MCS 352 2009-2010 Spring
Exercise Set XIII

1. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\oint_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, showing that the value of this integral tends to zero as N tends to infinity, point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

2. Let $f(z)$ be such that along the path C_N, described in Exercise 1, $|f(z)| \leq \dfrac{M}{|z|^k}$ where $k > 1$ and M are constants independent of N. Prove that

$$\sum_{n=-\infty}^{\infty} f(n) = -\{\text{sum of residues of } \pi \cot \pi z f(z) \text{ at the poles of } f(z)\}$$

3. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{a} \coth \pi a$ where $a > 0$.

4. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2}$ where $a > 0$.

5. If $f(z)$ satisfies the same conditions given in Exercise 2, prove that

$$\sum_{n=-\infty}^{\infty} (-1)^n f(n) = -\{\text{sum of residues of } \pi \csc \pi z f(z) \text{ at the poles of } f(z)\}$$

6. Prove that $\displaystyle\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2} = \frac{\pi^2 \cos \pi a}{\sin^2 \pi a}$ where a is real and different from $0, \pm 1, \pm 2, \cdots$.

7. Prove that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \cdots = \dfrac{\pi^3}{32}$.

8. Prove that

$$\frac{\coth \pi}{1^3} + \frac{\coth 2\pi}{2^3} + \frac{\coth 3\pi}{3^3} + \cdots = \frac{7\pi^3}{180}.$$

9. Prove that

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n \sin n\theta}{n^2+\alpha^2} = \frac{\pi}{2}\frac{\sinh \alpha\theta}{\sinh \alpha\pi}, \ -\pi < \theta < \pi$.

(e) $\displaystyle\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$.

(f) $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^4+n^2+1} = \frac{\pi\sqrt{3}}{3}\tanh\left(\frac{\pi\sqrt{3}}{2}\right)$.

(g) $\displaystyle\frac{1}{1^5} - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} + \cdots = \frac{5\pi^5}{1536}$.

10. Use the theorem involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

(a) $\displaystyle\frac{z^5}{1-z^3}$.

(b) $\displaystyle\frac{1}{1+z^2}$.

(c) $\displaystyle\frac{1}{z}$.

11. Let the degrees of the polynomials

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \ a_n \neq 0$$

and

$$Q(z) = b_0 + b_1 z + b_2 z^2 + \cdots + b_m z^m, \ b_m \neq 0$$

be such that $m \geq n+2$. Use the theorem involving a single residue, to show that if all of the zeros of $Q(z)$ are interior to a simple closed contour C, then

$$\oint_C \frac{P(z)}{Q(z)}\, dz = 0.$$

12. Use the theorem involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

(a) $\displaystyle f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$.

(b) $\displaystyle f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$.

(c) $\displaystyle f(z) = \frac{z^3 e^{\frac{1}{z}}}{1+z^3}$.

13. Prove that the sum of the residues of the function $\displaystyle\frac{2z^5-4z^2+5}{3z^6-8z+10}$ at all the poles is $\frac{2}{3}$.

14. Evaluate $\displaystyle\oint_{|z|=3} \frac{z^5 \cos \frac{1}{z-2}}{(z^2+3)^4(z-5)}\, dz.$

15. Evaluate $\displaystyle\oint_{|z|=1} \frac{z^6}{1-2z^8}\, dz.$

16. Evaluate $\displaystyle\oint_{|z|=528} \frac{z^{99}}{(z-1)(z-2)\cdots(z-100)}\, dz.$

17. Let $F(s)$ be the Laplace transform of $f(t)$. Using residues, find $f(t)$ whenever

(a) $F(s) = \dfrac{2s^3}{s^4-4}.$

(b) $F(s) = \dfrac{2s-2}{(s+1)(s^2+2s+5)}.$

(c) $F(s) = \dfrac{12}{s^3+8}.$

(d) $F(s) = \dfrac{s^2-a^2}{(s^2+a^2)^2},\ a > 0.$

(e) $F(s) = \dfrac{8a^3s^2}{(s^2+a^2)^3},\ a > 0.$

(f) $F(s) = \dfrac{\sinh(xs)}{s^2 \cosh s},\ 0 < x < 1.$

(g) $F(s) = \dfrac{1}{s \cosh(s^{\frac{1}{2}})}.$

(h) $F(s) = \dfrac{\coth(\pi s/2)}{s^2+1}.$

(i) $F(s) = \dfrac{\sinh(xs^{\frac{1}{2}})}{s^2 \sinh(s^{\frac{1}{2}})},\ 0 < x < 1.$

(j) $F(s) = \dfrac{1}{s^2} - \dfrac{1}{s \sinh s}.$

(k) $F(s) = \dfrac{\sinh(xs)}{s(s^2+\omega^2)\cosh s},\ 0 < x < 1$, where $\omega > 0$ and $\omega \neq \omega_n = \dfrac{(2n-1)\pi}{2},\ n = 1, 2, \cdots.$

(l) $F(s) = \dfrac{4s+3}{s^3+2s^2+s+2}.$

(m) $F(s) = \dfrac{1}{s^2+4}.$

(n) $F(s) = \dfrac{s+3}{(s-2)(s^2+1)}.$

(o) $F(s) = \dfrac{s^3+s^2-s+3}{s^5-s}.$

(p) $F(s) = \dfrac{s^3+2s^2-s+2}{s^5-s}.$

(q) $F(s) = \dfrac{s^3+3s^2-s+1}{s^5-s}.$

(r) $F(s) = \dfrac{s^3+s^2+s+3}{s^5-s}.$

(s) $F(s) = \dfrac{s^3+2s^2+4s+2}{(s^2+1)(s^2+4)}.$