MCS 352 2009-2010 Spring
Exercise Set I

1. Reduce each of these quantities to a real number.

   (a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$.

   (b) $\dfrac{5i}{(1-i)(2-i)(3-i)}$.

   (c) $(1-i)^4$.

2. In each case, sketch the set of points determined by the given condition.

   (a) $|z-1+i|=1$.

   (b) $|z+i|\leq 3$.

   (c) $|z-4i|\geq 4$.

   (d) $|z-4i|+|z+4i|=10$.

   (e) $|z-1|=|z+i|$.

3. Show that.

   (a) $\overline{\overline{z}+3i}=z-3i$.

   (b) $\overline{iz}=-i\overline{z}$.

   (c) $\overline{(2+i)^2}=3-4i$.

   (d) $|(2\overline{z}+5)(\sqrt{2}-i)|=\sqrt{3}|2z+5|$.

4. In each case, sketch the set of points determined by the given condition.

   (a) $\mathrm{Re}(\overline{z}-i)=2$.

   (b) $|2z-i|=4$.

5. Show that

$$\left|\frac{z_1+z_2}{z_3+z_4}\right|\leq\frac{|z_1|+|z_2|}{||z_3|-|z_4||}$$

   whenever $|z_3|\neq|z_4|$.

6. Show that

$$\left|\frac{1}{z^4-4z^2+3}\right|\leq\frac{1}{3}$$

   if z lies on the circle $|z|=2$.

7. Find $\mathrm{Arg}\, z$, if

   (a) $z=\dfrac{i}{-2-2i}$.

   (b) $z=(\sqrt{3}-i)^6$.

8. Find a value of θ in the interval $0\leq\theta<2\pi$ that satisfies the equation $|e^{i\theta}-1|=2$.

9. Use de Moivre's formula to derive the following trigonometric identities.

   (a) $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$.

   (b) $\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$.

10. By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that

    (a) $i(1-\sqrt{3}i)(\sqrt{3}+i)=2(1+\sqrt{3}i)$.

    (b) $\dfrac{5i}{2+i}=1+2i$.

    (c) $(-1+i)^7=-8(1+i)$.

    (d) $(1+\sqrt{3}i)^{-10}=2^{-11}(-1+\sqrt{3}i)$.

11. Find the square roots of

    (a) $2i$.

    (b) $1-\sqrt{3}i$.

    and express them in rectangular coordinates.

12. In each case, find all of the roots in rectangular coordinates and exhibit them as vertices of certain squares.

    (a) $(-16)^{\frac{1}{4}}$.

    (b) $(-8-8\sqrt{3}i)^{\frac{1}{4}}$.

13. Sketch the following sets and determine which are domains.

    (a) $|z-2+i|\leq 1$.

    (b) $|2z+3|>4$.

    (c) $\mathrm{Im}\, z>1$.

    (d) $\mathrm{Im}\, z=1$.

    (e) $0\leq\arg z\leq\dfrac{\pi}{4}$ $(z\neq 0)$.

    (f) $|z-4|\geq|z|$.

14. Which sets in Exercise 13 are neither open nor closed?

15. Which sets in Exercise 13 are bounded?

16. Determine the accumulation points of each of the following sets.

(a) $z_n = i^n,\ n = 1, 2, \cdots$.

(b) $z_n = \dfrac{i^n}{n},\ n = 1, 2, \cdots$.

(c) $0 \le \arg z < \dfrac{\pi}{2}\ (z \ne 0)$.

(d) $z_n = (-1)^n (1+i)\dfrac{n-1}{n},\ n = 1, 2, \cdots$.

17. Express in the form $a + bi$.

(a) $\dfrac{1}{6+2i}$.

(b) $\dfrac{(2+i)(3+2i)}{1-i}$.

(c) $\left(-\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^4$.

(d) $i^n,\ n \in \mathbb{Z}$.

18. Find (in rectangular form) the two values of $\sqrt{-8+6i}$.

19. Show that the n-th roots of 1 (aside from 1) satisfy the cyclotomic equation

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = 0.$$

20. Describe the sets whose points satisfy the following relations.

(a) $|z - i| \le 1$.

(b) $\left|\dfrac{z-1}{z+1}\right| = 1$.

(c) $|z - 2| > |z - 3|$.

(d) $\dfrac{1}{z} = \overline{z}$.

(e) $|z| < 1$ and $\operatorname{Im} z > 0$.

(f) $|z|^2 = \operatorname{Im} z$.

(g) $|z^2 - 1| < 1$.

21. Perform the required calculations and express your answers in the form $a + bi$.

(a) i^{275}.

(b) $\dfrac{1}{i^5}$.

(c) $\operatorname{Re}(i)$.

(d) $\operatorname{Im}(2)$.

(e) $(i-1)^3$.

(f) $(7-2i)(3i+5)$.

(g) $\operatorname{Re}(7+6i) + \operatorname{Im}(5-4i)$.

(h) $\operatorname{Im}\left(\dfrac{1+2i}{3-4i}\right)$.

(i) $\dfrac{(4-i)(1-3i)}{-1+2i}$.

(j) $\overline{(1+i\sqrt{3})(i+\sqrt{3})}$.

22. Evaluate the following quantities.

(a) $\overline{(1+i)(2+i)}(3+i)$.

(b) $\dfrac{3+i}{\overline{2+i}}$.

(c) $\operatorname{Re}((i-1)^3)$.

(d) $\operatorname{Im}((1+i)^{-2})$.

(e) $\dfrac{1+2i}{3-4i} - \dfrac{4-3i}{2-i}$.

(f) $(1+i)^{-2}$.

(g) $\operatorname{Re}((x-iy)^2)$.

(h) $\operatorname{Im}\left(\dfrac{1}{x-iy}\right)$.

(i) $\operatorname{Re}((x+iy)(x-iy))$.

(j) $\operatorname{Im}((x+iy)^3)$.

23. Evaluate the following quantities.

(a) $|(1+i)(2+i)|$.

(b) $\left|\dfrac{4-3i}{2-i}\right|$.

(c) $|z\overline{z}|$, where $z = x + iy$.

(d) $|z-1|^2$, where $z = x + iy$.

24. Which of the following points lie inside the circle $|z - i| = 2$? Explain.

(a) $\dfrac{1}{2} + i$.

(b) $\sqrt{2} + i(\sqrt{2}+1)$.

(c) $2 + 3i$.

(d) $-\dfrac{1}{2} + i\sqrt{3}$.

25. Prove that $\sqrt{2}|z| \ge |\operatorname{Re}(z)| + |\operatorname{Im}(z)|$.

26. Show that $|z_1 - z_2| \le |z_1| + |z_2|$.

27. Show that $||z_1| - |z_2|| \le |z_1 - z_2|$.

28. Find $\operatorname{Arg} z$ for the following values of z.

(a) $1 - i$.

(b) $-\sqrt{3} + i$.

(c) $(-1 - i\sqrt{3})^2$.

(d) $(1-i)^3$.

(e) $\dfrac{2}{1+i\sqrt{3}}$.

(f) $\dfrac{2}{i-1}$.

(g) $\dfrac{1+i\sqrt{3}}{(1+i)^2}$.

(h) $(1+i\sqrt{3})(1+i)$.

29. Use exponential notation to show that

(a) $(\sqrt{3}-i)(1+i\sqrt{3}) = 2\sqrt{3} + 2i$.

(b) $(1+i)^3 = -2 + 2i$.

(c) $2i(\sqrt{3}+i)(1+i\sqrt{3}) = -8$.

(d) $\dfrac{8}{1+i} = 4-4i$.

30. Represent the following complex numbers in polar form.

(a) -4.

(b) $6-6i$.

(c) $-7i$.

(d) $-2\sqrt{3}-2i$.

(e) $\dfrac{1}{(1-i)^2}$.

(f) $\dfrac{6}{i+\sqrt{3}}$.

(g) $3+4i$.

(h) $(5+5i)^3$.

31. Express the following in $a+bi$ form.

(a) $e^{\frac{i\pi}{2}}$.

(b) $4e^{-\frac{i\pi}{2}}$.

(c) $8e^{\frac{7\pi i}{3}}$.

(d) $-2e^{\frac{5\pi i}{6}}$.

(e) $2ie^{-\frac{3\pi i}{4}}$.

(f) $6e^{\frac{2\pi i}{3}}e^{\pi i}$.

(g) $e^2e^{\pi i}$.

(h) $e^{\frac{\pi i}{4}}e^{-\pi i}$.

32. Calculate the following.

(a) $(1-i\sqrt{3})^3(\sqrt{3}+i)^2$.

(b) $\dfrac{(1+i)^3}{(1-i)^5}$.

(c) $(\sqrt{3}+i)^6$.

33. Let z be any nonzero complex number and let n be an integer. Show that $z^n + (\bar{z})^n$ is a real number.

34. Find all the roots in both polar and Cartesian form for each expression.

(a) $(-2+2i)^{\frac{1}{3}}$.

(b) $(-1)^{\frac{1}{5}}$.

(c) $(-64)^{\frac{1}{4}}$.

(d) $(8)^{\frac{1}{6}}$.

(e) $(16i)^{\frac{1}{4}}$.

35. Find the three solutions to $z^{\frac{3}{2}} = 4\sqrt{2}+i4\sqrt{2}$.

36. Find a parametrization of the line that

(a) joins the origin to the point $1+i$.

(b) joins the point 1 to the point $1+i$.

(c) joins the point i to the point $1+i$.

(d) joins the point 2 to the point $1+i$.

37. Sketch the curve $z(t) = t^2+2t+i(t+1)$

(a) for $-1 \le t \le 0$.

(b) for $1 \le t \le 2$.

38. Find a parametrization of the curve that is a portion of the parabola $y = x^2$ that

(a) joins the origin to the point $2+4i$.

(b) joins the point $-1+i$ to the origin.

(c) joins the point $1+i$ to the origin.

39. Find a parametrization of the curve that is a portion of the circle $|z| = 1$ that joins the point $-i$ to i if

(a) the curve is the right semicircle.

(b) the curve is the left semicircle.

40. Find a parametrization of the curve that is a portion of the circle $|z| = 1$ that joins the point 1 to i if

(a) the parametrization is counterclockwise along the quarter circle.

(b) the parametrization is clockwise.

41. Consider the following sets.

(i) $\{z : \operatorname{Re}(z) > 1\}$.

(ii) $\{z : -1 < \operatorname{Im}(z) \le 2\}$.

(iii) $\{z : |z-2-i| \le 2\}$.

(iv) $\{z : |z+3i| > 1\}$.

(v) $\left\{re^{i\theta} : 0 < r < 1, -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$.

(vi) $\left\{re^{i\theta} : r > 1, -\frac{\pi}{4} < \theta < \frac{\pi}{3}\right\}$.

(vii) $\{z : |z| < 1 \text{ or } |z-4| < 1\}$.

(a) Sketch each set.

(b) State, with reasons, which of the following terms apply to the above sets: open; connected; domain; region; closed region; bounded.