MCS 352 2009-2010 Spring
Exercise Set VIII

1. Let $C_\rho(z_0)$ denotes the circle $\{z : |z - z_0| = \rho\}$. Find

   (a) $\displaystyle\oint_{C_1(0)} \frac{e^z + \cos z}{z}\, dz.$

   (b) $\displaystyle\oint_{C_1(1)} (z+1)^{-1}(z-1)^{-1}\, dz.$

   (c) $\displaystyle\oint_{C_1(1)} (z+1)^{-1}(z-1)^{-2}\, dz.$

   (d) $\displaystyle\oint_{C_1(1)} (z^3-1)^{-1}\, dz.$

   (e) $\displaystyle\oint_{C_1(0)} z^{-4} \sin z\, dz.$

   (f) $\displaystyle\oint_{C_1(0)} (z \cos z)^{-1}\, dz.$

   (g) $\displaystyle\oint_{C_1(0)} z^{-3} \sinh(z^2)\, dz.$

   (h) $\displaystyle\oint_{C_1(\frac{\pi}{2})} z^{-2} \sin z\, dz.$

   (i) $\displaystyle\oint_{C_1(\frac{\pi}{4})} z^{-2} \sin z\, dz.$

   (j) $\displaystyle\oint_{C_1(0)} z^{-n} e^z\, dz$ where $n \in \mathbb{Z}_+$.

   (k) $\displaystyle\oint_{C_1(0)} z^{-2}(z^2-16)^{-1} e^z\, dz.$

   (l) $\displaystyle\oint_{C_1(4)} z^{-2}(z^2-16)^{-1} e^z\, dz.$

   (m) $\displaystyle\oint_{C_1(1+i)} (z^4+4)^{-1}\, dz.$

   (n) $\displaystyle\oint_{C_{\frac{1}{2}}(0)} z^{-1}(z-1)^{-1} e^z\, dz.$

   (o) $\displaystyle\oint_{C_2(0)} z^{-1}(z-1)^{-1} e^z\, dz.$

   (p) $\displaystyle\oint_{C_1(i)} (z^2+1)^{-1} \sin z\, dz.$

   (q) $\displaystyle\oint_{C_1(-i)} (z^2+1)^{-1} \sin z\, dz.$

(r) $\oint_{C_1(i)} (z^2+1)^{-2}\, dz$.

(s) $\oint_{C_1(i)} (z^2+1)^{-1}\, dz$.

(t) $\oint_{C_1(-i)} (z^2+1)^{-1}\, dz$.

2. Let $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Find $\oint_{C_1(0)} P(z) z^{-n}\, dz$, where n is a positive integer.

3. Let z_1 and z_2 be two complex numbers that lie interior to the simple closed contour C with positive orientation. Evaluate $\oint_C (z-z_1)^{-1}(z-z_2)^{-1}\, dz$.

4. Let f be analytic in the simply connected domain D and let z_1 and z_2 be two complex numbers that lie interior to the simple closed contour C having positive orientation that lies in D. Show that

$$\frac{f(z_2)-f(z_1)}{z_2-z_1} = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_1)(z-z_2)}\, dz.$$

State what happens when $z_2 \to z_1$.

5. Compute $\oint_{|z|=1} \frac{e^z}{z}\, dz$.

6. Compute $\oint_{|z|=2} \frac{dz}{z^2+1}$.

7. Compute $\oint_{|z|=1} e^z z^{-n}\, dz$.

8. Compute $\oint_{|z|=2} z^n (1-z)^m\, dz$.

9. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals.

(a) $\oint_C \frac{e^{-z}}{z - \frac{\pi i}{2}}\, dz$.

(b) $\oint_C \frac{\cos z}{z(z^2+8)}\, dz$.

(c) $\oint_C \frac{\cosh z}{z^4}\, dz$.

(d) $\oint_C \frac{\tan \frac{z}{2}}{(z-x_0)^2}\, dz,\ -2 < x_0 < 2$.

10. Find the value of the integral of $g(z)$ around the circle $|z-i| = 2$ in the positive sense when

(a) $g(z) = \frac{1}{z^2+4}$.

(b) $g(z) = \frac{1}{(z^2+4)^2}$.

11. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(w) = \oint_C \frac{2z^2 - z - 2}{z - w}\, dz, \quad |w| \neq 3,$$

then $g(2) = 8\pi i$. What is the value of $g(w)$ when $|w| > 3$?

12. Let C be any simple closed contour, described in the positive sense in the z-plane, and write

$$g(w) = \oint_C \frac{z^3 + 2z}{(z-w)^3}\, dz.$$

Show that $g(w) = 6\pi i w$ when w is inside C and that $g(w) = 0$ when w is outside C.

13. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then

$$\int_C \frac{f'(z)}{z - z_0}\, dz = \int_C \frac{f(z)}{(z-z_0)^2}\, dz.$$

14. Let C be the unit circle $z = e^{i\theta}$, $-\pi \le \theta \le \pi$. First show that, for any real constant a,

$$\int_C \frac{e^{az}}{z}\, dz = 2\pi i.$$

Then write this integral in terms of θ to derive the integration formula

$$\int_0^{\pi} e^{a\cos\theta} \cos(a \sin\theta)\, d\theta = \pi.$$

15. (a) With the aid of the binomial formula, show that, for each value of n, the function

$$P_n(z) = \frac{1}{n! 2^n} \frac{d^n}{dz^n}(z^2 - 1)^n, \quad n = 0, 1, 2 \cdots,$$

is a polynomial of degree n.

(b) Let C denote any positively oriented simple closed contour surrounding a fixed point z. With the aid of the integral representation for the nth derivative of an analytic function, show that the polynomials in part (a) can be expressed in the form

$$P_n(z) = \frac{1}{2^{n+1}\pi i} \oint_C \frac{(s^2-1)^n}{(s-z)^{n+1}}\, ds, \quad n = 0, 1, 2, \cdots.$$

(c) Point out how the integrand in the representation for $P_n(z)$ in part (b) can be written $\dfrac{(s+1)^n}{s-1}$ if $z = 1$. Then apply the Cauchy integral formula to show that

$$P_n(1) = 1, \quad n = 0, 1, 2, \cdots.$$

Similarly, show that

$$P_n(-1) = (-1)^n, \quad n = 0, 1, 2, \cdots.$$

16. Factor each polynomial as a product of linear factors.

(a) $P(z) = z^4 + 4$.

(b) $P(z) = z^2 + (1+i)z + 5i$.

(c) $P(z) = z^4 - 4z^3 + 6z^2 - 4z + 5$.

(d) $P(z) = z^3 - (3+3i)z^2 + (-1+6i)z + 3 - i$. *Hint*: Show that $P(i) = 0$.

17. Let $f(z) = az^n + b$. Show that $\max\limits_{|z|\le 1} |f(z)| = |a| + |b|$.

18. Show that $\cos z$ is not a bounded function.

19. Let $f(z) = z^2$ and $R = \{z = x + iy : 2 \le x \le 3,\ 1 \le y \le 3\}$. Evaluate.

(a) $\max\limits_{z\in R} |f(z)|$.

(b) $\min_{z \in R} |f(z)|$.

(c) $\max_{z \in R} \operatorname{Re}(f(z))$.

(d) $\min_{z \in R} \operatorname{Im}(f(z))$.

20. Let $D_\rho(z_0)$ denotes the disk $\{z : |z - z_0| < \rho\}$ and $C_\rho(z_0)$ denotes the circle $\{z : |z - z_0| = \rho\}$. Let f be analytic in the disk $D_5(0)$ and suppose that $|f(z)| \leq 10$ for $z \in C_3(1)$.

    (a) Find a bound for $|f^{(4)}(1)|$.

    (b) Find a bound for $|f^{(4)}(0)|$. *Hint*: $\overline{D}_2(0) \subseteq \overline{D}_3(1)$.

21. Let f be an entire function such that $|f(z)| \leq M|z|$ for all $z \in \mathbb{C}$.

    (a) Show that, for $n \geq 2$, $f^{(n)}(z) = 0$ for all $z \in \mathbb{C}$.

    (b) Use part (a) to show that $f(z) = az + b$.

22. Establish the following minimum modulus principle.

    (a) Let f be analytic and nonconstant in the domain D. If $|f(z)| \geq m$ for all z in D, where $m > 0$, then $|f(z)|$ does not attain a minimum value at any point z_0 in D.

    (b) Show that the requirement $m > 0$ in part (a) is necessary by finding an example of a function defined on D for which $m = 0$, and yet whose minimum is attained somewhere in D.

23. Let $u(x, y)$ be harmonic for all (x, y). Show that

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + R\cos\theta, y_0 + R\sin\theta)\, d\theta,$$

where $R > 0$. *Hint*: Let $f(z) = u(x, y) + iv(x, y)$, where v is a harmonic conjugate of u.

24. Establish the following maximum principle for harmonic functions. Let $u(x, y)$ be harmonic and nonconstant in the simply connected domain D. Then u does not have a maximum value at any point (x_0, y_0) in D.

25. Let f be an entire function with the property that $|f(z)| \geq 1$ for all z. Show that f is constant.

26. Let f be a nonconstant analytic function in the closed disk $\{z : |z| \leq 1\}$. Suppose that $|f(z)| = K$ for $z \in \{z : |z| = 1\}$. Show that f has a zero in D.

27. Prove that a function which is analytic in the whole plane and satisfies an inequality $|f(z)| < |z|^n$ for some n and all sufficiently large $|z|$ reduces to a polynomial.

28. Suppose that $f(z)$ is entire and that the harmonic function $u(x, y) = \operatorname{Re}(f(z))$ has an upper bound u_0; that is, $u(x, y) \leq u_0$ for all points (x, y) in the xy plane. Show that $u(x, y)$ must be constant throughout the plane. *Hint*: Apply Liouville's theorem to the function $g(z) = e^{f(z)}$.

29. Let a function f be continuous in a closed bounded region R, and let it be analytic and not constant throughout the interior of R. Assuming that $f(z) \neq 0$ anywhere in R, prove that $|f(z)|$ has a minimum value m in R which occurs on the boundary of R and never in the interior. Do this by applying the corresponding result for maximum values to the function $g(z) = \dfrac{1}{z}$.

30. Use the function $f(z) = z$ to show that in Exercise 29 the condition $f(z) \neq 0$ anywhere in R is necessary in order to obtain the result of that exercise. That is, show that $|f(z)|$ can reach its minimum value at an interior point when that minimum value is zero.

31. Consider the function $f(z) = (z+1)^2$ and the closed triangular region R with vertices at the points $z = 0$, $z = 2$, and $z = i$. Find points in R where $|f(z)|$ has its maximum and minimum values. *Hint*: Interpret $|f(z)|$ as the square of the distance between z and -1.

32. Let $f(z) = u(x, y) + iv(x, y)$ be a function that is continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $u(x, y)$ has maximum and minimum values in R which occurs on the boundary of R and never in the interior.

33. Let $f(z) = e^z$ and $R = \{z = x + iy : 0 \le x \le 1,\ 0 \le y \le \pi\}$. Find points in R where the component function $u(x, y) = \mathrm{Re}(f(z))$ reaches its maximum and minimum values.

34. Let $f(z) = u(x, y)+iv(x, y)$ be continuous on a closed bounded region R and analytic and not constant throughout the interior of R. Prove that the component function $v(x, y)$ has maximum and minimum values in R which are reached on the boundary of R and never in the interior, where it is harmonic.

35. Suppose a polynomial is bounded by 1 in the unit disc. Show that all its coefficients are bounded by 1.

36. Suppose f is entire and $|f(z)| \le A + B|z|^{\frac{3}{2}}$. Show that f is a linear polynomial.

37. Suppose f is entire and $|f'(z)| \le |z|$ for all z. Show that $f(z) = a + bz^2$ with $|b| \le \dfrac{1}{2}$.

38. Prove that a nonconstant entire function cannot satisfy the two equations

$$f(z + 1) = f(z) \quad \text{and} \quad f(z + i) = f(z)$$

for all z. *Hint*: Show that a function satisfying both equalities would be bounded.

39. A *real polynomial* is a polynomial whose coefficients are all real. Prove that a real polynomial of odd degree must have a real zero.

40. Show that every real polynomial is equal to a product of linear and quadratic factors.

41. Suppose P is a polynomial such that $P(z)$ is real if and only if z is real. Prove that P is linear. *Hint*: Set $P = u + iv$, $z = x + iy$ and note that $v = 0$ if and only if $y = 0$. Conclude that (i) either $v_y \ge 0$ throughout the real axis or $v_y \le 0$ throughout the real axis; (ii) either $u_x \ge 0$ or $u_x \le 0$ for all real values and hence u is monotonic along the real axis; (iii) $P(z) = \alpha$ has only one solution for real-valued α.

42. Show that α is a zero of multiplicity k if and only if

$$P(\alpha) = P'(\alpha) = \cdots = P^{k-1}(\alpha) = 0, \quad P^k(\alpha) \ne 0.$$

43. Suppose that f is entire and that for each z, either $|f(z)| \le 1$ or $|f'(z)| \le 1$. Prove that f is a linear polynomial. *Hint*: Use a contour integral to show $|f(z)| \le A + |z|$ where $A = \max\{1, |f(0)|\}$.

44. Suppose that f is analytic in $|z| \le 1$, $|f(z)| \le 2$ for $|z| = 1$, $\mathrm{Im}(z) \ge 0$ and $|f(z)| \le 3$ for $|z| = 1$, $\mathrm{Im}(z) \le 0$. Show then that $|f(0)| \le \sqrt{6}$. *Hint*: Consider $f(z) \cdot f(-z)$.

45. Show directly that the maximum and minimum moduli of e^z are always assumed on the boundary of the compact domain.

46. Find the maximum and minimum moduli of $z^2 - z$ in the disc $|z| \le 1$.

47. Suppose that f and g are both analytic in a compact domain D. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. *Hint*: Consider $f(z)e^{i\alpha} + g(z)e^{i\beta}$ for appropriate α and β.

48. Suppose $P_n(z) = a_0 + a_1(z) + \cdots + a_n z^n$ is bounded by 1 for $|z| \le 1$. Show that $|P(z)| \le |z|^n$ for all $|z| \ge 1$. *Hint*: Use Exercise 35 to show $|a_n| \le 1$ and then consider $\dfrac{P(z)}{z^n}$ in the annulus: $1 \le |z| \le R$ for "large" R.