MCS 352 2009-2010 Spring
Exercise Set III

1. Find the values of
   (a) $(1+2i)^3$.
   (b) $\dfrac{5}{-3+4i}$.
   (c) $\left(\dfrac{2+i}{3-2i}\right)^2$.
   (d) $(1+i)^n+(1-i)^n$.

2. If $z=x+iy$ (x and y real), find the real and imaginary parts of
   (a) z^4.
   (b) $\dfrac{1}{z}$.
   (c) $\dfrac{z-1}{z+1}$.
   (d) $\dfrac{1}{z^2}$.

3. Show that
   (a) $\left(\dfrac{-1\pm i\sqrt{3}}{2}\right)^3=1$.
   (b) $\left(\dfrac{\pm 1\pm i\sqrt{3}}{2}\right)^6=1$.

   for all combinations of signs.

4. Compute
   (a) $\sqrt{i}$.
   (b) $\sqrt{-i}$.
   (c) $\sqrt{1+i}$.
   (d) $\sqrt{\dfrac{1-i\sqrt{3}}{2}}$.

5. Find the four values of $\sqrt[4]{-1}$.

6. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.

7. Solve the quadratic equation $z^2+(\alpha+i\beta)z+\gamma+i\delta=0$.

8. Verify by calculation that the values of $\dfrac{z}{z^2+1}$ for $z=x+iy$ and $z=x-iy$ are conjugate.

9. Find the absolute values of
   (a) $-2i(3+i)(2+4i)(1+i)$.
   (b) $\dfrac{(3+4i)(-1+2i)}{(-1-i)(3-i)}$.

10. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right|=1$$
if either $|a|=1$ or $|b|=1$. What exception must be made if $|a|=|b|=1$?

11. Find the conditions under which the equation $az+b\bar{z}+c=0$ in one complex unknown has exactly one solution, and compute that solution.

12. Prove that
$$\left|\frac{a-b}{1-\bar{a}b}\right|<1$$
if $|a|<1$ and $|b|<1$.

13. Express $\cos 3\theta$, $\cos 4\theta$, and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin 3\theta$.

14. Simplify $1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta$ and $1+\sin\theta+\sin 2\theta+\cdots+\sin n\theta$.

15. Use definition of limit to prove that
    (a) $\lim_{z\to z_0} \operatorname{Re}(z)=\operatorname{Re}(z_0)$.
    (b) $\lim_{z\to z_0} \bar{z}=\bar{z}_0$.
    (c) $\lim_{z\to 0} \dfrac{\bar{z}^2}{z}=0$.

16. Let a, b, and c denote complex constants. Then use definition of limit to show that
    (a) $\lim_{z\to z_0}(az+b)=az_0+b$.
    (b) $\lim_{z\to z_0}(z^2+c)=z_0^2+c$.
    (c) $\lim_{z\to 1-i}[x+i(2x+y)]=1+i \quad (z=x+iy)$.

17. Find $\lim_{z\to i}\dfrac{iz^3-1}{z+i}$.

18. Show that the limit of the function
$$f(z)=\left(\frac{z}{\bar{z}}\right)^2$$
as z tends to 0 does not exist.

19. Use definition of limit to prove that: "If $\lim_{z\to z_0} f(z)=w_0$ then $\lim_{z\to z_0}|f(z)|=|w_0|$".

20. Prove the followings:
    (a) $\lim_{z\to z_0} f(z)=\infty$ if and only if $\lim_{z\to z_0}\dfrac{1}{f(z)}=0$.
    (b) $\lim_{z\to\infty} f(z)=w_0$ if and only if $\lim_{z\to 0} f\left(\dfrac{1}{z}\right)=w_0$.

(c) $\lim_{z\to\infty} f(z) = \infty$ if and only if $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$.

21. Use Exercise 20 to show that

(a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$.

(b) $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$.

(c) $\lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$.

22. Use Exercise 20 to show that when

$$T(z) = \frac{az+b}{cz+d} \quad (ad-bc \neq 0),$$

(a) $\lim_{z\to\infty} T(z) = \infty$ if $c = 0$.

(b) $\lim_{z\to\infty} T(z) = \frac{a}{c}$ and $\lim_{z\to -\frac{d}{c}} T(z) = \infty$ if $c \neq 0$.

23. Find $f'(z)$ when

(a) $f(z) = 3z^2 - 2z + 4$.

(b) $f(z) = (1-4z^2)^3$.

(c) $\frac{z-1}{2z+1}, \quad z \neq -\frac{1}{2}$.

(d) $f(z) = \frac{(1+z^2)^4}{z^2}, \quad z \neq 0$.

24. If f and g are entire functions, which of the following are necessarily entire?

(a) $(f(z))^3$.

(b) $f(z)g(z)$.

(c) $\frac{f(z)}{g(z)}$.

(d) $f\left(\frac{1}{z}\right)$.

(e) $f(z-1)$.

(f) $f(g(z))$.

25. In each case determine where the following functions are analytic.

(a) $f(z) = \frac{2z+1}{z(z^2+1)}$.

(b) $f(z) = \frac{z^3+i}{z^2-3z+2}$.

(c) $f(z) = \frac{z^2+1}{(z+2)(z^2+2z+2)}$.

26. Use L'Hôpital's rule to find the following limits.

(a) $\lim_{z\to i} \frac{z^4-1}{z-i}$.

(b) $\lim_{z\to -i} \frac{z^6+1}{z^2+1}$.

(c) $\lim_{z\to 1+i} \frac{z^4+4}{z^2-2z+2}$.

(d) $\lim_{z\to 1+i\sqrt{3}} \frac{z^6-64}{z^3+8}$.

(e) $\lim_{z\to -1+i\sqrt{3}} \frac{z^9-512}{z^3-8}$.

27. Show that the following functions are nowhere differentiable.

(a) $f(z) = \mathrm{Re}(z)$.

(b) $f(z) = \mathrm{Im}(z)$.

28. Show that $f'(z)$ does not exist at any point if

(a) $f(z) = z - \bar{z}$.

(b) $f(z) = 2x + ixy^2$.

(c) $f(z) = e^x e^{-iy}$.

29. Show that $f'(z)$ and its derivative $f''(z)$ exist everywhere, and find $f''(z)$ when

(a) $f(z) = iz + 2$.

(b) $f(z) = e^{-x}e^{-iy}$.

(c) $f(z) = z^3$.

(d) $f(z) = \cos x \cosh y - i \sin x \sinh y$.

30. Determine where $f'(z)$ exists and find its value when

(a) $f(z) = \frac{1}{z}$.

(b) $f(z) = x^2 + iy^2$.

(c) $f(z) = z\,\mathrm{Im}(z)$.

31. Show that each of these functions is differentiable in the indicated domain of definition, and find $f'(z)$.

(a) $f(z) = \frac{1}{z^4}$, $z \neq 0$.

(b) $f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$.

(c) $f(z) = e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)$, $r > 0$, $0 < \theta < 2\pi$.

32. Use the Cauchy-Riemann conditions to determine where the following functions are differentiable, and evaluate the derivatives at those points where they exist.

(a) $f(z) = iz + 4i$.

(b) $f(z) = f(x,y) = \frac{y+ix}{x^2+y^2}$.

(c) $f(z) = -2(xy+x) + i(x^2-2y-y^2)$.

(d) $f(z) = x^3 - 3x^2 - 3xy^2 + 3y^2 + i(3x^2y - 6xy - y^3)$.

(e) $f(z) = x^3 + i(1-y)^3$.

(f) $f(z) = z^2 + z$.

(g) $f(z) = x^2 + y^2 + i2xy$.

(h) $f(z) = |z - (2+i)^2|$.

33. Find the constants a and b such that $f(z) = (2x - y) + i(ax + by)$ is differentiable for all z.

34. Use any method to show that the following functions are nowhere differentiable.

(a) $h(z) = e^y \cos x + ie^y \sin x$.

(b) $g(z) = z + \bar{z}$.

35. Determine where the following functions are differentiable and where they are analytic. Explain!

   (a) $f(z) = x^3 + 3xy^2 + i(y^3 + 3x^2y)$.

   (b) $f(z) = 8x - x^3 - xy^2 + i(x^2y + y^3 - 8y)$.

   (c) $f(z) = x^2 - y^2 + i2|xy|$.

36. Let f be a nonconstant analytic function in the domain D. Show that the function $g(z) = \overline{f(z)}$ is not analytic in D.

37. Show that each of the following functions is entire.

   (a) $f(z) = 3x + y + i(3y - x)$.

   (b) $f(z) = \sin x \cosh y + i \cos x \sinh y$.

   (c) $f(z) = e^{-y} \sin x - ie^{-y} \cos x$.

   (d) $f(z) = (z^2 - 2)e^{-x}e^{-iy}$.

38. Show that each of these functions is nowhere analytic.

   (a) $f(z) = xy + iy$.

   (b) $f(z) = 2xy + i(x^2 - y^2)$.

   (c) $f(z) = e^y e^{ix}$.

39. Does an analytic function $f(z) = u(x, y) + iv(x, y)$ exist for which $v(x, y) = x^3 + y^3$? Why or why not?

40. (a) Show that $f(z) = x^2 + iy^2$ is differentiable at all points on the line $y = x$.

   (b) Show that it is nowhere analytic.

41. Assume that f is analytic in a region and that at every point, either $f = 0$ or $f' = 0$. Show that f is constant.

42. Show that a nonconstant analytic function cannot map a region into a straight line or into a circular arc.

43. Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

44. Suppose f is an entire function of the form

$$f(x, y) = u(x) + iv(y).$$

   Show that f is a linear polynomial.

45. Determine where the following functions are harmonic.

   (a) $u(x, y) = e^x \cos y$ and $v(x, y) = e^x \sin y$.

   (b) $u(x, y) = \ln(x^2 + y^2)$ for $(x, y) \neq (0, 0)$.

46. Let a, b, and c be real constants. Determine a relation among the coefficients that will guarantee that the function $\phi(x, y) = ax^2 + bxy + cy^2$ is harmonic.

47. Let $v(x, y) = \arctan \dfrac{y}{x}$ for $x \neq 0$. Compute the partial derivatives of v, and verify that v satisfies Laplace's equation.

48. Find an analytic function $f(z) = u(x, y) + iv(x, y)$ for the following expressions.

   (a) $u(x, y) = y^3 - 3x^2y$.

   (b) $u(x, y) = \sin y \sinh x$.

   (c) $v(x, y) = e^y \sin x$.

   (d) $v(x, y) = \sin x \cosh y$.

49. Assume that $u(x, y)$ is harmonic on a region D that is symmetric about the line $y = 0$. Show that $U(x, y) = u(x, -y)$ is harmonic on D.

50. Let v be a harmonic conjugate of u. Show that $-u$ is a harmonic conjugate of v.

51. Let v be a harmonic conjugate of u. Show that $u^2 - v^2$ is a harmonic function.

52. Suppose that v is a harmonic conjugate of u and that u is a harmonic conjugate of v. Show that u and v must be constants functions.

53. Use the polar form of Laplace's equation to show that the following functions are harmonic.

   (a) $u(r, \theta) = \left(r + \dfrac{1}{r}\right) \cos \theta$

   (b) $v(r, \theta) = \left(r - \dfrac{1}{r}\right) \sin \theta$.

   (c) $u(r, \theta) = r^n \cos n\theta$

   (d) $v(r, \theta) = r^n \sin n\theta$.

54. Show that $u(x, y)$ is harmonic in some domain and find a harmonic conjugate $v(x, y)$ when

   (a) $u(x, y) = 2x(1 - y)$.

   (b) $u(x, y) = 2x - x^3 + 3xy^2$.

   (c) $u(x, y) = \sinh x \sin y$.

   (d) $u(x, y) = \dfrac{y}{x^2 + y^2}$.

55. Show that if v_1 and v_2 are harmonic conjugates of u in a domain D, then $v_1(x, y)$ and $v_2(x, y)$ can differ at most by an additive constant.

56. Verify that the function $u(r, \theta) = \ln r$ is harmonic in the domain $r > 0$, $0 < \theta < 2\pi$ by showing that it satisfies the polar form of Laplace's equation. Then use a technique similar to the one used for harmonic functions in Cartesian coordinates, but involving the Cauchy-Riemann equations in polar form to derive a harmonic conjugate.

57. Find all analytic functions $f = u + iv$ with $u(x, y) = x^2 - y^2$.