MCS 352 2009-2010 Spring
Exercise Set XII

1. Find the residue at $z = 0$ of the function

   (a) $\dfrac{1}{z+z^2}$.

   (b) $z \cos\left(\dfrac{1}{z}\right)$.

   (c) $\dfrac{z - \sin z}{z}$.

   (d) $\dfrac{\cot z}{z^4}$.

   (e) $\dfrac{\sinh z}{z^4(1-z^2)}$.

2. Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

   (a) $\dfrac{e^{-z}}{z^2}$.

   (b) $\dfrac{e^{-z}}{(z-1)^2}$.

   (c) $z^2 e^{\frac{1}{z}}$.

   (d) $\dfrac{z+1}{z^2-2z}$.

3. Let C denote the circle $|z| = 1$, taken counterclockwise, and follow the steps below to show that

$$\oint_C \exp\left(z + \frac{1}{z}\right) dz = 2\pi i \sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}.$$

   (a) By using the Maclaurin series for e^z and the termwise integration, write the above integral as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \oint_C z^n \exp\left(\frac{1}{z}\right) dz.$$

   (b) Apply Cauchy's residue theorem to evaluate the integrals appearing in part (a) to arrive at the desired result.

4. In each case, find the isolated singularity of the given function and determine whether that point is a pole, a removable singular point, or an essential singular point:

   (a) $ze^{\frac{1}{z}}$.

   (b) $\dfrac{z^2}{1+z}$.

   (c) $\dfrac{\sin z}{z}$.

   (d) $\dfrac{\cos z}{z}$.

   (e) $\dfrac{1}{(2-z)^3}$.

5. Show that the singular point of each of the following functions is a pole. Determine the order of that pole and the corresponding residue.

   (a) $\dfrac{1-\cosh z}{z^3}$.

   (b) $\dfrac{1-e^{2z}}{z^4}$.

   (c) $\dfrac{e^{2z}}{(z-1)^2}$.

6. Suppose that a function f is analytic at z_0, write $g(z) = \dfrac{f(z)}{z-z_0}$. Show that

   (a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$.

   (b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

7. Write the function

$$f(z) = \frac{8a^3z^2}{(z^2+a^2)^3}, \ a > 0$$

   as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3z^2}{(z+ai)^3}.$$

   Point out why $\phi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of f at that point is

$$\frac{\phi''(ai)/2}{z-ai} + \frac{\phi'(ai)}{(z-ai)^2} + \frac{\phi(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}.$$

8. In each case, show that any singular point of the function is a pole. Determine the order of each pole, and find the corresponding residue.

   (a) $\dfrac{z^2+2}{z-1}$.

   (b) $\left(\dfrac{z}{2z+1}\right)^3$.

   (c) $\dfrac{e^z}{z^2+\pi^2}$.

9. Show that

   (a) $\operatorname*{Res}_{z=-1} \dfrac{z^{\frac{1}{4}}}{z+1} = \dfrac{1+i}{\sqrt{2}}$, $0 < \arg z < 2\pi$.

   (b) $\operatorname*{Res}_{z=i} \dfrac{\operatorname{Log} z}{(z^2+1)^2} = \dfrac{\pi+2i}{8}$.

   (c) $\operatorname*{Res}_{z=i} \dfrac{z^{\frac{1}{2}}}{(z^2+1)^2} = \dfrac{1-i}{8\sqrt{2}}$, $0 < \arg z < 2\pi$.

10. Find the value of the integral

$$\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\,dz,$$

taken counterclockwise around the circle

   (a) $|z-2| = 2$.

   (b) $|z| = 4$.

11. Find the value of the integral

$$\oint_C \frac{1}{z^3(z+4)}\,dz,$$

taken counterclockwise around the circle

   (a) $|z| = 2$.

   (b) $|z+2| = 3$.

12. Evaluate the integral

$$\oint_C \frac{\cosh \pi z}{z(z^2+1)}\,dz$$

where C is the circle $|z| = 2$, described in the positive sense.

13. Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity.

14. Show that

   (a) $\operatorname*{Res}_{z=\pi i} \dfrac{z-\sinh z}{z^2 \sinh z} = \dfrac{i}{\pi}$.

   (b) $\operatorname*{Res}_{z=\pi i} \dfrac{e^{zt}}{\sinh z} + \operatorname*{Res}_{z=-\pi i} \dfrac{e^{zt}}{\sinh z} = -2\cos \pi t$.

15. Show that

   (a) $\operatorname*{Res}_{z=z_n} (z \sec z) = (-1)^{n+1} z_n$, where $z_n = \dfrac{\pi}{2} + n\pi$, $n = 0, \pm 1, \pm 2, \cdots$.

   (b) $\operatorname*{Res}_{z=z_n} (\tanh z) = 1$, where $z_n = \left(\dfrac{\pi}{2} + n\pi\right) i$, $n = 0, \pm 1, \pm 2, \cdots$.

16. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

   (a) $\displaystyle\oint_C \tan z\,dz$.

   (b) $\displaystyle\oint_C \frac{1}{\sinh 2z}\,dz$.

17. Let C_N denote the positively oriented boundary of the square whose edges lie along the lines

$$x = \pm\left(N + \frac{1}{2}\right)\pi \quad \text{and} \quad y = \pm\left(N + \frac{1}{2}\right)\pi,$$

where N is a positive integer. Show that

$$\oint_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \left[\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right].$$

Then, showing that the value of this integral tends to zero as N tends to infinity, point out how it follows that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

18. Show that

$$\oint_C \frac{dz}{(z^2-1)^2+3} = \frac{\pi}{2\sqrt{2}},$$

where C is the positively oriented boundary of the rectangle whose sides lie along the lines $x = \pm 2$, $y = 0$, and $y = 1$.

19. Consider the function

$$f(z) = \frac{1}{[q(z)]^2},$$

where q is analytic at z_0, $q(z_0) = 0$, and $q'(z_0) \neq 0$. Show that z_0 is a pole of order 2 of the function f, with residue $-\dfrac{q''(z_0)}{[q'(z_0)]^3}$.

20. Use the result in Exercise 19 to find the residue at $z = 0$ of the function

(a) $f(z) = \csc^2 z$.

(b) $f(z) = \dfrac{1}{(z+z^2)^2}$.

21. Use residues to evaluate the improper integrals.

(a) $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$.

(b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$.

(c) $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$.

(d) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx$.

(e) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx$.

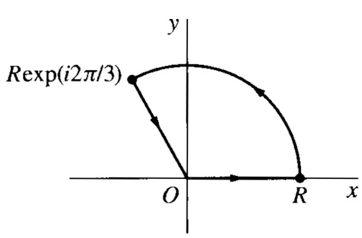
22. Use residues to find the Cauchy principal values of the integrals.

(a) $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$.

(b) $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+1)(x^2+2x+2)}\,dx$.

23. Use residues and the contour shown below, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$



24. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n}\csc\left(\frac{2m+1}{2n}\pi\right).$$

(a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right],$$

$k = 0, 1, 2, \cdots, n-1$, and that there are none on that axis.

(b) Show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha},$$

$k = 0, 1, 2, \cdots, n-1$, where c_k are the zeros found in part (a) and

$$\alpha = \frac{2m+1}{2n}\pi.$$

Then use the summation formula

$$\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}, \quad z \ne 1$$

to obtain the expression

$$2\pi i \sum_{k=0}^{n-1} \operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n\sin\alpha}.$$

(c) Use the final result in part (b) to complete the derivation of the integration formula.

25. The integration formula

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}[(2a^2+3)\sqrt{A+a}+a\sqrt{A-a}],$$

where a is any real number and $A = \sqrt{a^2+1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating. Follow the steps below to derive it.

(a) Point out why the four zeros of the polynomial

$$q(z) = (z^2-a)^2+1$$

are the square roots of the numbers $a \pm i$. Then, showing that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a}+i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a+i$, verify that $\pm\overline{z_0}$ are the square roots of $a-i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half-plane $\operatorname{Im} z \ge 0$.

(b) Using the method derived in Exercise 19, and keeping in mind that $z_0^2 = a+i$ for purposes of simplification, show that the point z_0 in part (a) is a pole of order 2 of the function $f(z) = \dfrac{1}{[q(z)]^2}$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a-i(2a^2+3)}{16A^2z_0}.$$

After observing that $q'(-\overline{z}) = -\overline{q'(z)}$ and $q''(-\overline{z}) = -\overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part (a) is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i}\,\mathrm{Im}\left[\frac{-a + i(2a^2+3)}{z_0}\right]$$

for the sum of the residues.

(c) Refer to part (a) and show that $q(z) \geq (R - |z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

26. Use residues to evaluate.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx,\ a > b > 0.$

(b) $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx,\ a > 0.$

(c) $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx,\ a > 0,\ b > 0.$

(d) $\displaystyle\int_{0}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx.$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx,\ a > 0.$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx,\ a > 0.$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)}\,dx.$

(h) $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin x}{(x^2+1)(x^2+9)}\,dx.$

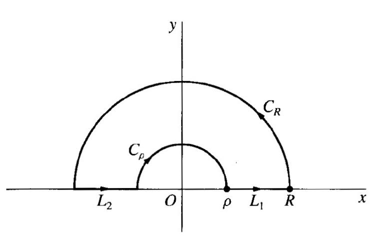
27. Use residues to find the Cauchy principal values of the improper integrals.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx.$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx.$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x+a)^2+b^2}\,dx,\ b > 0.$

28. Using the indented contour below



(a) derive the integration formula

$$\int_0^{\infty} \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a)$$

$a \geq 0,\ b \geq 0$. Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

(b) evaluate the improper integral

$$\int_0^{\infty} \frac{x^a}{(x^2+1)^2}\,dx,$$

where $-1 < a < 3$ and $x^a = \exp(a \ln x)$.

(c) use the function

$$f(z) = \frac{z^{\frac{1}{3}} \log z}{z^2+1} = \frac{e^{\frac{1}{3}\log z} \log z}{z^2+1},$$

$|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, to derive this pair of integration formulas:

$$\int_0^{\infty} \frac{\sqrt[3]{x} \ln x}{x^2+1}\,dx = \frac{\pi^2}{6},$$

and

$$\int_0^{\infty} \frac{\sqrt[3]{x}}{x^2+1}\,dx = \frac{\pi}{\sqrt{3}}.$$

(d) use the function

$$f(z) = \frac{(\log z)^2}{z^2+1},$$

$|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, to show that

$$\int_0^{\infty} \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8},$$

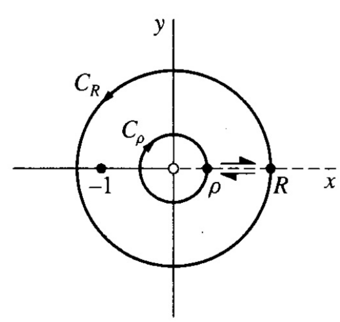
and

$$\int_0^{\infty} \frac{\ln x}{x^2+1}\,dx = 0.$$

29. Use the function

$$f(z) = \frac{z^{\frac{1}{3}}}{(z+a)(z+b)} = \frac{e^{\frac{1}{3}\log z}}{(z+a)(z+b)},$$

$|z| > 0,\ 0 < \arg z < 2\pi$, and closed contour similar to the one below to show formally that



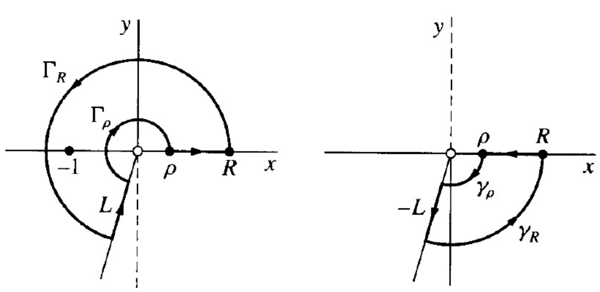
$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\,\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b},$$

$a > b > 0$.

30. Show that
$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$
by integrating an appropriate branch of the multivalued function
$$f(z) = \frac{z^{-\frac{1}{2}}}{z^2+1} = \frac{e^{-\frac{1}{2}\log z}}{z^2+1}$$
over
   (a) the indented path in Exercise 28.
   (b) the closed contour in Exercise 29.

31. The *beta function* is this function of two variables
$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt,$$
$p > 0,\ q > 0$. Make the substitution $t = \dfrac{1}{x+1}$ and use the result
$$\int_0^\infty \frac{x^a}{x+1} = -\frac{\pi}{\sin a\pi},\ -1 < a < 0,$$
obtained in class, to show that
$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)},\ 0 < p < 1.$$

32. Consider the two simple closed contours shown below and obtained by dividing into two pieces the annulus formed by the circles C_ρ and C_R in the picture used in Exercise 29. The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < \frac{3\pi}{2}$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.



   (a) Show how it follows from Cauchy's residue theorem that when the branch
$$f_1(z) = \frac{z^{-a}}{z+1},$$
$|z| > 0,\ -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$, of the multivalued function $\dfrac{z^{-a}}{z+1}$ is integrated around the closed contour on the left of figure above,
$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{\Gamma_R} f_1(z)\,dz + \int_L f_1(z)\,dz + \int_{\Gamma_\rho} f_1(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z).$$

   (b) Apply the Cauchy-Goursat theorem to the branch
$$f_2(z) = \frac{z^{-a}}{z+1},$$
$|z| > 0,\ \frac{\pi}{2} < \arg z < \frac{5\pi}{2}$, of $\dfrac{z^{-a}}{z+1}$, integrated around the closed contour on the right of figure above, to show that
$$-\int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{\gamma_\rho} f_2(z)\,dz - \int_L f_2(z)\,dz + \int_{\gamma_R} f_2(z)\,dz = 0.$$

   (c) Point out why, in the last lines in parts (a) and (b), the branches $f_1(z)$ and $f_2(z)$ of $\dfrac{z^{-a}}{z+1}$ can be replaced by the branch
$$f(z) = \frac{z^{-a}}{z+1}$$
$|z| > 0,\ 0 < \arg z < 2\pi$. Then, by adding corresponding sides of those two lines, derive equation
$$\int_\rho^R \frac{r^{-a}}{r+1}\,dr + \int_{C_R} f(z)\,dz - \int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}\,dr + \int_{C_\rho} f(z)\,dz = 2\pi i \operatorname*{Res}_{z=-1} f_1(z),$$
which was obtained only formally in class.

33. Use residues to evaluate
   (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.
   (b) $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$.
   (c) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\,d\theta}{5-4\cos 2\theta}$.

(d) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}$, $-1 < a < 1$.

(e) $\displaystyle\int_0^{\pi} \frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$, $-1 < a < 1$.

(f) $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$, $a > 1$.

(g) $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta$, $n = 1, 2, 3 \cdots$.

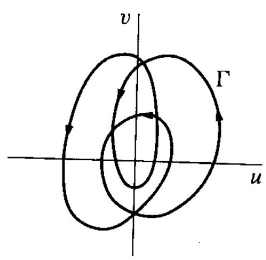
34. Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the value of $\triangle_C \arg f(z)$ when

(a) $f(z) = z^2$.

(b) $f(z) = \dfrac{z^3+2}{z}$.

(c) $f(z) = \dfrac{(2z-1)^7}{z^3}$.

35. Let f be a function which is analytic inside and on a simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown below. Determine the value of $\triangle_C \arg f(z)$ from that figure; and determine the number of zeros, counting multiplicities, of interior to C.



36. Suppose that a function f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Show that if f has n zeros z_k, $k = 1, 2, \cdots, n$, inside C, where each z_k is of multiplicity m_k, then

$$\oint_C \frac{zf'(z)}{f(z)}\, dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

37. Determine the number of zeros, counting multiplicities, of the polynomial

(a) $z^6 - 5z^4 + z^3 - 2z$.

(b) $2z^4 - 2z^3 + 2z^2 - 2z + 9$.

inside the circle $|z| = 1$.

38. Determine the number of zeros, counting multiplicities, of the polynomial

(a) $z^4 + 3z^3 + 6$.

(b) $z^4 - 2z^3 + 9z^2 + z - 1$.

(c) $z^5 + 3z^3 + z^2 + 1$.

inside the circle $|z| = 2$.

39. Determine the number of roots,counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| < 2$.

40. Show that if c is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has n roots, counting multiplicities, inside the circle $|z| = 1$.