MCS 352 2009-2010 Spring
Exercise Set XIV

1. Determine the angle of rotation at the point $z = 2+i$ when the transformation is $w = z^2$. Show that the scale factor of the transformation at that point is $2\sqrt{5}$.

2. What angle of rotation is produced by the transformation $w = \frac{1}{z}$ at the point $z = 1$? $z = i$?

3. Show that under the transformation $w = \frac{1}{z}$, the images of the lines $y = x - 1$ and $y = 0$ are the circle $u^2 + v^2 - u - v = 0$ and the line $v = 0$, respectively. Sketch all four curves, determine corresponding directions along them, and verify the conformality of the mapping at the point $z = 1$.

4. Show that the angle of rotation at a nonzero point $z_0 = r_0 \exp(i\theta_0)$ under the transformation $w = z^n$, $n \in \mathbb{N}$ is $(n-1)\theta_0$. Determine the scale factor of the transformation at that point.

5. Show that the transformation $w = \sin z$ is conformal at all points except $z = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

6. Find the local inverse of the transformation $w = z^2$ at the point $z_0 = 2$, $z_0 = -2$, $z_0 = -i$.

7. State where the following mappings are conformal.

   (a) $w = \exp z$.

   (b) $w = \sin z$.

   (c) $w = z^2 + 2z$.

   (d) $w = \exp(z^2 + 1)$.

   (e) $w = \frac{1}{z}$.

   (f) $w = \frac{z+1}{z-1}$.

8. Find the angle of rotation $\alpha = \operatorname{Arg} f'(z)$ and the scale factor $|f'(z)|$ of the mapping $w = f(z)$ at the indicated points.

   (a) $w = \frac{1}{z}$ at the points 1, $1+i$, and i.

   (b) $w = \ln r + i\theta$, where $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ at the points 1, $1+i$, i, and -1.

   (c) $w = r^{\frac{1}{2}} \cos\frac{\theta}{2} + i r^{\frac{1}{2}} \sin\frac{\theta}{2}$, where $-\pi < \theta < \pi$, at the points i, 1, $-i$, and $3+4i$.

   (d) $\sin z$ at the points $\frac{\pi}{2} + i$, 0, and $-\frac{\pi}{2} + i$.

9. If $w = S(z) = \frac{(1-i)z+2}{(1+i)z+2}$, find $S^{-1}(w)$.

10. If $w = S(z) = \frac{i+z}{i-z}$, find $S^{-1}(w)$.

11. Find the image of the right half plane $\operatorname{Re} z > 0$ under $\frac{i(1-z)}{1+z}$.

12. Show that the bilinear transformation $w = \frac{i(1-z)}{1+z}$ maps the portion of the disk $|z| < 1$ that lies in the upper half plane $\operatorname{Im} z > 0$ on to the first quadrant $u > 0$, $v > 0$.

13. Find the image of the upper half plane $\operatorname{Im} z > 0$ under the transformation $w = \frac{(1-i)z+2}{(1+i)z+2}$.

14. Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = 0$, $z_2 = i$, and $z_3 = -i$ onto $w_1 = -1$, $w_2 = 1$, and $w_3 = 0$, respectively.

15. Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = -i$, $z_2 = 0$, and $z_3 = i$ onto $w_1 = -1$, $w_2 = i$, and $w_3 = 1$, respectively.

16. Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = 0$, $z_2 = 1$, and $z_3 = 2$ onto $w_1 = 0$, $w_2 = 1$, and $w_3 = \infty$, respectively.

17. Find the bilinear transformation $w = S(z)$ that maps the points $z_1 = 1$, $z_2 = i$, and $z_3 = -1$ onto $w_1 = 0$, $w_2 = 1$, and $w_3 = \infty$, respectively.

18. Show that the transformation $w = \frac{i+z}{i-z}$ maps the unit disk $|z| < 1$ onto the right half plane $\operatorname{Re} z > 0$.

19. Find the image of the lower half plane $\operatorname{Im} z < 0$ under $w = \frac{i+z}{i-z}$.

20. If $S_1(z) = \dfrac{z-2}{z+1}$ and $S_2(z) = \dfrac{z}{z+3}$, find $S_1(S_2(z))$ and $S_2(S_1(z))$.

21. Find the image of the quadrant $x > 0,\ y > 0$ under $w = \dfrac{z-1}{z+1}$.

22. Find the image of the horizontal strip $0 < y < 2$ under $w = \dfrac{z}{z-i}$.

23. Show that the bilinear transformation $w = S(z) = \dfrac{az+b}{cz+d}$ is conformal at all points $z \neq -\dfrac{d}{c}$.

24. A *fixed point* of a mapping $w = f(z)$ is a point z_0 such that $f(z_0) = z_0$. Show that a bilinear transformation can have at most two fixed points.

25. Find the fixed points of $w = \dfrac{z-1}{z+1}$, $w = \dfrac{4z+3}{2z-1}$.

26. Find the image of the semi-infinite strip $0 < x < \dfrac{\pi}{2},\ y > 0$, under the transformation $w = \exp(iz)$.

27. Find the image of the rectangle $0 < x < \ln 2,\ 0 < y < \dfrac{\pi}{2}$, under the transformation $w = \exp z$.

28. Find the image of the first quadrant $x > 0,\ y > 0$, under $w = \dfrac{2}{\pi} \log z$.

29. Find the image of the annulus $1 < |z| < e$ under $w = \log z$.

30. Show that $w = \frac{2-z^2}{z^2}$ maps the portion of the right half plane $\operatorname{Re} z > 0$ that lies to the right of the hyperbola $x^2 - y^2 = 1$ onto the unit disk $|w| < 1$.

31. Show that the function $w = \dfrac{e^z - i}{e^z + i}$ maps the horizontal strip $-\pi < \operatorname{Im} z < 0$ onto the region $1 < |w|$.

32. Show that the function $w = \dfrac{e^z - i}{e^z + i}$ maps the horizontal strip $|y| < \dfrac{\pi}{2}$ onto the unit disk $|w| < 1$.

33. Find the image of the upper half plane $\operatorname{Im} z > 0$ under $w = \operatorname{Log} \frac{1+z}{1-z}$.

34. Find the image of the portion of the upper half plane $\operatorname{Im} z > 0$ that lies outside the circle $|z| = 1$ under the transformation $w = \operatorname{Log} \frac{1+z}{1-z}$.

35. Show that the function $w = \dfrac{(1+z)^2}{(1-z)^2}$ maps the portion of the disk $|z| < 1$ that lies in the first quadrant onto the portion of the upper half plane $\operatorname{Im} w > 0$ that lies outside the unit disk.

36. Find the image of the upper half plane $\operatorname{Im} z > 0$ under $w = \operatorname{Log}(1 - z^2)$.

37. Show that the transformation $w = \dfrac{z^2 - 1}{z^2 + 1}$ maps the portion of the first quadrant $x > 0,\ y > 0$, that lies outside the circle $|z| = 1$ onto the first quadrant $u > 0,\ v > 0$.

38. Find the image of the sector $r > 0,\ 0 < \theta < \dfrac{\pi}{4}$, under $w = \dfrac{i - z^4}{i + z^4}$.

39. Find the image of the infinite strip $\{z :\ 0 < \operatorname{Re} z < 1\}$ under the mapping $w = iz$.

40. Find the image of the half plane $\{z :\ \operatorname{Re} z > 0\}$ under the mapping $w = iz + i$.

41. Find the image of the half plane $\{z :\ \operatorname{Im} z > 0\}$ under the mapping $w = (1+i)z$.

42. Find the image of the half plane $\{z :\ \operatorname{Im} z > 1\}$ under the mapping $w = (1-i)z$.

43. Find the image of the semi-infinite strip $\{z :\ \operatorname{Re} z > 0,\ 0 < \operatorname{Im} z < 2\}$ under the mapping $w = iz + 1$.

44. Show that when $c_1 < 0$, the image of the half plane $\{z :\ \operatorname{Re} z < c_1\}$ under the transformation $w = \dfrac{1}{z}$ is the interior of a circle. What is the image when $c_1 = 0$?

45. Show that the image of the half plane $\{z :\ \operatorname{Im} z > c_2\}$ under the transformation $w = \dfrac{1}{z}$ is the interior of a circle, provided $c_2 > 0$. Find the image when $c_2 < 0$; also find it when $c_2 = 0$.

46. Find the image of the infinite strip $0 < \operatorname{Im} z < \dfrac{1}{2c}$ under the transformation $w = \dfrac{1}{z}$. Sketch the strip and its image.

47. Find the image of the quadrant $\operatorname{Re} z > 1,\ \operatorname{Im} z > 0$ under the transformation $w = \dfrac{1}{z}$.

48. Find the image of the semi-infinite strip $\operatorname{Re} z > 0$, $0 < \operatorname{Im} z < 1$ when $w = \dfrac{i}{z}$. Sketch the strip and its image.

49. Find the bilinear transformation that maps the points $z_1 = 2,\ z_2 = i,\ z_3 = -2$ onto the points $w_1 = 1$, $w_2 = i,\ w_3 = -1$.

50. Find the bilinear transformation that maps the points $z_1 = -i,\ z_2 = 0,\ z_3 = i$ onto the points $w_1 = -1$, $w_2 = i,\ w_3 = 1$. Into what curve is the imaginary axis $\operatorname{Re} z = 0$ transformed?

51. Find the bilinear transformation that maps the points $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ onto the points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

52. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$.

53. Show that a composition of two bilinear transformations is again a bilinear transformation.

54. Show that $\dfrac{i-z}{i+z}$ maps the half plane $\operatorname{Im} z > 0$ onto the disk $|w| < 1$.

55. Show that $\dfrac{z-1}{z+1}$ maps the half plane $\operatorname{Re} z > 0$ onto the disk $|w| < 1$.

56. By finding the inverse of the transformation $\dfrac{i-z}{i+z}$, show that $i\dfrac{1-z}{1+z}$ maps the disk $|z| < 1$ onto the half plane $\operatorname{Im} w > 0$.

57. Show that the bilinear transformation $\dfrac{z-2}{z}$ can be written $z_1 = z - 1$, $z_2 = i\dfrac{1-z_1}{1+z_1}$, $w = iz_2$. Then with the aid of the result in Exercise 56, verify that it maps the disk $|z-1| < 1$ onto the left half plane $\operatorname{Re} w < 0$.

58. Map $\mathbb{C}\backslash\{z : \ -1 \leq z \leq 1\}$ onto the upper half plane $\{w : \ \operatorname{Im} w > 0\}$.

59. Find the linear fractional transformation that maps the region $\{z : \ |z-2| < 2\}$ onto the right half plane $\{w : \ \operatorname{Re} z > 0\}$ in such a way that 1 and 0 are the fixed points of the transformation.

60. Find the linear fractional transformation $w = w(z)$ that maps $\operatorname{Re} z > 1$ onto $|w-2| < 2$ so that $z = 2$ is a fixed point and $w(1) = 0$.