MCS 352 2009-2010 Spring
Exercise Set X

1. Obtain the Maclaurin series representation
$$z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}, \quad z \in \mathbb{C}.$$

2. Obtain the Taylor series
$$e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}, \quad z \in \mathbb{C}$$
for the function $f(z) = e^z$ by
   (a) using $f^{(n)}(1)$, $n = 0, 1, 2, 3 \ldots,$
   (b) writing $e^z = e^{z-1}e$.

3. Find the Maclaurin series expansion of the function
$$f(z) = \frac{z}{z^4 + 9}.$$

4. Write the Maclaurin series representation of the function $f(z) = \sin(z^2)$, and point out how it follows that
$$f^{(4n)}(0) = 0 \quad \text{and} \quad f^{(2n+1)}(0) = 0, \; n = 0, 1, 2, \ldots.$$

5. With the aid of the identity
$$\cos z = -\sin\left(z - \frac{\pi}{2}\right),$$
expand $\cos z$ into a Taylor series about the point $z_0 = \frac{\pi}{2}$.

6. Use the identity $\sinh(z + \pi i) = -\sinh z$, and the fact $\sinh z$ is periodic with period $2\pi i$ to find the Taylor series for $\sinh z$ about the point $z_0 = \pi i$.

7. What is the largest circle within which the Maclaurin series for the function $\tanh z$ converges to $\tanh z$? Write the first two nonzero terms of that series.

8. Show that when $z \neq 0$,
   (a) $\dfrac{e^z}{z^2} = \dfrac{1}{z^2} + \dfrac{1}{z} + \dfrac{1}{2!} + \dfrac{z}{3!} + \dfrac{z^2}{4!} + \cdots,$
   (b) $\dfrac{\sin(z^2)}{z^4} = \dfrac{1}{z^2} - \dfrac{z^2}{3!} + \dfrac{z^6}{5!} - \dfrac{z^{10}}{7!} + \cdots.$

9. Derive the expansions
   (a) $\dfrac{\sinh z}{z^2} = \dfrac{1}{z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}, \; |z| > 0,$

(b) $z^3 \cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}}, \ |z| > 0.$

10. Show that when $0 < |z| < 4$,

$$\frac{1}{4z - z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}.$$

11. Find the Laurent series that represent the function

$$f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$$

in the domain $0 < |z| < \infty$.

12. Derive the Laurent series representation

$$\frac{e^z}{(z+1)^2} = \frac{1}{e}\left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2}\right], \ 0 < |z+1| < \infty.$$

13. Find a representation for the function

$$f(z) = \frac{1}{1+z}$$

in negative powers of z that is valid when $1 < |z| < \infty$.

14. Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(1-z)},$$

and specify the regions in which those expansions are valid.

15. Represent the function

$$f(z) = \frac{z+1}{z-1}$$

(a) by its Maclaurin series, and state where the representation is valid,

(b) by its Laurent series in the domain $1 < |z| < \infty$.

16. Show that when $0 < |z-1| < 2$,

$$\frac{z}{(z-1)(z-3)} = -3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}.$$

17. Write the two Laurent series in powers of z that represents the function

$$f(z) = \frac{1}{z(1+z^2)}$$

in certain domains, and specify those domains.

18. (a) Let a denote a real number, where $-1 < a < 1$, and derive the Laurent series representation

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n}, \quad |a| < |z| < \infty.$$

(b) Write $z = e^{i\theta}$ in the equation obtained in part (a) and then equate real parts and imaginary parts on each side of the result to derive the summation formulas

$$\sum_{n=1}^{\infty} a^n \cos n\theta = \frac{a\cos\theta - a^2}{1 - 2a\cos\theta + a^2} \quad \text{and} \quad \sum_{n=1}^{\infty} a^n \sin n\theta = \frac{a\sin\theta}{1 - 2a\cos\theta + a^2},$$

where $-1 < a < 1$.

19. By differentiating the Maclaurin series representation

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \ |z| < 1,$$

obtain the expansions

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n, \ |z| < 1,$$

and

$$\frac{2}{(1-z)^3} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n, \ |z| < 1.$$

20. By substituting $\dfrac{1}{1-z}$ for z in the expansion

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1) z^n, \ |z| < 1,$$

found in Exercise 19, derive the Laurent series representation

$$\frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n}, \ 1 < |z-1| < \infty.$$

21. Find the Taylor series for the function

$$\frac{1}{z} = \frac{1}{2 + (z-2)} = \frac{1}{2} \cdot \frac{1}{1 + (z-2)/2}$$

about the point $z_0 = 2$. Then, by differentiating that series term by term, show that

$$\frac{1}{z^2} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2} \right)^n, \ |z-2| < 2.$$

22. With the aid of series, prove that the function f defined by means of the equations

$$f(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{when} \quad z \neq 0 \\ 1 & \text{when} \quad z = 0 \end{cases}$$

is entire.

23. Prove that if

$$f(z) = \begin{cases} \dfrac{\cos z}{z^2 - (\pi/2)^2} & \text{when} \quad z \neq \pm\pi/2 \\ -\dfrac{1}{\pi} & \text{when} \quad z = \pm\pi/2, \end{cases}$$

then f is an entire function.

24. In the w plane, integrate the Taylor series expansion

$$\frac{1}{w} = \sum_{n=0}^{\infty}(-1)^n(w-1)^n, \; |w-1| < 1$$

along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$\operatorname{Log} z = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(z-1)^n, \; |z-1| < 1.$$

25. Use the result in Exercise 24 to show that if

$$f(z) = \begin{cases} \dfrac{\operatorname{Log} z}{z-1} & \text{when} \quad z \neq 1 \\ 1 & \text{when} \quad z = 1 \end{cases}$$

then f is analytic throughout the domain $0 < |z| < \infty, \; -\pi < \operatorname{Arg} z < \pi$.

26. Prove that if f is analytic at z_0 and $f(z_0) = f'(z_0) = \cdots = f^{(m)}(z_0) = 0$, then the function g defined by the equations

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-z_0)^{m+1}} & \text{when} \quad z \neq z_0 \\ \dfrac{f^{(m+1)}(z_0)}{(m+1)!} & \text{when} \quad z = z_0 \end{cases}$$

is analytic at z_0.

27. Use multiplication of series to show that

$$\frac{e^z}{z(z^2+1)} = \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \cdots \quad 0 < |z| < 1.$$

28. By writing $\csc z = \dfrac{1}{\sin z}$ and then using division, show that

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \left[\frac{1}{(3!)^2} - \frac{1}{5!}\right]z^3 + \cdots \quad 0 < |z| < \pi.$$

29. Use division to obtain the Laurent series representation

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z - \frac{1}{720}z^3 + \cdots \quad 0 < |z| < 2\pi.$$

30. Use the expansion

$$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\cdot\frac{1}{z} + \frac{7}{360}z + \cdots \quad 0 < |z| < \pi,$$

to show that

$$\oint_C \frac{dz}{z^2 \sinh z} = -\frac{\pi i}{3},$$

when C is the unit circle $|z| = 1$.

31. Let $f(z)$ be an entire function that is represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad |z| < \infty.$$

(a) By differentiating the composite function $g(z) = f(f(z))$ successively, find the first three nonzero terms in the Maclaurin series for $g(z)$ and thus show that

$$f(f(z)) = z + 2a_2 z^2 + 2(a_2^2 + a_3)z^3 + \cdots \quad |z| < \infty.$$

(b) Obtain the result in part (a) in a formal manner by writing

$$f(f(z)) = f(z) + a_2(f(z))^2 + a_3(f(z))^3 + \cdots ,$$

replacing $f(z)$ on the right-hand side here by its series representation, and then collecting terms in like powers of z.

(c) By applying the result in part (a) to the function $f(z) = \sin z$, show that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \cdots \quad |z| < \infty.$$

32. The Euler numbers are the numbers $E_n,\ n = 0, 1, 2, \cdots$ in the Maclaurin series representation

$$\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n, \quad |z| < \frac{\pi}{2}.$$

Point out why this representation is valid in the indicated disk and why

$$E_{2n+1} = 0, \quad n = 0, 1, 2, \cdots .$$

Then show that

$$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61.$$