MCS 352 2009-2010 Spring
Exercise Set VI

1. Evaluate.

   (a) $\int_0^1 (3t-i)^2\,dt.$

   (b) $\int_0^1 (t+2i)^3\,dt.$

   (c) $\int_0^{\frac{\pi}{2}} \cosh(it)\,dt.$

   (d) $\int_0^2 \frac{t}{t+i}\,dt.$

   (e) $\int_0^{\frac{\pi}{4}} te^{it}\,dt.$

2. Let m and n be integers. Show that

$$\int_0^{2\pi} e^{imt}e^{-int} = \begin{cases} 0 & \text{when} \quad m \neq n, \\ 2\pi & \text{when} \quad m = n. \end{cases}$$

3. Show that $\int_0^\infty e^{-zt}\,dt = \frac{1}{z}$ provided $\mathrm{Re}(z) > 0$.

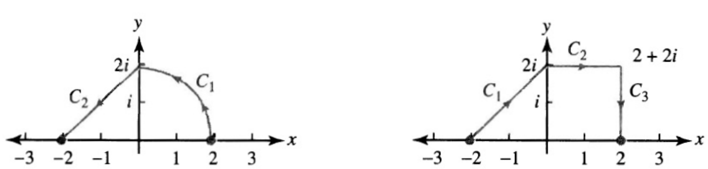
4. Let $f(t) = u(t) + iv(t)$, where u and v are differentiable. Show that

$$\int_a^b f(t)f'(t)\,dt = \frac{1}{2}(f(b))^2 - \frac{1}{2}(f(a))^2.$$

5. Give a parametrization of each contour

   (a) $C = C_1 + C_2$, as indicated on the left hand side of the figure below.

   (b) $C = C_1 + C_2 + c_3$, as indicated on the right hand side of the figure below.



6. Sketch the following curves.

   (a) $z(t) = t^2 - 1 + i(t+4)$, for $1 \leq t \leq 3$.

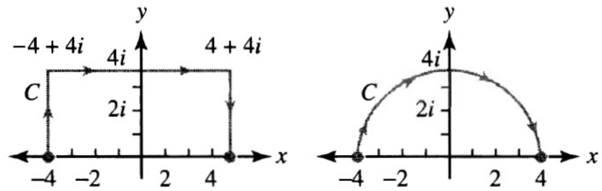
   (b) $z(t) = \sin t + i\cos 2t$, for $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

(c) $z(t) = 5\cos t - i3\sin t$, for $\frac{\pi}{2} \le t \le 2\pi$.

7. Evaluate $\displaystyle\int_C x\,dz$ from -4 to 4 along the following contours, as shown in the figures below.

   (a) The polygonal path C with vertices -4, $-4+4i$, $4+4i$, and 4.

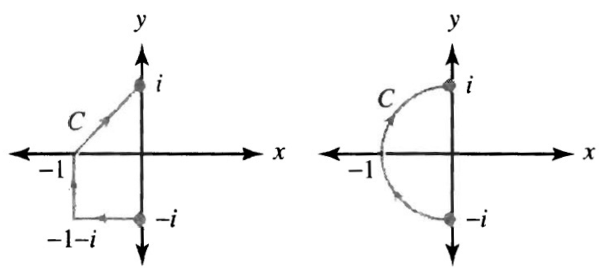
   (b) The contour C that is the upper half of the circle $|z| = 4$, oriented clockwise.



8. Evaluate $\displaystyle\int_C y\,dz$ for $-i$ to i along the following contours, as shown in the figures below.

   (a) The polygonal path C with vertices $-i$, $-1-i$, -1, and i.

   (b) The contour C that is the left half of the circle $|z| = 1$, oriented clockwise.



9. Let $C_r^+(a)$ denotes the circle of radius r centered at a, oriented counterclockwise. Evaluate

   (a) $\displaystyle\int_{C_4^+(0)} z\,dz$.

   (b) $\displaystyle\int_{C_4^+(0)} \overline{z}\,dz$.

   (c) $\displaystyle\int_{C_2^-(0)} \frac{1}{z}\,dz$. (The minus sign means clockwise orientation.)

   (d) $\displaystyle\int_{C_2^-(0)} \frac{1}{\overline{z}}\,dz$.

   (e) $\displaystyle\int_C (z+1)\,dz$, where C is $C_1^+(0)$ in the first quadrant.

   (f) $\displaystyle\int_C (x^2 - iy^2)\,dz$, where C is the upper half of $C_1^+(0)$.

   (g) $\displaystyle\int_C |z-1|^2\,dz$, where C is the upper half of $C_1^+(0)$.

10. Show that

   (a) $\displaystyle\left|\int_C \frac{1}{z^2-1}\,dz\right| \le \frac{\pi}{3}$, where C is the first quadrant portion of $C_2^+(0)$.

(b) $\left|\int_{C_R^+(0)} \frac{\text{Log}(z)}{z^2}\,dz\right| \leq 2\pi\left(\frac{\pi+\ln R}{R}\right)$.

11. Evaluate $\int_C z^2\,dz$ where C is the line segment from 1 to $1+i$.

12. Evaluate $\int_C |z^2|\,dz$ where C is given by $C:\ z(t)=t+it^2$, for $0\leq t\leq 1$.

13. Evaluate $\int_C e^z\,dz$, where C is the straight-line segment joining 1 to $1+i\pi$.

14. Evaluate $\int_C \overline{z}e^z\,dz$, where C is the square with vertices 0,1,1+i, and i taken with the counterclockwise orientation.

15. Evaluate $\int_C e^z\,dz$, where C is the straight-line segment joining 0 to $1+i$.

16. Evaluate $\int_C \cos z\,dz$, where C is the polygonal path from 0 to $1+i$ that consists of the line segments from 0 to 1 and 1 to $1+i$.

17. Determine the domain of analyticity for the following functions and evaluate $\oint_C f(z)\,dz$, where $C=\{z:\ |z|=1\}$, if

    (a) $f(z)=\dfrac{z}{z^2+2}$.

    (b) $f(z)=\dfrac{1}{z^2+2z+2}$.

    (c) $f(z)=\tan z$.

    (d) $f(z)=\text{Log}(z+5)$.

18. Show that $\oint_C z^{-1}\,dz=2\pi i$, where C is the square with vertices $1\pm i$ and $-1\pm i$.

19. Show that $\oint_C (4z^2-4z+5)^{-1}\,dz=0$, where $C=\{z:\ |z|=1\}$.

20. Find $\oint_C (2z-1)(z^2-z)^{-1}\,dz$ for the circle

    (a) $C=\{z:\ |z|=2\}$.

    (b) $C=\{z:\ |z|=\frac{1}{2}\}$.

21. Let C be the triangle with vertices 0, 1, and i and having positive orientation. Parametrize C and show that

    (a) $\oint_C 1\,dz=0$.

    (b) $\oint_C z\,dz=0$.

22. Evaluate $\oint_C (4z^2+4z-3)^{-1}\,dz=\oint_C (2z-1)^{-1}(2z+3)^{-1}\,dz$ for the circle

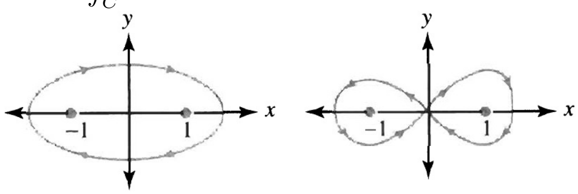
    (a) $C=\{z:\ |z|=1\}$.

    (b) $C=\{z:\ |z+\frac{2}{3}|=1\}$.

    (c) $C=\{z:\ |z|=3\}$.

23. Parametrize $C = \{z : |z| = 1$ with $z(t) = \cos t + i \sin t$, for $-\pi \le t\pi$. Use the principal branch of the square root function: $z^{\frac{1}{2}} = r^{\frac{1}{2}} \cos \dfrac{\theta}{2} + i r^{\frac{1}{2}} \sin \dfrac{\theta}{2}$, for $-\pi < \theta \le \pi$, to find $\oint_C z^{\frac{1}{2}}\, dz$. *Hint*: Take limits as $t \to -\pi$.

24. Evaluate $\int_C (z^2 - 1)^{-1}\, dz$ for the contours shown below:



(a) (b)

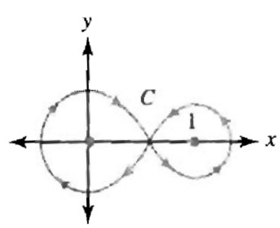
25. Evaluate $\oint_C |z|^2 e^z\, dz$, where $C = \{z : |z| = 1\}$.

26. Suppose that $f(z) = u(r, \theta) + iv(r, \theta)$ is analytic for all values of $z = re^{i\theta}$. Show that

$$\int_0^{2\pi} [u(r, \theta) \cos \theta - v(r, \theta) \sin \theta]\, d\theta = 0.$$

Hint: Integrate f around the circle $C = \{z : |z| = 1\}$.

27. If C is the figure eight contour shown below,



evaluate

(a) $\int_C (z^2 - z)^{-1}\, dz$.

(b) $\int_C (2z - 1)(z^2 - z)^{-1}\, dz$.

28. Evaluate.

(a) $\int_C z^2\, dz$, where C is the line segment from $1 + i$ to $2 + i$.

(b) $\int_C \cos z\, dz$, where C is the line segment from $-i$ to $1 + i$.

(c) $\int_C e^z\, dz$, where C is the line segment from 2 to $i\frac{\pi}{2}$.

(d) $\int_C ze^z\, dz$, where C is the line segment from $-1 - i\frac{\pi}{2}$ to $2 + i\pi$.

(e) $\int_C \dfrac{1 + z}{z}\, dz$, where C is the line segment from 1 to i.

(f) $\int_C \sin \dfrac{z}{2}\, dz$, where C is the line segment from 0 to $\pi - 2i$.

(g) $\int_C (z^2 + z^{-2})\, dz$, where C is the line segment from i to $1+i$.

(h) $\int_C ze^{z^2}\, dz$, where C is the line segment from $1-2i$ to $1+2i$.

(i) $\int_C z\cos z\, dz$, where C is the line segment from 0 to i.

(j) $\int_C \sin^2 z\, dz$, where C is the line segment from 0 to i.

(k) $\int_C \operatorname{Log} z\, dz$, where C is the line segment from 1 to $1+i$.

(l) $\int_C \frac{1}{z^2 - z}\, dz$, where C is the line segment from 2 to $2+i$.

(m) $\int_C \frac{2z-1}{z^2 - z}\, dz$, where C is the line segment from 2 to $2+i$.

(n) $\int_C \frac{z-2}{z^2 - z}\, dz$, where C is the line segment from 2 to $2+i$.

29. Show that $\int_C 1\, dz = z_2 - z_1$, where C is the line segment from z_1 to z_2, by parametrizing C.

30. Let z_1 and z_2 be points in the right half-plane and let C be the line segment joining them. Show that $\int_C \frac{dz}{z^2} = \frac{1}{z_1} - \frac{1}{z_2}$.

31. Let $z^{\frac{1}{2}}$ be the principal branch of the square root function.

(a) Evaluate $\int_C \frac{dz}{2z^{\frac{1}{2}}}$, where C is the line segment joining 9 to $3+4i$.

(b) Evaluate $\int_C z^{\frac{1}{2}}$, where C is the right half of the circle $\{z : |z| = 2\}$ joining $-2i$ to $2i$.

32. Using partial fraction decomposition, show that if z lies in the right half-plane and C is the line segment joining 0 to z, then

$$\int_C \frac{d\xi}{\xi^2 + 1} = \operatorname{Arctan} z = \frac{i}{2}\operatorname{Log}(z+i) - \frac{i}{2}\operatorname{Log}(z-i) + \frac{\pi}{2}.$$

33. Let f' and g' be analytic for all z and let C be any contour joining the points z_1 and z_2. Show that

$$\int_C f(z)g'(z)\, dz = f(z_2)g(z_2) - f(z_1)g(z_1) - \int_C f'(z)g(z)\, dz.$$

34. Show that $\int_C z^i\, dz = (i-1)\frac{1+e^{-\pi}}{2}$, where C is the upper half of $\{z : |z| = 1\}$.