MCS 352 2009-2010 Spring
Exercise Set XI

1. Find $\operatorname{Res}_{z=0} f(z)$ for
   (a) $f(z) = z^{-1}e^z$.
   (b) $f(z) = z^{-3}\cosh 4z$.
   (c) $f(z) = \csc z$.
   (d) $f(z) = \dfrac{z^2+4z+5}{z^2+z}$.
   (e) $f(z) = \cot z$.
   (f) $f(z) = z^{-3}\cos z$.
   (g) $f(z) = z^{-1}\sin z$.
   (h) $f(z) = \dfrac{z^2+4z+5}{z^3}$.
   (i) $f(z) = \exp\left(1+\dfrac{1}{z}\right)$.
   (j) $f(z) = z^4\sin\left(\dfrac{1}{z}\right)$.
   (k) $f(z) = z^{-1}\csc z$.
   (l) $f(z) = z^{-2}\csc z$.
   (m) $f(z) = \dfrac{e^{4z}-1}{\sin^2 z}$.
   (n) $f(z) = z^{-1}\csc^2 z$.

2. Evaluate
   (a) $\displaystyle\oint_{C_1(-1+i)} \frac{dz}{z^4+4}$.
   (b) $\displaystyle\oint_{C_2(i)} \frac{dz}{z(z^2-2z+2)}$.
   (c) $\displaystyle\oint_{C_2(0)} \frac{e^z}{z^3+z}\,dz$.
   (d) $\displaystyle\oint_{C_2(0)} \frac{\sin z}{4z^2-\pi^2}\,dz$.
   (e) $\displaystyle\oint_{C_2(0)} \frac{\sin z}{z^2+1}\,dz$.
   (f) $\displaystyle\oint_{C_1(0)} \frac{dz}{z^2\sin z}$.
   (g) $\displaystyle\oint_{C_1(0)} \frac{dz}{z\sin^2 z}$.

3. Let f and g be analytic at z_0. If $f(z_0)\neq 0$ and g has a simple zero at z_0, then show that $\operatorname{Res}_{z=z_0}\dfrac{f(z)}{g(z)} = \dfrac{f(z_0)}{g'(z_0)}$.

4. Find $\displaystyle\oint_C (z-1)^{-2}(z^2+4)^{-1}\,dz$ when
   (a) $C = C_1(1)$.
   (b) $C = C_4(0)$.

5. Find $\displaystyle\oint_C (z^6+1)^{-1}\,dz$ when
   (a) $C = C_{\frac{1}{2}}(i)$.
   (b) $C = C_1(\frac{1+i}{2})$. *Hint*: If z_0 is a singularity of $f(z) = \dfrac{1}{z^6+1}$, show that $\operatorname{Res}_{z=z_0} f(z) = -\frac{1}{6}z_0$.

6. Find $\displaystyle\oint_C (3z^4+10z^2+3)^{-1}\,dz$ when
   (a) $C = C_1(i\sqrt{3})$.
   (b) $C = C_1(\frac{i}{\sqrt{3}})$.

7. Find $\displaystyle\oint_C (z^4-z^3-2z^2)^{-1}\,dz$ when
   (a) $C = C_{\frac{1}{2}}(0)$.
   (b) $C = C_{\frac{3}{2}}(0)$.

8. Use residues to find the partial fraction representations of
   (a) $\dfrac{1}{z^2+3z+2}$.
   (b) $\dfrac{3z-3}{z^2-z-2}$.
   (c) $\dfrac{z^2-7z+4}{z^2(z+4)}$.
   (d) $\dfrac{10z}{(z^2+4)(z^2+9)}$.
   (e) $\dfrac{2z^2-3z-1}{(z-1)^3}$.
   (f) $\dfrac{z^3+3z^2-z+1}{z(z+1)^2(z^2+1)}$.

9. Let f be analytic in a simply connected domain D, and let C be a simple closed positively oriented contour in D. If z_0 is the only zero of f in D and z_0 lies interior to C, then show that $\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = k$, where k is the order of the zero at z_0.

10. Let f be analytic at the points $0, \pm 1, \pm 2, \cdots$. If $g(z) = \pi f(z)\cot \pi z$, then show that $\operatorname{Res}_{z=n} g(z) = f(n)$ for $n = 0, \pm 1, \pm 2, \cdots$.

11. Use residues to find

(a) $\displaystyle\int_0^{2\pi} \frac{1}{3\cos\theta + 5}\,d\theta.$

(b) $\displaystyle\int_0^{2\pi} \frac{1}{4\sin\theta + 5}\,d\theta.$

(c) $\displaystyle\int_0^{2\pi} \frac{1}{15\sin^2\theta + 1}\,d\theta.$

(d) $\displaystyle\int_0^{2\pi} \frac{1}{5\cos^2\theta + 4}\,d\theta.$

(e) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 + 4\cos\theta}\,d\theta.$

(f) $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 3\cos\theta}\,d\theta.$

(g) $\displaystyle\int_0^{2\pi} \frac{1}{(5 + 3\cos\theta)^2}\,d\theta.$

(h) $\displaystyle\int_0^{2\pi} \frac{1}{(5 + 4\cos\theta)^2}\,d\theta.$

(i) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5 + 3\cos\theta}\,d\theta.$

(j) $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{13 - 12\cos\theta}\,d\theta.$

(k) $\displaystyle\int_0^{2\pi} \frac{1}{(1 + 3\cos^2\theta)^2}\,d\theta.$

(l) $\displaystyle\int_0^{2\pi} \frac{1}{(1 + 8\cos^2\theta)^2}\,d\theta.$

(m) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\,d\theta.$

(n) $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 3\cos 2\theta}\,d\theta.$

(o) $\displaystyle\int_0^{2\pi} \frac{1}{a\cos\theta + b\sin\theta + d}\,d\theta$, where a, b, and d are real and $a^2 + b^2 < d^2$.

(p) $\displaystyle\int_0^{2\pi} \frac{1}{a\cos^2\theta + b\sin^2\theta + d}\,d\theta$, where a, b, and d are real and $a > d$ and $b > d$.

12. Use residues to evaluate

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+16)^2}\,dx.$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+16}\,dx.$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+9)^2}\,dx.$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{x+3}{(x^2+9)^2}\,dx.$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{2x^2+3}{(x^2+9)^2}\,dx.$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4+4}\,dx.$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+4}\,dx.$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^3}\,dx.$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2(x^2+4)}\,dx.$

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{x+2}{(x^2+4)(x^2+9)}\,dx.$

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{3x^2+2}{(x^2+4)(x^2+9)}\,dx.$

(l) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx.$

(m) $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{x^6+1}\,dx.$

(n) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)(x^2+b^2)}\,dx$, where $a > 0$ and $b > 0$.

(o) $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}\,dx$, where $a > 0$.

13. Use residues to find the Cauchy principal value of

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+9}\,dx$ and $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+9}\,dx.$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\cos x}{x^2+9}\,dx$ and $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}\,dx.$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+4)^2}\,dx.$

(d) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)^2}\,dx.$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+9)}\,dx.$

(f) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\,dx.$

(g) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+5}\,dx.$

(h) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 4x + 5}\, dx$.

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^4 + 4}\, dx$.

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4 + 4}\, dx$.

(k) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 2x + 2}\, dx$.

(l) $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin 2x}{x^4 + 4}\, dx$.

14. Use residues to compute

(a) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^3 + x}\, dx$.

(b) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^3 + 1}\, dx$.

(c) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^3 + 1}\, dx$.

(d) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4 - 1}\, dx$.

(e) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x^4}{x^6 - 1}\, dx$.

(f) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$.

(g) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - x}\, dx$.

(h) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(\pi^2 - x^2)}\, dx$.

(i) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{\pi^2 - 4x^2}\, dx$.

(j) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2 + 1)}\, dx$.

(k) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{x \cos x}{x^2 + 3x + 2}\, dx$.

(l) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x(1 - x^2)}\, dx$.

(m) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{a^2 - x^2}\, dx$.

(n) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$. *Hint*: Use trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

15. Use residues to compute

(a) P.V. $\displaystyle\int_{0}^{\infty} \frac{1}{x^{\frac{2}{3}}(1 + x)}\, dx$.

(b) P.V. $\displaystyle\int_{0}^{\infty} \frac{1}{x^{\frac{1}{2}}(1 + x)}\, dx$.

(c) P.V. $\displaystyle\int_{0}^{\infty} \frac{x^{\frac{1}{2}}}{(1 + x)^2}\, dx$.

(d) P.V. $\displaystyle\int_{0}^{\infty} \frac{x^{\frac{1}{2}}}{1 + x^2}\, dx$.

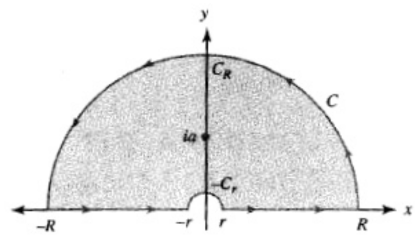
(e) P.V. $\displaystyle\int_{0}^{\infty} \frac{\ln(x^2 + 1)}{x^2 + 1}\, dx$. *Hint*: Use the integrand $f(z) = \dfrac{\log(z + i)}{z^2 + 1}$.

(f) P.V. $\displaystyle\int_{0}^{\infty} \frac{\ln x}{(1 + x^2)^2}\, dx$.

(g) P.V. $\displaystyle\int_{0}^{\infty} \frac{\ln(1 + x)}{x^{1+a}}\, dx$, where $0 < a < 1$.

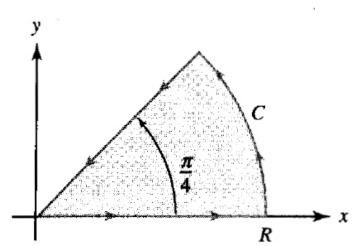
(h) P.V. $\displaystyle\int_{0}^{\infty} \frac{\ln x}{(x + a)^2}\, dx$, where $a > 0$.

(i) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$. *Hint*: Use the integrand $f(z) = \dfrac{e^{iz}}{z}$ and the contour C shown below. Let $r \to 0$ and $R \to \infty$.



(j) P.V. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$. *Hint*: Use the integrand $f(z) = \dfrac{1 - e^{i2z}}{z^2}$ and the contour C shown above. Let $r \to 0$ and $R \to \infty$.

(k) The Fresnel integrals $\displaystyle\int_0^{\infty} \cos(x^2)\, dx$ and $\displaystyle\int_0^{\infty} \sin(x^2)\, dx$ are important in the study of optics. Use the integrand $f(z) = e^{-z^2}$ and the contour shown below, and let $R \to \infty$ to get the value of these integrals. Use the fact from calculus that $\displaystyle\int_0^{\infty} e^{-x^2}\, dx = \sqrt{\frac{\pi}{2}}$.



16. Let $f(z) = z^5 - z$. Find the number of times the image $f(C)$ winds around the origin if

(a) $C = C_{\frac{1}{2}}(0)$.

(b) C is the rectangle with vertices $\pm\dfrac{1}{2} \pm 3i$.

(c) $C = C_2(0)$.

(d) $C = C_{1.25}(i)$.

17. Show that four of the five roots of the equation $z^5 + 15z + 1 = 0$ belong to the annulus $A(0, \frac{3}{2}, 2) = \{z : \frac{3}{2} < |z| < 2\}$.

18. Let $g(z) = z^5 + 4z - 15$.

    (a) Show that there are no zeros in $D_1(0)$.

    (b) Show that there are five zeros in $D_2(0)$. *Hint*: Consider $f(z) = -z^5$.

19. Let $g(z) = z^3 + 9z + 27$.

    (a) Show that there are no zeros in $D_2(0)$.

    (b) Show that there are three zeros in $D_4(0)$.

20. Let $g(z) = z^5 + 6z^2 + 2z + 1$.

    (a) Show that there are two zeros in $D_1(0)$.

    (b) Show that there are five zeros in $D_2(0)$.

21. Let $g(z) = z^6 - 5z^4 + 10$.

    (a) Show that there are no zeros in $|z| < 1$.

    (b) Show that there are four zeros in $|z| < 2$.

    (c) Show that there are six zeros in $|z| < 3$.

22. Let $g(z) = 3z^3 - 2iz^2 + iz - 7$.

    (a) Show that there are no zeros in $|z| < 1$.

    (b) Show that there are three zeros in $|z| < 2$.

23. Use Rouché's theorem to prove the fundamental theorem of algebra. *Hint*: For the polynomial $g(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$, let $f(z) = -a_n z^n$. Show that, for points z on the circle $C_R(0)$,

$$\left|\frac{f(z) + g(z)}{f(z)}\right| < \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n|R},$$

    and conclude that the right side of this inequality is less than 1 when R is large.

24. Suppose that $h(z)$ is analytic and nonzero and $|h(z)| < 1$ for $z \in D_1(0)$. Prove that the function $g(z) = h(z) - z^n$ has n zeros inside the unit circle $C_1(0)$.

25. Suppose that $f(z)$ is analytic inside and on the simple closed contour C. If $f(z)$ is a one-to-one function at points z on C, then prove that $f(z)$ is one-to-one inside C. *Hint*: Consider the image of C.