+z)^\beta &= 1 + \beta z + \frac{\beta(\beta-1)}{2!} z^2 + \frac{\beta(\beta-1)(\beta-2)}{3!} z^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\beta(\beta-1)(\beta-2)\cdots(\beta-n+1)}{n!} z^n.\end{aligned}$$

12. Find $f^{(3)}(0)$ for

   (a) $f(z) = \displaystyle\sum_{n=0}^{\infty} (3+(-1)^n)^n z^n$.

   (b) $g(z) = \displaystyle\sum_{n=1}^{\infty} \frac{(1+i)^n}{n} z^n$.

   (c) $h(z) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(\sqrt{3}+i)^n}$.

13. Suppose that $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n$ is an entire function.

   (a) Find a series representation for $\overline{f(\overline{z})}$, using powers of $\overline{z}$.

   (b) Show that $\overline{f(\overline{z})}$ is an entire function.

   (c) Does $\overline{f(\overline{z})} = f(z)$? Why or why not?

14. Let $f(z) = \displaystyle\sum_{n=0}^{\infty} c_n z^n = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 + \cdots$, where the coefficients c_n are the Fibonacci numbers defined by $c_0 = 1$, $c_1 = 1$, and $c_n = c_{n-1} + c_{n-2}$, for $n \geq 2$.

   (a) Show that $f(z) = \dfrac{1}{1-z-z^2}$, for all $z \in D_R(0)$ for some number R.

   (b) Find the value of R in part (a) for which the series representation is valid.

15. Use the Maclaurin series and the Cauchy product to verify that $\sin 2z = 2\cos z \sin z$ up to terms involving z^5.

16. Compute the Taylor series for the principal logarithm $f(z) = \operatorname{Log} z$ expanded about the center $z_0 = -1 + i$.

17. The Fresnel integrals $C(z)$ and $S(z)$ are defined by

$$C(z) = \int_0^z \cos(\xi^2)\,d\xi \quad \text{and} \quad S(z) = \int_0^z \sin(\xi^2)\,d\xi.$$

We define $F(z)$ by $F(z) = C(z) + iS(z)$.

(a) Verify the identity $F(z) = \displaystyle\int_0^z e^{i\xi^2}\,d\xi$.

(b) Integrate the power series for $e^{i\xi^2}$ and obtain the power series for $F(z)$.

(c) Use the partial sum involving terms up to z^9 to find approximations to $C(1.0)$ and $S(1.0)$.

18. Let f be defined in a domain that contains the origin. The function f is said to be even if $f(-z) = f(z)$, and it is called odd if $f(-z) = -f(z)$.

(a) Show that the derivative of an odd function is an even function.

(b) Show that the derivative of an even function is an odd function.

(c) If f is even, show that all the coefficients of the odd powers of z in the Maclaurin series are zero.

(d) If f is odd, show that all the coefficients of the even powers of z in the Maclaurin series are zero.

19. Find two Laurent series expansions for $f(z) = \dfrac{1}{z^3 - z^4}$ that involve powers of z.

20. Show that $f(z) = \dfrac{1}{1-z}$ has a Laurent series representation about the point $z_0 = i$ given by

$$f(z) = \frac{1}{1-z} = -\sum_{n=1}^{\infty} \frac{(1-i)^{n-1}}{(z-i)^n} \quad \text{valid for } |z-i| > \sqrt{2}).$$

21. Find the Laurent series for $f(z) = \dfrac{\sin 2z}{z^4}$ that involves powers of z.

22. Show that $\dfrac{1-z}{z-2} = -\displaystyle\sum_{n=0}^{\infty} \frac{1}{(z-1)^n}$ is valid for $|z-1| > 1$.

23. Find the Laurent series for $\sin\left(\frac{1}{z}\right)$ centered at $\alpha = 0$. Where is the series valid?

24. Show that $\dfrac{1-z}{z-3} = -\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{(z-1)^n}$ is valid for $|z-1| > 2$.

25. Find the Laurent series for $f(z) = \dfrac{\cosh z - \cos z}{z^5}$ that involves powers of z.

26. Find the Laurent series for $f(z) = \dfrac{1}{z^4(1-z)^2}$ that involves powers of z and is valid for $|z| > 1$.

27. Find two Laurent series for $z^{-1}(4-z)^{-2}$ involving powers of z and state where they are valid.

28. Find three Laurent series for $(z^2 - 5z + 6)^{-1}$ centered at $\alpha = 0$.

29. Find the Laurent series for $\operatorname{Log}\left(\dfrac{z-a}{z-b}\right)$, where a and b are positive real numbers with $b > a > 1$, and state where the series is valid. *Hint*: For these conditions, show that $\operatorname{Log}\left(\frac{z-a}{z-b}\right) = \operatorname{Log}\left(1 - \frac{a}{z}\right) - \operatorname{Log}\left(1 - \frac{b}{z}\right)$.

30. Can $\operatorname{Log} z$ be represented by a Maclaurin series or a Laurent series about the point $\alpha = 0$? Explain your answer.

31. Use the Maclaurin series for $\sin z$ and then long division to get the Laurent series for $\csc z$ with $\alpha = 0$.

32. Show that $\cosh\left(z+\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} a_n z^n$, where the coefficients can be expressed in the form $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos n\theta \cosh(2\cos\theta)\, d\theta$. *Hint*: Let the path of integration be the circle $C_1(0)$.

33. Consider the real-valued function $u(\theta) = \frac{1}{5-4\cos\theta}$.

   (a) Use the substitution $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ and obtain

$$u(\theta) = f(z) = \frac{-z}{(z-2)(2z-1)}.$$

   (b) Expand the function $f(z)$ in part (a) in a Laurent series that is valid in the annulus $A(0, \frac{1}{2}, 2)$.

   (c) Use the substitution $\cos n\theta = \frac{1}{2}\left(z^n + z^{-n}\right)$ in part (b) and obtain the Fourier series for $(\theta) :\ u(\theta) = \frac{1}{3} + \frac{1}{3}\sum_{n=1}^{\infty} 2^{-n+1}\cos(n\theta)$.

34. The Bessel function $J_n(z)$ is sometimes defined by the generating function

$$\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)t^n.$$

Use the circle $C_1(0)$ as the contour of integration and show that

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - z\sin\theta)\, d\theta.$$

35. Suppose that the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ converges in the annulus $A(0, r_1, r_2)$, where $r_1 < 1 < r_2$. Consider the real-valued function $u(\theta) = f(e^{i\theta})$ and show that $u(\theta)$ has the Fourier series expansion

$$u(\theta) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\phi} f(e^{i\phi})\, d\phi.$$

36. Locate the zeros of the following functions and determine their order.

   (a) $(1+z^2)^4$.
   (b) $\sin^2 z$.
   (c) z^2+2z+2.
   (d) $\sin z^2$.
   (e) z^4+10z^2+9.
   (f) $1+e^z$.
   (g) z^6+1.
   (h) $z^3 e^{z-1}$.
   (i) z^6+2z^3+1.
   (j) $z^3\cos^2 z$.
   (k) z^8+z^4.
   (l) $z^2\cosh z$.

37. Locate the poles of the following functions and determine their order.

(a) $(z^2+1)^{-3}(z-1)^{-4}$.
(b) $z^{-1}(z^2-2z+2)^{-2}$.
(c) $(z^6+1)^{-1}$.
(d) $(z^4+z^3-2z^2)^{-1}$.
(e) $(3z^4+10z^2+3)^{-1}$.
(f) $\left(i+\frac{2}{z}\right)^{-1}\left(3+\frac{4}{z}\right)^{-1}$.
(g) $z\cot z$.
(h) $z^{-5}\sin z$.
(i) $(z^2\sin z)^{-1}$.
(j) $z^{-1}\csc z$.
(k) $(1-e^z)^{-1}$.
(l) $z^{-5}\sinh z$.

38. Locate the singularities of the following functions and determine their type.

(a) $\dfrac{z^2}{z-\sin z}$.
(b) $\sin\left(\dfrac{1}{z}\right)$.
(c) $ze^{\frac{1}{z}}$.
(d) $\tan z$
(e) $(z^2+z)^{-1}\sin z$. .
(f) $\dfrac{z}{\sin z}$.
(g) $\dfrac{e^z-1}{z}$.
(h) $\dfrac{\cos z-\cos(2z)}{z^4}$.

39. Suppose that f has a removable singularity at z_0. Show that the function $\dfrac{1}{f}$ has either a removable singularity or a pole at z_0.

40. Let f be analytic and have zero of order k at z_0. Show that f' has a zero of order $k-1$ at z_0.

41. Let f and g be analytic at z_0 and have zeros of order m and n, respectively, at z_0. What can you say about the zero of $f+g$ at z_0?

42. Let f and g have poles of order m and n, respectively, at z_0. Show that $f+g$ has either a pole or a removable singularity at z_0.

43. Let f be analytic and have a zero of order k at z_0. Show that the function $\dfrac{f'}{f}$ has a simple pole at z_0.

44. Let f have a pole of order k at z_0. Show that f' has a pole of order $k+1$ at z_0.

45. Find the singularities of the following functions.

(a) $\dfrac{1}{\sin\left(\frac{1}{z}\right)}$.
(b) $\operatorname{Log} z^2$.
(c) $\cot z-\dfrac{1}{z}$

46. Determine whether there exists a function f that is analytic at 0 such that for $n=1,2,3,\ldots$,

(a) $f\left(\frac{1}{2n}\right) = 0$ and $f\left(\frac{1}{2n-1}\right) = 0$.

(b) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2}$.

(c) $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3}$.

47. Consider the function $f(z) = z \sin\left(\frac{1}{z}\right)$.

(a) Show that there is a sequence $\{z_n\}$ of points converging to 0 such that $f(z_n) = 0$ for $n = 1, 2, 3, \ldots$.

(b) Does this result contradict the following result? Why or why not?
Suppose that f is analytic in the domain D and that $\alpha \in D$. If there exists a sequence of points $\{z_n\}$ in D such that $z_n \to \alpha$, and $f(z_n) = 0$, then $f(z) = 0$ for all $z \in D$.

48. Let $f(z) = \tan z$.

(a) Find the first few terms of the Maclaurin series for $f(z)$ if $|z| < \frac{\pi}{2}$.

(b) What are the values of $f^{(6)}(0)$ and $f^{(7)}(0)$?

49. Show that the real function f defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{when} \quad x \neq 0 \\ 0 & \text{when} \quad x = 0 \end{cases}$$

is continuous at $x = 0$ but the corresponding function $g(z)$ defined by

$$g(z) = \begin{cases} z \sin\left(\frac{1}{z}\right) & \text{when} \quad z \neq 0 \\ 0 & \text{when} \quad z = 0 \end{cases}$$

is not continuous at $z = 0$.

50. Use L'Hôpital's rule to find the following limits.

(a) $\displaystyle\lim_{z \to 1+i} \frac{z - 1 - i}{z^4 + 4}$.

(b) $\displaystyle\lim_{z \to i} \frac{z^2 - 2iz - 1}{z^4 + 2z^2 + 1}$.

(c) $\displaystyle\lim_{z \to i} \frac{1 + z^6}{1 + z^2}$.

(d) $\displaystyle\lim_{z \to 0} \frac{\sin z + \sinh z - 2z}{z^5}$.