MCS 352 2009-2010 Spring
Exercise Set V

1. Express e^z in the form $u + iv$ for the following values of z.
   (a) $-\frac{\pi}{3}$.
   (b) $\frac{1}{2} - i\frac{\pi}{4}$.
   (c) $-4 + 5i$.
   (d) $-1 + i\frac{3\pi}{2}$.
   (e) $1 + i\frac{5\pi}{4}$.
   (f) $\frac{\pi}{3} - 2i$.
2. Find the real and imaginary parts of
   (a) e^{e^z}.
   (b) z^z.
3. Find all values of z for which the following equations hold.
   (a) $e^z = -4$.
   (b) $e^z = 2 + 2i$.
   (c) $e^z = \sqrt{3} - i$.
   (d) $e^z = -1 + i\sqrt{3}$.
   (e) $e^z = -2$.
   (f) $e^z = 1 + i\sqrt{3}$.
   (g) $e^{2z-1} = 1$.
4. Prove that $e^{z+i\pi} = e^{z-i\pi}$ holds for all z.
5. Express e^{z^2} and $e^{\frac{1}{z}}$ in the Cartesian form $u(x, y) + iv(x, y)$.
6. Show that $\left|e^{-z}\right| < 1$ if and only if $\mathrm{Re}(z) > 0$.
7. Show that $f(z) = ze^z$ is analytic for all z by showing that its real and imaginary parts satisfy the Cauchy-Riemann sufficient conditions for differentiability.
8. Find the derivatives of the following.
   (a) e^{iz}.
   (b) $z^4 e^{z^3}$.
   (c) $e^{(a+ib)z}$.
   (d) $e^{\frac{1}{z}}$.
9. Let n be a positive integer. Show that
   (a) $(e^z)^n = e^{nz}$.
   (b) $\frac{1}{(e^z)^n} = e^{-nz}$.
10. Show that $\sum_{n=0}^{\infty} e^{inz}$ converges for $\mathrm{Im}(z) > 0$.
11. Use the fact e^{z^2} is analytic to show that $e^{x^2-y^2} \sin 2xy$ is a harmonic function.
12. Show the following concerning the exponential map.
    (a) The image of the line $\{(x, y) : x = t,\ y = 2\pi + t\}$, where $-\infty < t < \infty$ is a spiral.
    (b) The image of the first quadrant $\{(x, y) : x > 0,\ y > 0\}$ is the region $\{w : |w| > 1\}$.
    (c) If a is a real constant, the horizontal strip $\{(x, y) : a < y \leq a + 2\pi\}$ is mapped one-to-one and onto the nonzero complex numbers.
    (d) The image of the vertical line segment $\{(x, y) : x = 2,\ y = t\}$ where $\frac{\pi}{6} < t < \frac{7\pi}{6}$ is a half circle.
    (e) The image of the horizontal ray $\left\{(x, y) : x > 0,\ y = \frac{\pi}{3}\right\}$ is a ray.
13. Show that
    (a) $e^{2\pm 3\pi i} = -e^2$.
    (b) $e^{\frac{2+\pi i}{4}} = \sqrt{\frac{e}{2}}(1 + i)$.
    (c) $e^{z+\pi i} = -e^z$.
14. Write $|e^{2z+i}|$ and $|e^{iz^2}|$ in terms of x and y. Then show that
    $$|e^{2z+i} + e^{iz^2}| \leq e^{2x} + e^{-2xy}.$$
15. Show that $\overline{e^{iz}} = e^{i\bar{z}}$ if and only if $z = n\pi,\ n \in \mathbb{Z}$.
16. (a) Show that if e^z is real, then $\mathrm{Im}(z) = n\pi,\ n \in \mathbb{Z}$.
    (b) If e^z is pure imaginary, what restriction is placed on z?
17. Describe the behavior of $e^z = e^x e^{iy}$ as
    (a) x tends to $-\infty$;
    (b) y tends to ∞;
18. Write $\mathrm{Re}(e^{\frac{1}{z}})$ in terms of x and y. Why is this function harmonic in every domain that does not contain the origin?
19. Find all values for
    (a) $\mathrm{Log}(ie^2)$.
    (b) $\mathrm{Log}(\sqrt{3} - i)$.
    (c) $\mathrm{Log}(i\sqrt{2} - \sqrt{2})$.

(d) $\text{Log}((1+i)^4)$.

(e) $\log(-3)$.

(f) $\log 8$.

(g) $\log(4i)$.

(h) $\log(-\sqrt{3}-i)$.

20. Find all the values of z for which each equation holds.

(a) $\text{Log}(z) = 1 - i\frac{\pi}{4}$.

(b) $\text{Log}(z-1) = i\frac{\pi}{2}$.

(c) $e^z = -ie$.

(d) $e^{z+1} = i$.

21. Show that the following functions are harmonic in the right half-plane $\{z : \text{Re}\, z > 0\}$.

(a) $u(x,y) = \ln(x^2+y^2)$.

(b) $\arctan\left(\frac{y}{x}\right)$.

22. Show that $z^n = e^{n\log_\alpha(z)}$, where n is an integer and $\log_\alpha(z)$ is any branch of the logarithm.

23. Construct branches of $f(z) = \log(z+2)$ that are analytic at all points in the plane except at points on the following rays.

(a) $\{(x,y) : x \geq -2,\ y = 0\}$.

(b) $\{(x,y) : x = -2,\ y \geq 0\}$.

(c) $\{(x,y) : x = -2,\ y \leq 0\}$.

24. Show that the mapping $w = \text{Log}\, z$ maps

(a) the ray $\left\{z = re^{i\theta} : r > 0,\ \theta = \frac{\pi}{3}\right\}$ one-to-one and onto the horizontal line $\left\{(u,v) : v = \frac{\pi}{3}\right\}$.

(b) the semicircle $\left\{z = 2e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right\}$ one-to-one and onto the vertical line segment $\left\{(\ln 2, v) : -\frac{\pi}{2} \leq v \leq \frac{\pi}{2}\right\}$.

25. Show that

(a) $\text{Log}(-ei) = 1 - \frac{\pi}{2}i$.

(b) $\text{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$.

26. Verify that when $n = 0, \pm1, \pm2, \cdots$.

(a) $\log e = 1 + 2n\pi i$.

(b) $\log i = \left(2n + \frac{1}{2}\right)\pi i$.

(c) $\log(-1+\sqrt{3}i) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$.

27. Show that

(a) $\text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$.

(b) $\text{Log}(-1+i)^2 \neq 2\,\text{Log}(-1+i)$.

28. Show that

(a) $\log(i^2) = 2\log i$ when

$$\log z = \ln r + i\theta, \quad r > 0,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}.$$

(b) $\log(i^2) \neq 2\log i$ when

$$\log z = \ln r + i\theta, \quad r > 0,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}.$$

29. Show that

(a) the set of values of $\log(i^{\frac{1}{2}})$ is $(n + \frac{1}{4})\pi i$, $n = 0, \pm1, \pm2, \cdots$ and that the same is true of $\frac{1}{2}\log i$.

(b) the set of values of $\log(i^2)$ is not the same as the set of values of $2\log i$.

30. Find all roots of the equation $\log z = i\frac{\pi}{2}$.

31. Show that

(a) the function $\text{Log}(z-i)$ is analytic everywhere except on the half line $y = 1,\ x \leq 0$.

(b) the function

$$\frac{\text{Log}(z+4)}{z^2+i}$$

is analytic everywhere except at the points $\pm\frac{1-i}{\sqrt{2}}$ and on the portion $x \leq -4$ of the real axis.

32. Show that

$$\text{Re}(\log(z-1)) = \frac{1}{2}\ln((x-1)^2+y^2),\ z \neq 1.$$

Why must this function satisfy Laplace's equation when $z \neq 1$?

33. Show that if $rez_1 > 0$ and $rez_2 > 0$, then

$$\text{Log}(z_1z_2) = \text{Log}\, z_1 + \text{Log}\, z_2.$$

34. Show that, for any two nonzero complex numbers z_1 and z_2,

$$\text{Log}(z_1z_2) = \text{Log}\, z_1 + \text{Log}\, z_2 + 2N\pi i$$

where N has one of the values $0, \pm1$.

35. Show that when $n = 0, \pm1, \pm2, \cdots$

(a) $(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right)\exp\left(\frac{i}{2}\ln 2\right)$.

(b) $(-1)^{\frac{1}{\pi}} = e^{(2n+1)i}$.

36. Find the principal value of

(a) 4^i.

(b) $(1+i)^{\pi i}$.

(c) $(-1)^{\frac{1}{\pi}}$.

(d) $(1+i\sqrt{3})^{\frac{i}{2}}$.

(e) $\left(\frac{e}{2}(-1-\sqrt{3}i)\right)^{3\pi i}$.

(f) $(1-i)^{4i}$.

37. Find all values of

(a) i^i.

(b) $(-1)^{\sqrt{2}}$.

(c) $(i)^{\frac{2}{\pi}}$.

(d) $(1+i)^{2-i}$.

(e) $(-1)^{\frac{3}{4}}$.

(f) $(i)^{\frac{2}{3}}$.

38. Show that $(-1+i\sqrt{3})^{\frac{3}{2}} = \pm 2\sqrt{2}$.

39. Show that the result in Exercise 38 could have obtained by writing

(a) $(-1+i\sqrt{3})^{\frac{3}{2}} = \left((-1+i\sqrt{3})^{\frac{1}{2}}\right)^3$ and first finding the square roots of $-1+i\sqrt{3}$.

(b) $(-1+i\sqrt{3})^{\frac{3}{2}} = \left((-1+i\sqrt{3})^3\right)^{\frac{1}{2}}$ and first cubing $-1+i\sqrt{3}$.

40. Let $c = a + bi$ be a fixed complex number, where $c \neq 0, \pm 1, \pm 2, \cdots$, and note that i^c is multiple-valued. What restriction must be placed on the constant c so that the values of $|i^c|$ are all the same?

41. For $z = re^{i\theta} \neq 0$, show that the principal branch of

(a) z^i is given by the equation

$$z^i = e^{-\theta}(\cos(\ln r) + i \sin(\ln r)),$$

where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) z^α (α a real number) is given by the equation

$$z^\alpha = r^\alpha \cos \alpha\theta + ir^\alpha \sin \alpha\theta,$$

where $r > 0$ and $-\pi < \theta \leq \pi$.

42. Write $\tan z$ in the form $u(x,y) + iv(x,y)$.

43. Show that for all z,

(a) $\sin(\pi - z) = \sin z$.

(b) $\sin\left(\frac{\pi}{2} - z\right) = \cos z$.

(c) $\sinh(z + i\pi) = -\sinh z$.

(d) $\tanh(z + i\pi) = \tanh z$.

(e) $\sin(iz) = i \sinh z$.

(f) $\cosh(iz) = \cos z$.

44. Express the following quantities in $u + iv$ form.

(a) $\cos(1+i)$.

(b) $\sin\left(\frac{\pi + 4i}{4}\right)$.

(c) $\sin 2i$.

(d) $\cos(-2+i)$.

(e) $\tan\left(\frac{\pi + 2i}{4}\right)$.

(f) $\tan\left(\frac{\pi + i}{2}\right)$.

(g) $\sinh(1 + i\pi)$.

(h) $\cosh\left(\frac{i\pi}{2}\right)$.

(i) $\cosh\left(\frac{4 - i\pi}{4}\right)$.

45. Find the derivatives of the following, and state where they are defined.

(a) $\sin\left(\frac{1}{z}\right)$.

(b) $z \tan z$.

(c) $\sec z^2$.

(d) $z \csc^2 z$.

(e) $z \sinh z$.

(f) $\cosh z^2$.

(g) $z \tan z$.

46. Show that

(a) $\sin \overline{z} = \overline{\sin z}$ holds for all z.

(b) $\sin \overline{z}$ is nowhere analytic.

(c) $\cos \overline{z} = \overline{\cos z}$ holds for all z.

(d) $\cos \overline{z}$ is nowhere analytic.

(e) $\sinh \overline{z} = \overline{\sinh z}$ holds for all z.

(f) $\sinh \overline{z}$ is nowhere analytic.

(g) $\cosh \overline{z} = \overline{\cosh z}$ holds for all z.

(h) $\cosh \overline{z}$ is nowhere analytic.

47. Show that $\overline{\tanh z} = \tanh \overline{z}$ at points where $\cosh z \neq 0$.

48. Show that

(a) $\lim_{z \to 0} \frac{\cos z - 1}{z} = 0$.

(b) $\lim_{y \to \infty} \tan(x_0 + iy) = i$, where x_0 is any fixed real number.

49. Find all values of z for which each equation holds.

(a) $\sin z = \cosh 4$.

(b) $\cos z = 2$.

(c) $\sin z = i \sinh 1$.

(d) $\sinh z = \frac{i}{2}$.

(e) $\cosh z = 1$.

(f) $\sinh z = i$.

(g) $\cosh z = \frac{1}{2}$.

(h) $\cosh z = -2$.

50. Show that for $z = x + iy$,

(a) $|\sin z| \geq |\sin x|$.

(b) $|\cos z| \geq |\cos x|$.

51. Show that for $z = x + iy$,

(a) $|\sinh y| \leq |\sin z| \leq \cosh y$.

(b) $|\sinh y| \leq |\cos z| \leq \cosh y$.

(c) $|\sinh x| \le |\cosh z| \le \cosh x$.

52. Show that $|\cos z|^2 + |\sin z|^2 \ge 1$ for all $z \in \mathbb{C}$, and the equality holds if and only if z is a real number.

53. Show that

(a) $\overline{\cos(iz)} = \cos(i\overline{z})$ for all z.

(b) $\overline{\sin(iz)} = \sin(i\overline{z})$ if and only if $z = n\pi i,\ n \in \mathbb{Z}$.

54. Give an elegant argument that explains why the following functions are harmonic.

(a) $h(x, y) = \sin x \cosh y$.

(b) $h(x, y) = \cos x \sinh y$.

(c) $h(x, y) = \sinh x \cos y$.

(d) $h(x, y) = \cosh x \sin y$.

55. Establish the following identities.

(a) $e^{iz} = \cos z + i \sin z$.

(b) $\cos z = \cos x \cosh y - i \sin x \sinh y$.

(c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$.

(d) $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

(e) $|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

(f) $|\sinh z|^2 = \sinh^2 x + \sin^2 y$.

(g) $\cosh z = \cosh x \cos y + i \sinh x \sin y$.

(h) $\cosh^2 z - \sinh^2 z = 1$.

(i) $\sinh z + \cosh z = e^z$.

(j) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$.

(k) $\sinh(z + \pi i) = -\sinh z$.

(l) $\cosh(z + \pi i) = -\cosh z$.

(m) $\tanh(z + \pi i) = \tanh z$.

56. Show that

(a) $\sinh z = 0$ if and only if $z = n\pi i,\ n \in \mathbb{Z}$.

(b) $\cosh z = 0$ if and only if $z = \left(\frac{\pi}{2} + n\pi\right) i,\ n \in \mathbb{Z}$.

57. Find all values of the following.

(a) $\arcsin \frac{5}{4}$.

(b) $\arccos \frac{5}{3}$.

(c) $\arcsin 3$.

(d) $\arccos 3i$.

(e) $\arctan 2i$.

(f) $\arctan i$.

(g) $\arctan(1 + i)$.

(h) $\operatorname{arcsinh} i$.

(i) $\operatorname{arcsinh} \frac{3}{4}$.

(j) $\operatorname{arccosh} i$.

(k) $\operatorname{arccosh}(-1)$.

(l) $\operatorname{arccosh} \frac{1}{2}$.

(m) $\operatorname{arctanh} i$.

(n) $\operatorname{arctanh} 0$.

(o) $\operatorname{arctanh} i\sqrt{3}$.

58. Solve the equation $\sin z = 2$ for z by

(a) equating real parts and imaginary parts in that equation.

(b) Using the formula for $\arcsin z$.

59. Establish the following identities.

(a) $\arccos z = -i \log\left(z + i(1 - z^2)^{\frac{1}{2}}\right)$.

(b) $\dfrac{d}{dz} \arccos z = -\dfrac{1}{(1 - z^2)^{\frac{1}{2}}}$.

(c) $\arctan z = \dfrac{i}{2} \log\left(\dfrac{i + z}{i - z}\right)$.

(d) $\dfrac{d}{dz} \arctan z = \dfrac{1}{1 + z^2}$.

(e) $\arcsin z + \arccos z = \dfrac{\pi}{2} + 2n\pi$, where $n \in \mathbb{Z}$.

(f) $\dfrac{d}{dz} \operatorname{arctanh} z = \dfrac{1}{1 - z^2}$.

(g) $\operatorname{arcsinh} z = \log\left(z + (z^2 + 1)^{\frac{1}{2}}\right)$.

(h) $\dfrac{d}{dz} \operatorname{arcsinh} z = \dfrac{1}{(z^2 + 1)^{\frac{1}{2}}}$.

(i) $\operatorname{arccosh} z = \log\left(z + (z^2 - 1)^{\frac{1}{2}}\right)$.

(j) $\dfrac{d}{dz} \operatorname{arccosh} z = \dfrac{1}{(z^2 - 1)^{\frac{1}{2}}}$.