MCS 352 2009-2010 Spring
Exercise Set IV

1. Find the following limits.

   (a) $\displaystyle\lim_{n\to\infty}\left(\frac{1}{2}+\frac{i}{4}\right)^n$.

   (b) $\displaystyle\lim_{n\to\infty}\frac{n^2+i2^n}{2^n}$.

2. Suppose that $\displaystyle\sum_{n=1}^{\infty} z_n = S$. Show that $\displaystyle\sum_{n=1}^{\infty}\overline{z_n}=\overline{S}$.

3. Evaluate $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{2+i}\right)^n$.

4. Show that $\displaystyle\sum_{n=0}^{\infty}\frac{(z+i)^n}{2^n}$ converges for all values of z in the disk $D_2(-i)=\{z : |z+i|<2\}$ and diverges if $|z+i|>2$.

5. Use the ratio test to show that the following series converge.

   (a) $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1+i}{2}\right)^n$.

   (b) $\displaystyle\sum_{n=1}^{\infty}\frac{(1+i)^n}{n2^n}$.

   (c) $\displaystyle\sum_{n=1}^{\infty}\frac{(1+i)^n}{n!}$.

6. Use the ratio test to find a disk in which the following series converge.

   (a) $\displaystyle\sum_{n=0}^{\infty}(1+i)^n z^n$.

   (b) $\displaystyle\sum_{n=0}^{\infty}\frac{(z-i)^n}{(3+4i)^n}$.

7. (a) Use the formula for geometric series with $z=re^{i\theta}$, where $r<1$, to show that

$$\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} r^n e^{in\theta} = \frac{1-r\cos\theta+ir\sin\theta}{1+r^2-2r\cos\theta}.$$

   (b) Use part (a) to obtain

$$\sum_{n=0}^{\infty} r^n\cos n\theta = \frac{1-r\cos\theta}{1+r^2-2r\cos\theta}$$

   and

$$\sum_{n=0}^{\infty} r^n\sin n\theta = \frac{r\sin\theta}{1+r^2-2r\cos\theta}.$$

8. Find the radius of convergence of the following.

   (a) $\displaystyle f(z)=\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(2n)!}$.

   (b) $\displaystyle f(z)=\sum_{n=0}^{\infty}\left(\frac{4n^2}{2n+1}-\frac{6n^2}{3n+4}\right)^n z^n$.

   (c) $\displaystyle f(z)=\sum_{n=0}^{\infty}(2-(-1)^n)^n z^n$.

   (d) $\displaystyle f(z)=\sum_{n=0}^{\infty}\left(\frac{3n+7}{4n+2}\right)^n z^n$.

   (e) $\displaystyle f(z)=\sum_{n=0}^{\infty} z^{2n}$.

   (f) $\displaystyle f(z)=\sum_{n=0}^{\infty}(1+(-1)^n)^n z^n$.

   (g) $\displaystyle f(z)=\sum_{n=1}^{\infty} nz^n$.

   (h) $\displaystyle f(z)=\sum_{n=0}^{\infty} z^{n^2}$.

9. Does there exist a power series $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ that converges at $z_1=4-i$ and diverges at $z_2=2+3i$? Why or why not?

10. Show in two ways that the sequence

$$z_n=-2+i\frac{(-1)^n}{n^2},\quad (n=1,2,\cdots)$$

    converges to -2.

11. Show that any series of the form $\displaystyle\sum_{n=0}^{\infty} c_n z^n$ with $c_n=\pm1$ for all n has radius of convergence 1.

12. Find the radius of convergence of the following power series:

    (a) $\displaystyle\sum_{n=0}^{\infty} z^{n!}$.

    (b) $\displaystyle\sum_{n=0}^{\infty}(n+2^n)z^n$.

13. Suppose $\sum c_n z^n$ has radius of convergence R. Find the radius of convergence of

    (a) $\sum n^p c_n z^n$.

(b) $\sum |c_n| z^n$.

(c) $\sum c_n^2 z^n$.

(d) $\sum c_n^2 z^{2n}$.

14. Find the radius of convergence of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!}$.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{n! z^n}{n^n}$.

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n z^n}{n!}$.

15. Find the domain of convergence of

(a) $\displaystyle\sum_{n=0}^{\infty} n(z-i)^n$.

(b) $\displaystyle\sum_{n=0}^{\infty} n^2(2z-1)^n$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}(z+1)^n$.

16. If $\lim_{n\to\infty} z_n = A$, prove that

$$\lim_{n\to\infty} \frac{1}{n}(z_1 + z_2 + \cdots + z_n) = A.$$

17. Expand $(1-z)^{-m}$, m a positive integer, in powers of z.

18. Expand $\dfrac{2z+3}{z+1}$, in powers of $z-1$. What is the radius of convergence?

19. If $f(z) = \sum a_n z^n$, what is $\sum n^3 a_n z^n$?

20. For what values of z is

$$\sum_{0}^{\infty} \left(\frac{z}{1+z}\right)^n$$

convergent?

21. For what values of z is

$$\sum_{0}^{\infty} \frac{z^n}{1+z^{2n}}$$

convergent?