MCS 352 2009-2010 Spring
Exercise Set II

1. Express the following functions of the form $u(x,y)+iv(x,y)$.
   (a) $f(z)=z^3$.
   (b) $f(z)=\bar{z}^2+(2-3i)z$.
   (c) $f(z)=\dfrac{1}{z^2}$.

2. Express the following functions in the polar coordinate form $u(r,\theta)+iv(r,\theta)$.
   (a) $f(z)=z^5+\bar{z}^5$.
   (b) $f(z)=z^5+\bar{z}^3$.
   (c) For what values of z are the above expressions valid? Why?

3. Let $w=f(z)=(3+4i)z-2+i$.
   (a) Find the image of the disk $|z-1|<1$.
   (b) Find the image of the line $x=t$, $y=1-2t$ for $-\infty<t<\infty$.
   (c) Find the image of the half-plane $\operatorname{Im} z>1$.

4. Let $w=(2+i)z-2i$. Find the triangle onto which the triangle with vertices $z_1=-2+i$, $z_2=-2+2i$, and $2+i$ is mapped.

5. Find the linear transformations $w=f(z)$ that satisfy the following conditions.
   (a) The points $z_1=2$ and $z_2=-3i$ map onto $w_1=1+i$ and $w_2=1$.
   (b) The circle $|z|=1$ maps onto the circle $|w-3+2i|=5$, and $f(-i)=3+3i$.
   (c) The triangle with vertices $-4+2i$, $-4+7i$, and $1+2i$ maps onto the triangle with vertices 1, 0, and $1+i$, respectively.

6. Write the function $f(z)=z^3+z+1$ in the form $f(z)=u(x,y)+iv(x,y)$.

7. Suppose that $f(z)=x^2-y^2-2y+i(2x-2xy)$, where $z=x+iy$. Use the expressions

$$x=\frac{z+\bar{z}}{2},\quad y=\frac{z-\bar{z}}{2i}$$

   to write $f(z)$ in terms of z, and simplify the result.

8. Find the images of the mapping $w=z^2$ in each case, and sketch the mapping.
   (a) The triangle with vertices 0, 2, $2+2i$.
   (b) The infinite strip $\{(x,y)\mid 1<x<2\}$.
   (c) The right half-plane region to the right of the hyperbola $x^2-y^2=1$.
   (d) The first quadrant region between the hyperbolas $xy=\dfrac{1}{2}$ and $xy=4$.

9. For what values of z does $(z^2)^{\frac{1}{2}}=z$ hold if the principle value of the square root is to be used?

10. Sketch the set of points satisfying the following relations.
    (a) $\operatorname{Re}(z^2)>4$.
    (b) $\operatorname{Im}(z^2)>6$.

11. Find and illustrate the images of the following sets under the mapping $w=z^{\frac{1}{2}}$.
    (a) $\left\{re^{i\theta}\mid r>1,\ \dfrac{\pi}{3}<\theta<\dfrac{\pi}{2}\right\}$.
    (b) $\left\{re^{i\theta}\mid 1<r<9,\ 0<\theta<\dfrac{2\pi}{3}\right\}$.
    (c) $\left\{re^{i\theta}\mid r<4,\ -\pi<\theta<\dfrac{\pi}{2}\right\}$.
    (d) The vertical line $\{(x,y)\mid x=4\}$.
    (e) The infinite strip $\{(x,y)\mid 2<y<6\}$.
    (f) The region to the right of the parabola $x=4-\dfrac{y^2}{16}$.

12. Find the image of the right half-plane $\operatorname{Re}(z)>1$ under the mapping $w=z^2+2z+1$.

13. Find the image of the following sets under the mapping $w=z^3$.
    (a) $\left\{re^{i\theta}\mid 1<r<2,\ \dfrac{\pi}{4}<\theta<\dfrac{\pi}{3}\right\}$.
    (b) $\left\{re^{i\theta}\mid r>3,\ \dfrac{2\pi}{3}<\theta<\dfrac{3\pi}{4}\right\}$.

14. Find the image of $\left\{re^{i\theta}\mid r>2,\ \dfrac{\pi}{4}<\theta<\dfrac{\pi}{3}\right\}$ under the following mappings.
    (a) z^3.
    (b) z^4.
    (c) z^6.

15. Find the image of the sector $r>0$, $-\pi<\theta<\dfrac{2\pi}{3}$ under the following mappings.
    (a) $z^{\frac{1}{2}}$.
    (b) $z^{\frac{1}{3}}$.
    (c) $z^{\frac{1}{4}}$.

16. Find the following limits.

(a) $\lim_{z \to 2+i} (z^2 - 4z + 2 + 5i)$.

(b) $\lim_{z \to 1+i} \dfrac{z^2 + z - 2 + i}{z^2 - 2z + 1}$.

(c) $\lim_{z \to 1+i} \dfrac{z^2 + z - 1 - 3i}{z^2 - 2z + 2}$.

17. Determine where the following functions are continuous.

(a) $z^4 - 9z^2 + iz - 2$.

(b) $\dfrac{z+1}{z^2+1}$.

(c) $\dfrac{z^2 + 6z + 5}{z^2 + 3z + 2}$.

(d) $\dfrac{z^4 + 1}{z^2 + 2z + 2}$.

(e) $\dfrac{x + iy}{x - 1}$.

(f) $\dfrac{x + iy}{|z| - 1}$.

18. Let $f(z) = \dfrac{z\,\mathrm{Re}(z)}{|z|}$ when $z \neq 0$, and let $f(0) = 0$. Show that $f(z)$ is continuous for all values of z.

19. Let $f(z) = \dfrac{z^2}{|z|^2} = \dfrac{x^2 - y^2 + i2xy}{x^2 + y^2}$.

(a) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = x$.

(b) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the line $y = 2x$.

(c) Find $\lim_{z \to 0} f(z)$ as $z \to 0$ along the parabola $y = x^2$.

(d) What can you conclude about the limit of $f(z)$ as $z \to 0$? Why?

20. Let $f(z) = f(x, y) = \dfrac{xy^3}{x^2 + 2y^6} + i\dfrac{x^3 y}{5x^6 + y^2}$ when $z \neq 0$, and let $f(0) = 0$.

(a) Show that $\lim_{z \to 0} f(z) = f(0) = 0$ if z approaches zero along any straight line that passes through the origin.

(b) Show that f is not continuous at the point 0.

21. Does $\lim_{z \to -4} \mathrm{Arg}(z)$ exist? Why?

22. Let $f(z) = \dfrac{x^2 + iy^2}{|z|^2}$ when $z \neq 0$, and let $f(0) = 1$. Show that $f(z)$ is not continuous at $z_0 = 0$.

23. Let $f(z) = xe^y + iy^2 e^{-x}$. Show that $f(z)$ is continuous for all values of z.

24. Let $f(z) = \dfrac{\mathrm{Re}(z)}{|z|}$ when $z \neq 0$, and let $f(0) = 1$. Is $f(z)$ continuous at the origin?

25. Let $f(z) = \dfrac{(\mathrm{Re}(z))^2}{|z|}$ when $z \neq 0$, and let $f(0) = 0$. Is $f(z)$ continuous at the origin?

26. Let $f_1(z) = |z|^{\frac{1}{3}} e^{i\frac{\mathrm{Arg}(z)}{3}} = r^{\frac{1}{3}} \cos\frac{\theta}{3} + ir^{\frac{1}{3}} \sin\frac{\theta}{3}$, where $|z| = r \neq 0$, and $\theta = \mathrm{Arg}(z)$. f_1 denotes the principal cube root function.

(a) Show that f_1 is a branch of the multivalued cube root $f(z) = z^{\frac{1}{3}}$.

(b) What is the range of f_1?

(c) Where is f_1 continuous?

27. Let $f_2(z) = r^{\frac{1}{3}} \cos\left(\frac{\theta + 2\pi}{3}\right) + ir^{\frac{1}{3}} \sin\left(\frac{\theta + 2\pi}{3}\right)$, where $r > 0$, and $-\pi < \theta \leq \pi$.

(a) Show that f_2 is a branch of the multivalued cube root $f(z) = z^{\frac{1}{3}}$.

(b) What is the range of f_2?

(c) Where is f_2 continuous?

(d) What is the branch point associated with f.

28. Find a branch of the multivalued cube root function that is different from those in Exercises 26 and 27. State the domain and range of the branch you find.

29. Let $f(z) = z^{\frac{1}{n}}$ denote the multivalued nth root, where n is a positive integer.

(a) Show that f is, in general an n-valued function.

(b) Write the principal nth root function.

(c) Write a branch of the multivalued function nth root function that is different from the one in part (b).

30. *Limits involving* ∞. The function $f(z)$ is said to have the limit L as z approaches ∞, and we write $\lim_{z \to \infty} f(z) = L$ if and only if for every $\epsilon > 0$ there exists an $R > 0$ such that $|f(z) - L| < \epsilon$ whenever $|z| > R$. Likewise, $\lim_{z \to z_0} f(z) = \infty$ if and only if for every $R > 0$ there exists $\delta > 0$ such that $|f(z)| > R$ whenever $0 < |z - z_0| < \delta$. Use this definition to

(a) show that $\lim_{z \to \infty} \dfrac{1}{z} = 0$.

(b) show that $\lim_{z \to 0} \dfrac{1}{z} = \infty$.

31. Show that the reciprocal transformation $w = \dfrac{1}{z}$ maps the vertical strip given by $0 < x < \dfrac{1}{2}$ onto the region in the right half-plane $\mathrm{Re}(w) > 0$ that lies outside the disk $\{w \,:\, |w - 1| < 1\}$.

32. Find the image of the quadrant $x > 1,\ y > 1$ under the mapping $\dfrac{1}{z}$.

33. Use the definition in Exercise 30 to prove that $\lim_{z \to \infty} \dfrac{z+1}{z-1} = 1$.

34. Show that $z = x + iy$ is mapped onto the point

$$\left(\frac{x}{x^2 + y^2 + 1}, \frac{y}{x^2 + y^2 + 1}, \frac{x^2 + y^2}{x^2 + y^2 + 1} \right)$$

on the Riemann sphere.