MCS 352 2009-2010 Spring
Exercise Set VII

1. Evaluate the following integrals

   (a) $\displaystyle\int_1^2 \left(\frac{1}{t} - i\right)^2 dt.$

   (b) $\displaystyle\int_0^{\frac{\pi}{6}} e^{i2t}\, dt.$

2. Compute $\displaystyle\int_\gamma x\, dz$ where γ is the directed line segment from 0 to $1 + i$.

3. Evaluate $\displaystyle\int_C f(z)\, dz$ where $f(z) = x^2 + iy^2$ and C is given by

   (a) $z(t) = t + it,\ 0 \le t \le 1.$

   (b) $z(t) = t^2 + it^2,\ 0 \le t \le 1.$

4. Evaluate $\displaystyle\int_C f(z)\, dz$ where $f(z) = \frac{1}{z}$ and C is given by

   (a) $z(t) = R\cos t + iR\sin t,\ 0 \le t \le 2\pi,\ R > 0.$

   (b) $z(t) = \sin t + i\cos t,\ 0 \le t \le 2\pi.$

5. Show that, if f is a continuous real-valued function and $|f| \le 1$, then

$$\left|\int_C f(z)\, dz\right| \le 4,$$

   where $C = \{z :\ |z| = 1\}$. *Hint*: Show that $\left|\int_C f(z)\, dz\right| \le \int_0^{2\pi} |\sin t|\, dt.$

6. Use parametric representations for C, or legs of C, to evaluate $\displaystyle\int_C f(z)\, dz$, if

   (a) $f(z) = \dfrac{z+2}{z}$ and C is

      i. the semicircle $z = 2e^{i\theta},\ 0 \le \theta \le \pi.$

      ii. the semicircle $z = 2e^{i\theta},\ \pi \le \theta \le 2\pi.$

      iii. the circle $z = 2e^{i\theta},\ 0 \le \theta \le 2\pi.$

   (b) $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

      i. the semicircle $z = 1 + e^{i\theta},\ \pi \le \theta \le 2\pi.$

      ii. the segment $0 \le x \le 2$ of the real axis.

   (c) $f(z) = \pi e^{\pi \bar{z}}$ and C is the boundary of the square with vertices at the points 0, 1, $1+i$, and i, the orientation of C being in the counterclockwise direction.

(d) $f(z)$ is defined by the equations

$$f(z) = \begin{cases} 1 & \text{when} \quad y < 0 \\ 4y & \text{when} \quad y > 0, \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

(e) $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the plane.

(f) $f(z)$ is the branch

$$z^{-1+i} = \exp((-1+i)\log z), \quad |z| > 0,\ 0 < \arg z < 2\pi$$

of the indicated power function, and C is the positively oriented unit circle $|z| = 1$.

7. With the aid of the result in Exercise 2, Set VI, evaluate the integral

$$\int_C z^m \overline{z}^n\, dz,$$

where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

8. Evaluate the integral $\int_C \overline{z}\, dz$ where C is the part of the circle $|z| = 2$ in the right half-plane from $z = -2i$ to $z = 2i$, by using the parametrization

   (a) $z = 2e^{i\theta}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}.$

   (b) $z = \sqrt{4 - y^2} + iy, \quad -2 \le y \le 2.$

9. Let C and C_0 denote the circles $z = Re^{i\theta},\ 0 \le \theta \le 2\pi$ and $z = z_0 + Re^{i\theta},\ 0 \le \theta \le 2\pi$, respectively. Use these parametric representations to show that

$$\int_C f(z)\, dz = \int_{C_0} f(z - z_0)\, dz$$

when f is piecewise continuous on C.

10. Show that

$$\int_{C_R(z_0)} (z - z_0)^{a-1}\, dz = i\frac{2R^a}{a}\sin(a\pi),$$

where $C_R(z_0) = \{z : |z - z_0| = R\}$ and a is any real number other than zero and the principal branch of the integrand and the principal value of R^a are taken.

11. Without evaluating the integral, show that

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \le \frac{\pi}{3}$$

where C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

12. Let C denote the line segment from $z = i$ to $z = 1$. By observing that, of all the points on that line segment, the midpoint is the closest to the origin, show that

$$\left| \int_C \frac{dz}{z^4} \right| \le 4\sqrt{2}$$

without evaluating the integral.

13. Show that if C is the boundary of the triangle with vertices at the points 0, $3i$, and -4, oriented in the counterclockwise direction, then

$$\left| \int_C (e^z - \overline{z})\, dz \right| \le 60.$$

14. Let C_R denote the upper half of the circle $|z| = R$, $R > 2$, taken in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\, dz \right| \leq \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

15. Let C_R be the circle $|z| = R$, $R > 1$, described in the counterclockwise direction. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

and then use l'Hospital's rule to show that the value of this integral tends to zero as R tends to infinity.

16. Let C_ρ denote the circle $|z| = \rho$, $0 < \rho < 1$, oriented in the counterclockwise direction, and suppose that $f(z)$ is analytic in the disk $|z| \leq 1$. Show that if $z^{-\frac{1}{2}}$ represents any particular branch of that power of z, then there is a nonnegative constant M, independent of ρ, such that

$$\left| \int_{C_\rho} z^{-\frac{1}{2}} f(z)\, dz \right| \leq 2\pi M \sqrt{\rho}.$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

17. Let C_N denote the boundary of the square formed by the lines

$$x = \pm \left(N + \frac{1}{2} \right) \pi \quad \text{and} \quad y = \pm \left(N + \frac{1}{2} \right) \pi,$$

where N is a positive integer, and let the orientation of C_N be counterclockwise.

(a) With the aid of the inequalities

$$|\sin z| \geq |\sin x| \quad \text{and} \quad |\sin z| \geq |\sinh y|,$$

show that $|\sin z| \geq 1$ on the vertical sides of the square and that $|\sin z| > \sinh \frac{\pi}{2}$ on the horizontal sides. Thus show that there is a positive constant A, independent of N, such that $|\sin z| \geq A$ for all points z lying on the contour C_N.

(b) Using the final result in part (a), show that

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z}\, dz \right| \leq \frac{16}{(2N + 1)\pi A}$$

and hence that the value of this integral tends to zero as N tends to infinity.

18. Compute $\displaystyle\int_{|z|=r} x\, dz$ for the positive sense of the circle in two ways: first, by use of a parameter, and second, by observing that $x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z} \right)$ on the circle.

19. Compute $\displaystyle\int_{|z|=2} \frac{dz}{z^2 - 1}$ for the positive sense of the circle.

20. Apply the Cauchy-Goursat theorem to show that

$$\int_C f(z)\, dz = 0$$

when the contour C is the circle $|z| = 1$, in either direction, and when

(a) $f(z) = \dfrac{z^2}{z-3}$.
(b) $f(z) = ze^{-z}$.
(c) $f(z) = \dfrac{1}{z^2+2z+2}$.
(d) $f(z) = \operatorname{sech} z$.
(e) $f(z) = \tan z$.
(f) $f(z) = \operatorname{Log}(z+2)$.

21. Let C_1 denote the positively oriented circle $|z| = 4$ and C_2 the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$ and $y = \pm 1$. With the aid of "deformation of contour" theorem, point out why

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

when

(a) $f(z) = \dfrac{1}{3z^2+1}$.
(b) $f(z) = \dfrac{z+2}{\sin\frac{z}{2}}$.
(c) $f(z) = \dfrac{z}{1-e^z}$.

22. Use the method described below to derive the integration formula

$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2},\ b > 0.$$

(a) Show that the sum of the integrals of e^{-z^2} along the lower and upper horizontal legs of the rectangular path in the figure below can be written

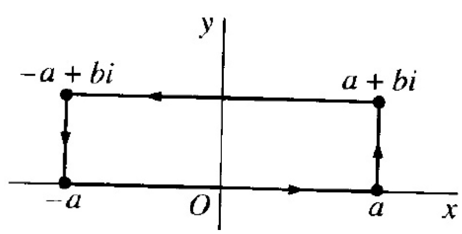
$$2\int_0^a e^{-x^2}\,dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx$$

and that the sum of the integrals along the vertical legs on the right and left can be written

$$ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}\,dy - ie^{-a^2}\int_0^b e^{y^2}e^{i2ay}\,dy.$$

Thus, with the aid of the Cauchy-Goursat theorem, show that

$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}\,dx + e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2ay\,dy.$$



(b) By accepting the fact that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and observing that

$$\left|\int_0^b e^{y^2}\sin 2ay\,dy\right| < \int_0^b e^{y^2}\,dy,$$

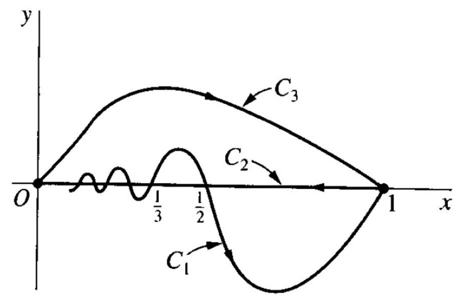
obtain the desired integration formula by letting a tend to infinity in the equation at the end of part (a).

23. Show that the path C_1 from the origin to the point $z = 1$ along the graph of the function defined by means of the equations

$$y(x) = \begin{cases} x^3 \sin \dfrac{\pi}{x} & \text{when} \quad 0 < x \le 1, \\ 0 & \text{when} \quad x = 0 \end{cases}$$

is a smooth arc that intersects the real axis an infinite number of times. Let C_2 denote the line segment along the real axis from $z = 1$ back to origin, and let C_3 denote any smooth arc from the origin to $z = 1$ that does not intersect itself and has only its end points in common with the arcs C_1 and C_2 (see the figure below). Apply the Cauchy-Goursat theorem to show that if a function f is entire, then

$$\int_{C_1} f(z)\,dz = \int_{C_3} f(z)\,dz \text{ and } \int_{C_2} f(z)\,dz = -\int_{C_3} f(z)\,dz.$$



Conclude that, even though the closed contour $C = C_1 + C_2$ intersects itself an infinite number of times,

$$\int_C f(z)\,dz = 0.$$

24. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1,\ 0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\frac{\theta}{2}}, \quad r > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2},$$

of the multi-valued function $z^{\frac{1}{2}}$. Show that

$$\int_C f(z)\,dz = 0.$$

by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy-Goursat theorem not apply here?

25. Show that if C is a positively oriented simple closed contour, then the area of the region enclosed by C can be written

$$\frac{1}{2i}\oint_C \bar{z}\,dz.$$

26. Evaluate $\displaystyle\int_C (z - i)\,dz$ where C is the parabolic segment:

$$z(t) = t + it^2, \quad -1 \le t \le 1$$

(a) by applying the integral formula involving an antiderivative of the integrand.

(b) by integrating along the straight line from $-1 + i$ to $1 + i$ and applying the Cauchy-Goursat Theorem.

27. Use an antiderivative to show that, for every contour C extending from a point z_1 to a point z_2,

$$\int_C z^n\,dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}), \quad n = 0, 1, 2, \cdots.$$

28. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

(a) $\displaystyle\int_{i}^{\frac{i}{2}} e^{\pi z}\,dz.$

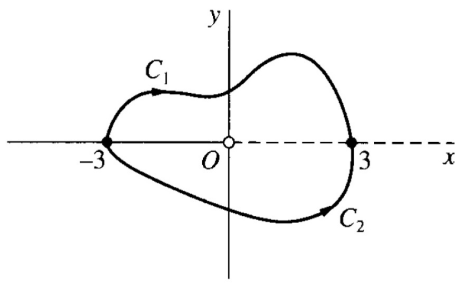
(b) $\displaystyle\int_{0}^{\pi+2i} \cos\frac{z}{2}\,dz.$

(c) $\displaystyle\int_{1}^{3} (z-2)^3\,dz.$

29. Evaluate the integral $\displaystyle\oint_C z^{\frac{1}{2}}\,dz$, where the integrand is the branch

$$z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\theta}{2}}, \quad r > 0,\ 0 < \theta < 2\pi,$$

of the square root function and $C = C_2 - C_1$ is the contour shown below.



30. Show that

$$\int_{-1}^{1} z^i\,dz = \frac{1+e^{-\pi}}{2}(1-i),$$

where z^i denotes the principal branch

$$z^i = e^{i\,\mathrm{Log}\,z}, \quad |z| > 0,\ -\pi < \mathrm{Arg}\,z < \pi$$

and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.